A remark on β-locally closed sets

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Abstract

The aim of this note is to show that every subset of a given topological space is the intersection of a preopen and a preclosed set, therefore β-locally closed, and that every topological space is β-submaximal.

1 Introduction

In a recent paper, Gnanambal and Balachandran [1] introduced the classes of β-locally closed sets, β-submaximal spaces and β-LC-continuous functions. The purpose of our note is to show that every subset of any topological space is the intersection of a preopen set and a preclosed set, hence β-locally closed, and therefore every function $f\colon (X,\tau) \to (Y,\sigma)$ is β-LC-continuous. We have felt the need to point out explicitly this observation since over the years several papers have investigated concepts like "pre-locally closed sets" or "β-locally closed sets" which do not have any nontrivial meaning. In addition, we will show that every space is β-submaximal and we will point out that most results of [1] are either trivial or false.

Let A be a subset of a topological space (X,τ). Following Kronheimer [2], we call the interior of the closure of A, denoted by A^+, the *consolidation* of A. Sets included in their consolidation are called *preopen* or *locally dense*. Complements of preopen sets are called *preclosed* and the preclosure of a set A, denoted by $\mathrm{pcl}(A)$, is the intersection of all preclosed supersets of A. Since union of preopen sets is also preopen, the preclosure of every set is in fact a preclosed set. If A is included in the closure of its consolidation, then A is called *β-open* or *semi-preopen*. Complements of β-open sets are called *β-closed*. The β-closure of A, denoted by $\mathrm{cl}_\beta(A)$ is the intersection of all β-closed supersets of A. In [1], Gnanambal and

Balachandran called a set A *β-locally closed* if A is intersection of a β-open and a β-closed set. They defined a set A to be *β-dense* [1] if $\mathrm{cl}_\beta(A) = X$ and called a space X *β-submaximal* [1] if every β-dense subset is β-open. A function $f\colon (X,\tau) \to (Y,\sigma)$ is called *β-LC-continuous* [1] if the preimage of every open subset of Y is β-locally closed in X.

The following implications hold and none of them is reversible:

$$\text{dense} \Rightarrow \text{preopen} \Rightarrow \beta\text{-open} \Rightarrow \beta\text{-locally closed}$$

2 Every set is β-locally closed

PROPOSITION 2.1 *Every subset A of a topological space (X,τ) is the intersection of a preopen and a preclosed set, hence pre-locally closed.*

Proof. Let $A \subseteq (X,\tau)$. Set $A_1 = A \cup (X \setminus \mathrm{cl}(A))$. Since A_1 is dense in X, it is also preopen. Let A_2 be the preclosure of A, i.e., $A_2 = A \cup \mathrm{cl}(\mathrm{int}(A))$. Clearly, A_2 is a preclosed set. Note now that $A = A_1 \cap A_2$. $\Box$

COROLLARY 2.2 (i) *Every set is β-locally closed and every function is β-LC-continuous.*

(i) *Every topological space is β-submaximal.*

Proof. (i) Every preopen (resp. preclosed) set is β-open (resp. β-closed).

(ii) By [1, Corollary 3.24] a topological space is β-submaximal if and only if every set is β-locally closed.

REMARK 2.3 (i) Corollary 2.2 makes [1] trivial.

(ii) Example 3.4 from [1] is wrong as the subset $A = \{\frac{1}{n}\colon n = 1,2,\ldots\} \cup (2,3) \cup (3,4) \cup \{4\} \cup (5,6) \cup \{x\colon x \text{ is irrational and } 7 \le x < 8\}$ of the real line $\mathbb{R}$ is indeed β-locally closed.

(iii) Proposition 3.6 from [1] is wrong as every proper nonempty subset of the real line $\mathbb{R}$ with the indiscrete topology is β-open and preclosed but not semi-open.

(iv) Example 4.11 from [1] is wrong, since the space (X,τ), where $X = \{a,b,c,d\}$, $\tau = \{\emptyset, \{a,b\}, \{c,d\}, X\}$ is *not* an $\alpha\beta$-space. Note that $\{a\}$ is β-open but not α-open (an *α-open set* is a set which is the difference of an open and a nowhere dense set).

(v) An $\alpha\beta$-space [1] is in fact a strongly irresolvable, extremally disconnected space.

(vi) An α-locally closed set ([1, Definition 2.1 (x)]) is nothing else but a simly-open set.

References

[1] Y. Gnanambal and K. Balachandran, β-locally closed sets and β-LC-continuous functions, *Mem. Fac. Sci. Kochi Univ. Ser. A Math.*, **19** (1998), 35–44.

[2] E.H. Kronheimer, The topology of digital images, *Topology Appl.*, **46** (3) (1992), 279–303.

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