A REMARK ON $\beta$-LOCALLY CLOSED SETS

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Abstract

The aim of this note is to show that every subset of a given topological space is the intersection of a preopen and a preclosed set, therefore $\beta$-locally closed, and that every topological space is $\beta$-submaximal.

1 Introduction

In a recent paper, Gnanambal and Balachandran [1] introduced the classes of $\beta$-locally closed sets, $\beta$-submaximal spaces and $\beta$-LC-continuous functions. The purpose of our note is to show that every subset of any topological space is the intersection of a preopen set and a preclosed set, hence $\beta$-locally closed, and therefore every function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\beta$-LC-continuous. We have felt the need to point out explicitly this observation since over the years several papers have investigated concepts like ”pre-locally closed sets” or ”$\beta$-locally closed sets” which do not have any nontrivial meaning. In addition, we will show that every space is $\beta$-submaximal and we will point out that most results of [1] are either trivial or false.

Let $A$ be a subset of a topological space $(X, \tau)$. Following Kronheimer [2], we call the interior of the closure of $A$, denoted by $A^+$, the consolidation of $A$. Sets included in their consolidation are called preopen or locally dense. Complements of preopen sets are called preclosed and the preclosure of a set $A$, denoted by $\text{pcl}(A)$, is the intersection of all preclosed supersets of $A$. Since union of preopen sets is also preopen, the preclosure of every set is in fact a preclosed set. If $A$ is included in the closure of its consolidation, then $A$ is called $\beta$-open or semi-preopen. Complements of $\beta$-open sets are called $\beta$-closed. The $\beta$-closure of $A$, denoted by $\text{cl}_\beta(A)$ is the intersection of all $\beta$-closed supersets of $A$. In [1], Gnanambal and
Balachandran called a set $A$ $\beta$-locally closed if $A$ is intersection of a $\beta$-open and a $\beta$-closed set. They defined a set $A$ to be $\beta$-dense [1] if $\text{cl}_\beta(A) = X$ and called a space $X$ $\beta$-submaximal [1] if every $\beta$-dense subset is $\beta$-open. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\beta$-LC-continuous [1] if the preimage of every open subset of $Y$ is $\beta$-locally closed in $X$.

The following implications hold and none of them is reversible:

$$\text{dense} \Rightarrow \text{preopen} \Rightarrow \text{\beta-open} \Rightarrow \text{\beta-locally closed}$$

2 Every set is $\beta$-locally closed

**Proposition 2.1** Every subset $A$ of a topological space $(X, \tau)$ is the intersection of a pre-open and a preclosed set, hence pre-locally closed.

**Proof.** Let $A \subseteq (X, \tau)$. Set $A_1 = A \cup (X \setminus \text{cl}(A))$. Since $A_1$ is dense in $X$, it is also preopen. Let $A_2$ be the preclosure of $A$, i.e., $A_2 = A \cup \text{cl}(\text{int}(A))$. Clearly, $A_2$ is a preclosed set. Note now that $A = A_1 \cap A_2$. □

**Corollary 2.2** (i) Every set is $\beta$-locally closed and every function is $\beta$-LC-continuous.

(i) Every topological space is $\beta$-submaximal.

**Proof.** (i) Every preopen (resp. preclosed) set is $\beta$-open (resp. $\beta$-closed).

(ii) By [1, Corollary 3.24] a topological space is $\beta$-submaximal if and only if every set is $\beta$-locally closed.

**Remark 2.3** (i) Corollary 2.2 makes [1] trivial.

(ii) Example 3.4 from [1] is wrong as the subset $A = \{\frac{1}{n} : n = 1, 2, \ldots\} \cup (2, 3) \cup (3, 4) \cup \{4\} \cup (5, 6) \cup \{x : x \text{ is irrational and } 7 \leq x < 8\}$ of the real line $\mathbb{R}$ is indeed $\beta$-locally closed.

(iii) Proposition 3.6 from [1] is wrong as every proper nonempty subset of the real line $\mathbb{R}$ with the indiscrete topology is $\beta$-open and preclosed but not semi-open.

(iv) Example 4.11 from [1] is wrong, since the space $(X, \tau)$, where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a, b\}, \{c, d\}, X\}$ is not an $\alpha\beta$-space. Note that $\{a\}$ is $\beta$-open but not $\alpha$-open (an $\alpha$-open set is a set which is the difference of an open and a nowhere dense set).

(v) An $\alpha\beta$-space [1] is in fact a strongly irresolvable, extremally disconnected space.

(vi) An $\alpha$-locally closed set ([1, Definition 2.1 (x)]) is nothing else but a simly-open set.
References
