On p-closed spaces*

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Abstract

In this paper we will continue the study of p-closed spaces, i.e. spaces where every preopen cover has a finite subfamily whose pre-closures cover the space. This class of spaces is strictly placed between the class of strongly compact spaces and the class of quasi-H-closed spaces. We will provide new characterizations of p-closed spaces and investigate their relationships with some other classes of topological spaces.

1 Introduction and Preliminaries

The aim of this paper is to continue the study of p-closed spaces, which were introduced in 1989 by Abo-Khadra [1]. A topological space (X, τ) is called *p-closed* if every preopen cover of X has a finite subfamily whose pre-closures cover X.

Let A be a subset of a topological space (X, τ). Following Kronheimer [13], we call the interior of the closure of A, denoted by A^+, the *consolidation* of A. Sets included in their consolidation play a significant role in e.g. questions concerning covering properties, decompositions of continuity, etc. Such sets are called *preopen* [15] or *locally dense* [4]. A subset A of a space (X, τ) is called *preclosed* if its complement is preopen, i.e. if $\mathrm{cl}(\mathrm{int}A) \subseteq A$. The preclosure of $A \subseteq X$, denoted by $\mathrm{pcl}(A)$, is the intersection of all preclosed supersets of A. Since any union of preopen sets is also preopen, the preclosure of every set is preclosed. It is well known that $\mathrm{pcl}A = A \cup \mathrm{cl}(\mathrm{int}A)$ for any $A \subseteq X$.

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Another interesting property of preopen sets is the following: When a certain topological property is inherited by both open and dense sets, it is often then inherited by preopen sets.

Several important concepts in Topology are and can be defined in terms of preopen sets. Among the most well-known are Bourbaki's submaximal spaces (see [2]). A topological space is called *submaximal* if every (locally) dense subset is open or, equivalently, if every subset is locally closed,i.e. the intersection of an open set and a closed set. Another class of spaces commonly characterized in terms of preopen sets is the class of strongly irresolvable spaces introduced by Foran and Liebnitz in [9]. A topological space (X, τ) is called *strongly irresolvable* [9] if every open subspace of X is irresolvable, i.e. it cannot be represented as the disjoint union of two dense subsets. Subspaces that contain two disjoint dense subsets are called *resolvable*. Ganster [10] has pointed out that a space is strongly irresolvable if and only if every preopen set is semi-open, where a subset S of a space (X, τ) is called *semi-open* if $S \subseteq \mathrm{cl}(\mathrm{int}S)$. We will denote the families of preopen (resp. semi-open) sets of a space (X, τ) by $PO(X)$ (resp. $SO(X)$).

Many classical topological notions such as compactness and connectedness have been extended by using preopen sets instead of open sets. Among them are the class of *strongly compact spaces* [16] (= every preopen cover has a finite subcover) studied by Janković, Reilly and Vamanamurthy [12] and by Ganster [11], and the class of *preconnected spaces* (= spaces that cannot be represented as the disjoint union of two preopen subsets) introduced by Popa [19]. The study of topological properties via preopenness has gained significant importance in General Topology and one example for that is the fact four (out of the ten) articles in the 1998 Volume of "Memoirs of the Faculty of Science Kochi University Series A Mathematics" were more or less devoted to preopen sets.

A point $x \in X$ is called a *δ-cluster point* of a set A [25] if $A \cap U \neq \emptyset$ for every regular open set U containing x. The set of all δ-cluster points of A forms the *δ-closure* of A denoted by $\mathrm{cl}_\delta(A)$, and A is called *δ-closed* [25] if $A = \mathrm{cl}_\delta(A)$. If $A \subseteq \mathrm{int}(\mathrm{cl}_\delta(A))$, then A is said to be *δ-preopen* [21]. Complements of δ-preopen sets are called *δ-preclosed* and the *δ-preclosure* of a set A, denoted by δ-$\mathrm{pcl}(A)$, is the intersection of all δ-preclosed supersets of A.

Following [22], we will call a topological space (X, τ) *δp-closed* if for every δ-preopen cover $\{V_\alpha : \alpha \in A\}$ of X, there exists a finite subset A_0 of A such that $X = \cup\{\delta\text{-pcl}(V_\alpha) : \alpha \in A_0\}$.

2 p-closed spaces

Definition 1 A topological space (X, τ) is said to be *p-closed* [1] (resp. *quasi-H-closed* = QHC) if for every preopen (resp. open) cover $\{V_\alpha : \alpha \in A\}$ of X, there exists a finite subset A_0 of A such that $X = \cup\{\mathrm{pcl}(V_\alpha) : \alpha \in A_0\}$ (resp. $X = \cup\{\mathrm{cl}(V_\alpha) : \alpha \in A_0\}$.

It is clear that every strongly compact space is p-closed, and that every p-closed space is QHC. We also observe that a space (X, τ) is QHC if and only if every preopen cover has a finite dense subsystem (= finite subfamily whose union is a dense subset). Since every preopen set is δ-preopen, we have δ-pcl$S \subseteq$ pcl$S$ for every $S \subseteq X$. This implies that every δp-closed space is p-closed.

Theorem 2.1 *Let (X, τ) be QHC and strongly irresolvable. Then (X, τ) is p-closed.*

Proof. Let $\{S_i : i \in I\}$ be any preopen cover of X. Since X is QHC, there exists a finite subset A of I such that $X = \cup\{\mathrm{cl}(S_i) : i \in A\}$. Since X is strongly irresolvable, $S_i \in SO(X)$ and therefore $\mathrm{cl}(S_i) = \mathrm{cl}(\mathrm{int}(S_i)) = \mathrm{pcl}(S_i)$ for each $i \in I$. Hence X is p-closed. □

Corollary 2.2 *Let (X, τ) be strongly irresolvable (or submaximal). Then (X, τ) is p-closed if and only if it is QHC .*

Observe that a p-closed space need not be strongly irresolvable as any finite indiscrete space shows. However, we do have the following result.

Theorem 2.3 *Let (X, τ) be a p-closed T_0 space. Then (X, τ) is strongly irresolvable.*

Proof. Suppose that W is a nonempty, open and resolvable subspace of X. Then W is dense-in-itself and also infinite, since (X, τ) is T_0. Let $W = E_1 \cup E_2$, where E_1 and E_2 are disjoint dense subsets of W, and wlog. we may assume that E_1 is infinite. Moreover, let $A = \{x \in E_1 : \{x\} \in PO(X)\}$. Observe that for each $y \in E_1 \setminus A$, $\{y\}$ is nowhere dense. Now pick $y \in E_1 \setminus A$. If $S_y = (X \setminus W) \cup E_2 \cup \{y\}$ then S_y is dense and therefore preopen. If G is a nonempty open set contained in S_y, then $G \cap E_1 \subseteq \{y\}$ and so $G \cap W \subseteq \mathrm{cl}(E_1) \subseteq \mathrm{cl}\{y\}$.

Since $\{y\}$ is nowhere dense, $G \cap W$ is empty and so $\mathrm{cl}(\mathrm{int}(S_y)) \subseteq X \setminus W$, thus $pclS_y = S_y$. Now, observe that $\{\{x\}: x \in A\} \cup \{S_y: y \in E_1 \setminus A\}$ is a preopen cover of X. Hence there exists a finite subset A_1 of A and a finite subset A_2 of $E_1 \setminus A$ such that $X = \{\{x\}: x \in A_1\} \cup \{S_y: y \in A_2\}$. Then, $E_1 \subseteq A_1 \cup A_2$ which is a contradiction. Thus X is strongly irresolvable. $\square$

By combining the previous two results we immediately have:

Theorem 2.4 *Let (X, τ) be a T_0 space. Then (X, τ) is p-closed if and only if (X, τ) is QHC and strongly irresolvable.*

The following diagram exhibits the relationships between the class of p-closed spaces and some related classes of topological spaces. Note that none of the implications is reversible.

$$\begin{tikzcd}
\text{strongly compact} \arrow[r] \arrow[d] & \text{p-closed} \arrow[drr] & \delta\text{p-closed} \arrow[l] & \\
\alpha\text{-compact} \arrow[r] & \text{compact} \arrow[r] & \text{nearly compact} \arrow[r] & \text{QHC} \\
\text{semi-compact} \arrow[r] \arrow[u] & \text{s-closed} \arrow[r] \arrow[ur] & \text{S-closed} \arrow[ur] &
\end{tikzcd}$$

Example 2.5 (i) Recall that a space (X, τ) is called *α-scattered* [7] if it has a dense set of isolated points. Clearly every α-scattered space is strongly irresolvable and so, by Theorem 2.1, every α-scattered QHC space is p-closed. In particular, the Katetov extension $\kappa\mathbb{N}$ of the set of natural numbers $\mathbb{N}$ (see e.g. [20]) is p-closed and not compact, hence not strongly compact.

(ii) The unit interval $[0, 1]$ with the usual topology is compact, hence QHC, but not p-closed since it is resolvable.

(iii) Let $X = \mathbb{R}$, $\tau = \{\emptyset, \{0\}, X\}$. Then, X is p-closed and s-closed but not α-compact and hence not strongly compact (A space is *α-compact* if every cover by α-open sets has a finite subcover, where a set is *α-open* if it is the difference of an open and a nowhere dense set; clearly every α-open set is preopen but not vice versa). Additionally, this space is not δp-closed since every subset is δ-preopen.

We next discuss the relationship between p-closedness and compactness. Recall that a space (X,τ) is called *nearly compact* [24] if every cover of X by regular open sets has a finite subcover, i.e. the semiregularization (X,τ_s) of (X,τ) is compact. Example 4.8 (d) in [20] shows that there exists a Hausdorff, non-compact, semi-regular and QHC space with a dense set of isolated points. Such a space is p-closed but not nearly compact. Example 2.10 in [22] provides another such example.

For any infinite cardinal κ, a topological space (X,τ) is called *κ-extremally disconnected* (= κ-e.d.) [6] if the boundary of every regular open set has cardinality (strictly) less than κ. Several topological spaces share this property for $\kappa = \aleph_0$. Since there are finite spaces which fail to be extremally disconnected, clearly $\aleph_0$-extremal disconnectedness is a strictly weaker property than extremal disconnectedness.

Theorem 2.6 *If a topological space (X,τ) is p-closed and $\aleph_0$-extremally disconnected (resp. extremally disconnected), then (X,τ) is nearly compact (resp. s-closed).*

Proof. We first prove the case when the space is $\aleph_0$-extremally disconnected. Let $\{A_i\colon i \in I\}$ be any regular open cover of X. For each $i \in I$, we have $\mathrm{pcl}(A_i) = A_i \cup \mathrm{cl}(\mathrm{int}(A_i)) = \mathrm{cl}(A_i)$. Since X is p-closed, then there exists a finite $F \subseteq I$ such that $X = \cup_{i\in F}\mathrm{cl}(A_i)$. Note that for each A_i, we have $\mathrm{cl}(A_i) = B_i \cup C_i$, where $B_i = \mathrm{int}(\mathrm{cl}(A_i))$ and $C_i = \mathrm{cl}(A_i) \setminus \mathrm{int}(\mathrm{cl}(A_i))$. Since X is $\aleph_0$-extremally disconnected, then C_i is finite for each $i \in F$. Since $B_i = A_i$, for each $i \in F$, then $\cup_{i\in F}A_i$ covers X but a finite amount. Hence, X is nearly compact. The proof of the second part of the theorem is similar to the first one and hence omitted. $\square$

On the other hand (see e.g. [20], page 450) there exist dense-in-itself, compact and extremally disconnected Hausdorff spaces. Let Y be a dense-in-itself Hausdorff space and let X be the Stone space of $RO(X,\tau)$. Then X is dense-in-itself, compact and extremally disconnected. In addition, X is resolvable and hence cannot be p-closed.

A filter base $\mathcal{F}$ on a topological space (X,τ) is said to *pre-θ-converge* to a point $x \in X$ if for each $V \in PO(X,x)$, there exists $F \in \mathcal{F}$ such that $F \subseteq \mathrm{pcl}(V)$. A filter base $\mathcal{F}$ is said to *pre-θ-accumulate* at $x \in X$ if $\mathrm{pcl}(V) \cap F \neq \emptyset$ for every $V \in PO(X,x)$ and every $F \in \mathcal{F}$. The *preinterior* of a set A, denoted by $\mathrm{pint}(A)$, is the union of all preopen subsets of A.

Theorem 2.7 *For a topological space (X, τ) the following conditions are equivalent:*

(a) (X, τ) *is p-closed,*

(b) *every maximal filter base pre-θ-converges to some point of X,*

(c) *every filter base pre-θ-accumulates at some point of X,*

(d) *for every family $\{V_\alpha : \alpha \in A\}$ of preclosed subsets such that $\cap\{V_\alpha : \alpha \in A\} = \emptyset$, there exists a finite subset A_0 of A such that $\cap\{\mathrm{pint}(V_\alpha) : \alpha \in A_0\} = \emptyset$.*

Proof. (a) $\Rightarrow$ (b): Let $\mathcal{F}$ be a maximal filer base on X. Suppose that $\mathcal{F}$ does not pre-θ-converge to any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not pre-θ-accumulate at any point of X. For each $x \in X$, there exists $F_x \in \mathcal{F}$ and $V_x \in PO(X, x)$ such that $\mathrm{pcl}(V_x) \cap F_x = \emptyset$. The family $\{V_x : x \in X\}$ is a cover of X by preopen sets of X. By (a), there exists a finite number of points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup\{\mathrm{pcl}(V_{x_i}) : i = 1, 2, \ldots, n\}$. Since $\mathcal{F}$ is a filter base on X, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \cap\{F_{x_i} : i = 1, 2, \ldots, n\}$. Therefore, we obtain $F_0 = \emptyset$. This is a contradiction.

(b) $\Rightarrow$ (c): Let $\mathcal{F}$ be any filter base on X. Then, there exists a maximal filter base $\mathcal{F}_0$ such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (b), $\mathcal{F}_0$ pre-θ-converges to some point $x \in X$. For every $F \in \mathcal{F}$ and every $V \in PO(X, x)$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq \mathrm{pcl}(V)$; hence $\emptyset \neq F_0 \cap F \subseteq \mathrm{pcl}(V) \cap F$. This shows that $\mathcal{F}$ pre-θ-accumulates at x.

(c) $\Rightarrow$ (d): Let $\{V_\alpha : \alpha \in A\}$ be any family of preclosed subsets of X such that $\cap\{V_\alpha : \alpha \in A\} = \emptyset$. Let $\Gamma(A)$ denote the ideal of all finite subsets of A. Assume that $\cap\{\mathrm{pint}(V_\alpha) : \alpha \in \gamma\} \neq \emptyset$ for every $\gamma \in \Gamma(A)$. Then, the family $\mathcal{F} = \{\cap_{\alpha \in \gamma} \mathrm{pint}(V_\alpha) : \gamma \in \Gamma(A)\}$ is a filter base on X. By (c), $\mathcal{F}$ pre-θ-accumulates at some point $x \in X$. Since $\{X \setminus V_\alpha : \alpha \in A\}$ is a cover of X, $x \in X \setminus V_{\alpha_0}$ for some $\alpha_0 \in A$. Therefore, we obtain $X \setminus V_{\alpha_0} \in PO(X, x)$, $\mathrm{pint}(V_{\alpha_0}) \in \mathcal{F}$ and $\mathrm{pcl}(X \setminus V_{\alpha_0}) \cap \mathrm{pint}(V_{\alpha_0}) = \emptyset$. This is a contradiction.

(d) $\Rightarrow$ (a): Let $\{V_\alpha : \alpha \in A\}$ be a cover of X by preopen sets of X. Then $\{X \setminus V_\alpha : \alpha \in A\}$ is a family of preclosed subsets of X such that $\cap\{X \setminus V_\alpha : \alpha \in A\} = \emptyset$. By (d), there exists a finite subset A_0 of A such that $\cap\{\mathrm{pint}(X \setminus V_\alpha) : \alpha \in A_0\} = \emptyset$; hence $X = \cup\{\mathrm{pcl}(V_\alpha) : \alpha \in A_0\}$. This shows that X is p-closed. $\Box$

Definition 2 A topological space (X, τ) is said to be *strongly p-regular* (resp. *p-regular* [8], *almost p-regular* [14]) if for each point $x \in X$ and each preclosed set (resp. closed set, regular closed set) F such that $x \notin F$, there exist disjoint preopen sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 2.8 *If a topological space X is p-closed and strongly p-regular (resp. p-regular, almost p-regular), then X is strongly compact (resp. compact, nearly compact).*

Proof. We prove only the case of p-regular spaces. Let X be a p-closed and p-regular space. Let $\{V_\alpha : \alpha \in A\}$ be any open cover of X. For each $x \in X$, there exists an $\alpha(x) \in A$ such that $x \in V_{\alpha(x)}$. Since X is p-regular, there exists $U(x) \in PO(X)$ such that $x \in U(x) \subseteq \text{pcl}(U(x)) \subseteq V_{\alpha(x)}$ [8, Theorem 3.2]. Then, $\{U(x) : x \in X\}$ is a preopen cover of the p-closed space X and hence there exists a finite amount of points, say, $x_1, x_2, \ldots, x_n$ such that $X = \cup_{i=1}^{n} \text{pcl}(U(x_i)) = \cup_{i=1}^{n} V_{\alpha(x_i)}$. This shows that X is compact. $\square$

3 p-closed subspaces

Recall that a topological space (X, τ) is called *hyperconnected* if every open subset of X is dense. In the opposite case, X is called *hyperdisconnected*. A set A is called *semi-regular* [5] if it is both semi-open and semi-closed. Di Maio and Noiri [5] have shown that a set A is semi-regular if and only if there exists a regular open set U with $U \subseteq A \subseteq \text{cl}(U)$. Cameron [3] used the term regular semi-open for a semi-regular set.

Lemma 3.1 [17, Mashhour et al.] *Let A and B be subsets of a topological space (X, τ).*

(1) If $A \in PO(X)$ and $B \in SO(X)$, then $A \cap B \in PO(B)$.

(2) If $A \in PO(B)$ and $B \in PO(X)$, then $A \in PO(X)$.

Lemma 3.2 *Let $B \subseteq A \subseteq X$ and $A \in SO(X)$. Then,* $\text{pcl}_A(B) \subseteq \text{pcl}_X(B)$.

Theorem 3.3 *If every proper semi-regular subspace of a hyperdisconnected topological space (X, τ) is p-closed, then X is also p-closed.*

Proof. Since (X,τ) is not hyperconnected, then there exists a proper semi-regular set A. Let $\{A_i\}_{i\in I}$ be any preopen cover of X. Since A is semi-open, then by Lemma 3.1 $A_i \cap A = B_i \in PO(A,\tau|A)$. Then $\{B_i\}_{i\in I}$ is a preopen cover of the p-closed space $(A,\tau|A)$. Then, there exists a finite subset F of I such that $A = \cup_{i\in F}\mathrm{pcl}_A(B_i) \subseteq \cup_{i\in F}\mathrm{pcl}(B_i)$ (by Lemma 3.2). Therefore, we have $A \subseteq \cup_{i\in F}\mathrm{pcl}(A_i)$. Since A is semi-regular, $X \setminus A$ is also semi-regular and by a similar argument we can find a finite subset G of I such that $X \setminus A \subseteq \cup_{i\in G}\mathrm{pcl}(A_i)$. Hence, $X = \cup_{i\in F\cup G}\mathrm{pcl}(A_i)$. This shows that X is p-closed. □

Theorem 3.4 *If there exists a proper semi-regular subset A of a topological space (X,τ) such that A and $X \setminus A$ are p-closed subspaces, then X is also p-closed.*

Proof. Similar to the one of Theorem 3.3 and hence omitted. □

Lemma 3.5 *Let $A \subseteq B \subseteq X$ and $B \in PO(X)$. If $A \in PO(B)$, then* $\mathrm{pcl}(A) \subseteq \mathrm{pcl}_B(A)$.

Theorem 3.6 *If (X,τ) is a p-closed spaces and A is preregular (i.e. both preopen and preclosed), then $(A,\tau|A)$ is also p-closed (as a subspace).*

Proof. Let $\{A_i\}_{i\in I}$ be any preopen cover of $(A,\tau|A)$. By Lemma 3.1, $A_i \in PO(X)$ for each $i \in I$ and $\{A_i \colon i \in I\} \cup (X \setminus A) = X$. Since X is p-closed, there exists a finite subset F of I such that $X = (\cup_{i\in F}\mathrm{pcl}_X(A_i)) \cup (X \setminus A)$; hence $A = \cup_{i\in F}\mathrm{pcl}_X(A_i)$. For each $i \in F$, we have by Lemma 3.5, $\mathrm{pcl}_X(A_i) \subseteq \mathrm{pcl}_A(A_i)$ and $A = \cup_{i\in F}\mathrm{pcl}_A(A_i)$. Therefore, A is a p-closed subspace. □

Example 3.7 An open, even a δ-open subset of a p-closed space need not be p-closed (as a subspace). Consider any infinite set X with the point excluded topology. Since the only preopen set containing the excluded point is the whole space X, then the space in question is p-closed. However, the (infinite) set of isolated points of X is not p-closed.

4 Sets which are p-closed relative to a space

A subset S of a topological space (X, τ) is said to be *p-closed relative to* X if for every open cover $\{V_\alpha : \alpha \in A\}$ of S by preopen subsets of (X, τ), there exists a finite subset A_0 of A such that $X = \cup\{\mathrm{pcl}(V_\alpha) : \alpha \in A_0\}$.

Theorem 4.1 *For a topological space (X, τ) the following conditions are equivalent:*

(a) *S is p-closed relative to X,*

(b) *every maximal filter base on X which meets S pre-θ-converges to some point of S,*

(c) *every filter base on X which meets S pre-θ-accumulates at some point of S,*

(d) *for every family $\{V_\alpha : \alpha \in A\}$ of preclosed subsets of (X, τ) such that $[\cap\{V_\alpha : \alpha \in A\}] \cap S = \emptyset$, there exists a finite subset A_0 of A such that $[\cap\{\mathrm{pint}(V_\alpha) : \alpha \in A_0\}] \cap S = \emptyset$.*

A point $x \in X$ is said to be a *pre-θ-accumulation point* of a subset A of a topological space (X, τ) if $\mathrm{pcl}(U) \cap A \neq \emptyset$ for every $U \in PO(X, x)$. The set of all pre-θ-accumulation points of A is called the *pre-θ-closure* of A and is denoted by $\mathrm{pcl}_\theta(A)$. A subset A of a topological space (X, τ) is said to be *pre-θ-closed* if $\mathrm{pcl}_\theta(A) = A$. The complement of a pre-θ-closed set is called *pre-θ-open*.

Proposition 4.2 *Let A be a subset of a topological space (X, τ). Then:*

(i) *If $A \in PO(X)$, then $\mathrm{pcl}(A) = \mathrm{pcl}_\theta(A)$.*

(ii) *If A is preregular, then A is pre-θ-closed.*

(iii) *If $A \in SO(X)$, then $\mathrm{pcl}(A) = \mathrm{cl}(A)$.*

Theorem 4.3 *If X is a p-closed space, then every pre-θ-open cover of X has a finite subcover.*

Proof. Let $\{V_\alpha : \alpha \in A\}$ be any cover of X by pre-θ-open subsets of X. For each $x \in X$, there exists $\alpha(x) \in A$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is pre-θ-open, there exists $V_x \in PO(X)$ such that $x \in V_x \subseteq \mathrm{pcl}(V_x) \subseteq V_{\alpha(x)}$. The family $\{V_x : x \in X\}$ is a preopen cover of X. Since X is p-closed, there exists a finite number of points, say, $x_1, x_2, \ldots, x_n$ in X such that $X = \cup\{\mathrm{pcl}(V_{x_i}) : i = 1, 2, \ldots, n\}$. Therefore, we obtain that $X = \cup\{V_{\alpha(x_i)} : i = 1, 2, \ldots, n\}$. □

Question. Is the converse in Theorem 4.3 true?

Theorem 4.4 *Let A, B be subsets of a space X. If A is pre-θ-closed and B is p-closed relative to X, then $A \cap B$ is p-closed relative to X.*

Proof. Let $\{V_\alpha : \alpha \in A\}$ be any cover of $A \cap B$ by preopen subsets of X. Since $X \setminus A$ is pre-θ-open, for each $x \in B \setminus A$ there exists $W_x \in PO(X, x)$ such that $\mathrm{pcl}(W_x) \subseteq X \setminus A$. The family $\{W_x : x \in B \setminus A\} \cup \{V_\alpha : \alpha \in A\}$ is a cover of B by preopen sets of X. Since B is p-closed relative to X, there exist a finite number of points, say, $x_1, x_2, \ldots, x_n$ in $B \setminus A$ and a finite subset A_0 of A such that

$$B \subseteq [\cup_{i=1}^{n} \mathrm{pcl}(W_{x_i})] \cup [\cup_{\alpha \in A_0} \mathrm{pcl}(V_\alpha)].$$

Since $\mathrm{pcl}(W_{x_i}) \cap A = \emptyset$ for each i, we obtain that $A \cap B \subseteq \cup\{\mathrm{pcl}(V_\alpha) : \alpha \in A_0\}$. This shows that $A \cap B$ is p-closed relative to X. $\square$

Corollary 4.5 *If K is pre-θ-closed set of a p-closed space (X, τ), then K is p-closed relative to X.*

Question. If in a topological space (X, τ) every proper pre-θ-closed set is p-closed relative to X, is X necessarily p-closed?

A topological space (X, τ) is called *preconnected* [19] if X can not be expressed as the union of two disjoint preopen sets. In the opposite case, X is called *predisconnected*. Note that every preconnected space is irresolvable but not vice versa.

Theorem 4.6 *Let X be a predisconnected space. Then X is p-closed if and only if every preregular subset of X is p-closed relative to X.*

Proof. Necessity. Every preregular set is pre-θ-closed by Proposition 4.2. Since X is p-closed, the proof is completed by Corollary 4.5.

Sufficiency. Let $\{V_\alpha : \alpha \in A\}$ be a preopen cover of X. Since X is predisconnected, there exists a proper preregular subset A of X. By our hypothesis, A and $X \setminus A$ are p-closed relative to X. There exist finite subsets A_1 and A_2 of A such that

$$A \subseteq \cup_{\alpha \in A_1} \mathrm{pcl}(V_\alpha) \text{ and } X \setminus A \subseteq \cup_{\alpha \in A_2} \mathrm{pcl}(V_\alpha).$$

Therefore, we obtain that $X = \cup\{\mathrm{pcl}(V_\alpha : \alpha \in A_1 \cup A_2\}$. $\square$

Theorem 4.7 *If there exists a proper preregular subset A of a topological space (X, τ) such that A and $X \setminus A$ are p-closed relative to X, then X is p-closed.*

Proof. This proof is similar to the one of Theorem 4.6 and hence omitted. □

Theorem 4.8 *Let X_0 be a semi-open subset of a topological space (X, τ). If X_0 is a p-closed space, then it is p-closed relative to X.*

Proof. Let $\{V_\alpha \colon \alpha \in A\}$ be any cover of X_0 by preopen subsets of X. Since $X_0 \in SO(X)$, by Lemma 3.1 we have that $X_0 \cap V_\alpha = W_\alpha \in PO(X_0)$ for each $\alpha \in A$. Therefore, $\{W_\alpha \colon \alpha \in A\}$ is a preopen cover of X_0. since X_0 is p-closed, there exists a finite subset A_0 of A such that $X_0 = \cup\{\mathrm{pcl}_{X_0}(W_\alpha) \colon \alpha \in A_0\}$. By Lemma 3.2, we obtain that $X_0 \subseteq \cup\{\mathrm{pcl}(W_\alpha) \colon \alpha \in A_0\} \subseteq \cup\{\mathrm{pcl}(V_\alpha) \colon \alpha \in A_0\}$. This shows that X_0 is p-closed relative to X. □

Theorem 4.9 *Let X_0 be a preopen subset of a topological space (X, τ). If X_0 is p-closed relative to X, then it is a p-closed subspace of X.*

Proof. Let $\{V_\alpha \colon \alpha \in A\}$ be any cover of X_0 by preopen subsets of X_0. Since $X_0 \in PO(X)$, by Lemma 3.1, $V_\alpha \in PO(X)$ for each $\alpha \in A$. Since X_0 is p-closed relative to X, there exists a finite subset A_0 of A such that $X_0 \subseteq \cup\{\mathrm{pcl}(V_\alpha) \colon \alpha \in A_0\}$. Since $X_0 \in PO(X)$, by Lemma 3.5 we obtain $X_0 = \cup\{\mathrm{pcl}_{X_0}(V_\alpha) \colon \alpha \in A_0\}$. This shows that X_0 is p-closed subspace of X. □

Corollary 4.10 *Let X_0 be an (α-)open subset of a topological space (X, τ). Then X_0 is a p-closed subspace of X if and only if it is p-closed relative to X.*

Proof. This is an immediate consequence of Theorem 4.8 and Theorem 4.9. □

Recall that a function $f \colon (X, \tau) \to (Y, \sigma)$ is called *preirresolute* [23] (resp. *precontinuous* [15]) if $f^{-1}(V)$ is preopen in X for every preopen (resp. open) subsets V of Y.

Lemma 4.11 [18] *A function $f \colon (X, \tau) \to (Y, \sigma)$ is preirresolute (resp. precontinuous) if and only if for each subset A of X, $f(\mathrm{pcl}(A)) \subseteq \mathrm{pcl}(f(A))$ (resp. $f(\mathrm{pcl}(A)) \subseteq \mathrm{cl}(f(A))$).*

Theorem 4.12 *If a function $f:(X,\tau) \to (Y,\sigma)$ is a preirresolute (resp. precontinuous) surjection and K is p-closed relative to X, then $f(K)$ is p-closed (resp. QHC) relative to Y.*

Proof. Let $\{V_\alpha : \alpha \in A\}$ be any cover of $f(K)$ by preopen (resp. open) subsets of Y. Since f is preirresolute (resp. precontinuous), $\{f^{-1}(V_\alpha) : \alpha \in A\}$ is a cover of K by preopen subsets of X, where K is p-closed relative to X. Therefore, there exists a finite subset A_0 of A such that $K \subseteq \cup_{\alpha \in A_0} \mathrm{pcl}(f^{-1}(V_\alpha))$. Since f is preirresolute (resp. precontinuous) and surjective, by Lemma 4.11, we have

$$f(K) \subseteq \cup_{\alpha \in A_0} f(\mathrm{pcl}(f^{-1}(V_\alpha))) \subseteq \cup_{\alpha \in A_0} \mathrm{pcl}(V_\alpha)$$

$$(\text{resp. } f(K) \subseteq \cup_{\alpha \in A_0} f(\mathrm{pcl}(f^{-1}(V_\alpha))) \subseteq \cup_{\alpha \in A_0} \mathrm{cl}(V_\alpha)).$$

Corollary 4.13 *If a function $f:(X,\tau) \to (Y,\sigma)$ is a preirresolute (resp. continuous) surjection and X is p-closed, then Y is p-closed (resp. QHC).*

Corollary 4.14 (i) *The property "p-closed" is topological.*

(ii) *If the product space $\prod_{\alpha \in A} X_\alpha$ is p-closed, then X_α is p-closed for each $\alpha \in A$.*

Remark 4.15 Even finite product of p-closed spaces need not be p-closed; for consider the product of the space from Example 2.5 (i) with any two point indiscrete space. This product space shows that Theorem 3.4.3 from [1] is wrong, i.e., every proper preregular subset might be p-closed relative to the space and still the space might fail to be p-closed. Additionally, Example 3.4.1 from [1] is also false.

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