ON GENERALIZED CLOSED SETS

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Abstract

In this paper we study generalized closed sets in the sense of N. Levine. We will consider the question of when some classes of generalized closed sets coincide. Also, some lower separation axioms weaker than T_1 are investigated. We will provide characterizations of extremally disconnected spaces and sg-submaximal spaces by using various kinds of generalized closed sets.

1 Introduction

Closed sets are fundamental objects in a topological space. For example, one can define the topology on a set by using either the axioms for the closed sets or the Kuratowski closure axioms. In 1970, N. Levine [16] initiated the study of so-called generalized closed sets. By definition, a subset S of a topological space X is called *generalized closed* if $\mathrm{cl}A \subseteq U$ whenever $A \subseteq U$ and U is open. This notion has been studied extensively in recent years by many topologists because generalized closed sets are not only natural generalizations of closed sets. More importantly, they also suggest several new properties of topological spaces. Most of these new properties are separation axioms weaker than T_1, some of which have been found to be useful in computer science and digital topology. For example, the well-known digital line is $T_{3/4}$ but not T_1. Other new properties are defined by variations of the property of submaximality. Furthermore, the study of generalized closed sets also provides new characterizations of some known classes of spaces, for example, the class of extremally disconnected spaces.

[0]1991 Mathematics subject classification: 54A05, 54D10, 54F65, 54G05.

Key Words and Phrases: Generalized closed set, T_{gs}-space, submaximal, extremally disconnected.

For the sake of convenience, we begin with some basic concepts, although most of these concepts can be found from the references of this paper. A subset S of a topological space X is called *α-open* [resp. *semi-open, preopen, semi-preopen*] if $S \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}\ S))$ [resp. $S \subseteq \mathrm{cl}(\mathrm{int}\ S)$, $S \subseteq \mathrm{int}(\mathrm{cl}\ S)$, $S \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}\ S))$]. Moreover, S is said to be *α-closed* [resp. *semi-closed, preclosed, semi-preclosed*] if $X \backslash S$ is α-open [resp. semi-open, preopen, semi-preopen] or, equivalently, if $\mathrm{cl}(\mathrm{int}(\mathrm{cl}\ S)) \subseteq S$ [resp. $\mathrm{int}(\mathrm{cl}\ S) \subseteq S$, $\mathrm{cl}(\mathrm{int}\ S) \subseteq S$, $\mathrm{int}(\mathrm{cl}(\mathrm{int}\ S)) \subseteq S$]. The *$\alpha$-closure* [resp. *semi-closure, preclosure, semi-preclosure*] of $S \subseteq X$ is the smallest α-closed [resp. semi-closed, preclosed, semi-preclosed] set containing S. It is well-known that α-cl $S = S \bigcup \mathrm{cl}(\mathrm{int}(\mathrm{cl}\ S))$ and scl $S = S \bigcup \mathrm{int}(\mathrm{cl}\ S)$, pcl $S = S \bigcup \mathrm{cl}(\mathrm{int}\ S)$ and spcl $S = S \bigcup \mathrm{int}(\mathrm{cl}(\mathrm{int}\ S))$. The *$\alpha$-interior* of $S \subseteq X$ is the largest α-open set contained in S, and we have α-int $S = S \bigcap \mathrm{int}(\mathrm{cl}(\mathrm{int}\ S))$. It is worth mentioning that the collection $\alpha(X)$ of all α-open subsets of X is a topology on X [21] which is finer than the original one, and that a subset S of X is α-open if and only if S is semi-open and preopen [23].

Definition 1. Let X be a topological space. A subset A of X is called

(1) *generalized closed* (briefly, *g-closed*) [16] if cl $A \subseteq U$, whenever $A \subseteq U$ and U is open;

(2) *semi-generalized closed* (briefly, *sg-closed*) [2], if scl $A \subseteq U$ whenever $A \subseteq U$ and U is semi-open;

(3) *generalized semiclosed* (briefly, *gs-closed*) [1] if scl $A \subseteq U$ whenever $A \subseteq U$ and U is open;

(4) *generalized α-closed* (briefly, *$g\alpha$-closed*) [18], if α-cl $A \subseteq U$ whenever $A \subseteq U$ and U is α-open, or equivalently, if A is g-closed with respect to $\alpha(X)$;

(5) *α-generalized closed* (briefly, *αg-closed*) [19] if α-cl $A \subseteq U$ whenever $A \subseteq U$ and U is open;

(6) *generalized semi-preclosed* (briefly, *gsp-closed*) [6] if spcl $A \subseteq U$ whenever $A \subseteq U$ and U is open;

(7) *regular generalized closed* (briefly, *r-g-closed*) [22] if cl $A \subseteq U$ whenever $A \subseteq U$ and U is regular open.

In addition to Definition 1 above, a subset A of X is *g-open* [16] [*sg-open* [2]] if $X \setminus A$ is g-closed [*sg*-closed]. Other classes of generalized open sets can be defined in a similar manner. Recall that a space X is said to be *submaximal* if every dense subset of X is open. As variations of submaximality, we obtain the notions of α-submaximality, g-submaximality and

sg-submaximality. A space X is *α-submaximal* [resp. *g-submaximal*, *sg-submaximal*] if every dense subset is α-open (resp. g-open, sg-open). α-submaximal spaces have been studied by Ganster in [12]. Obviously, every submaximal space is g-submaximal, and it has been pointed out in [4] , Corollary 3.4., that if $(X, \alpha(X))$ is g-submaximal, then $(X, \alpha(X))$ is also sg-submaximal. Note that any indiscrete space with at least two points is g-submaximal but not submaximal, and an sg-submaximal space which is not g-submaximal was given in [4].

In [7], Dontchev summarized the fundamental relationships between various types of generalized closed sets in the following diagram. It was noted in [7] that, in general, none of the implications in that diagram is reversible.

$$
\begin{tikzcd}
\text{closed set} \arrow[r] \arrow[d] & \text{g-closed set} \arrow[r] \arrow[dr] \arrow[drr] & \alpha\text{g-closed set} \arrow[d] \arrow[ddr] & \\
\alpha\text{-closed set} \arrow[r] \arrow[d] & \text{g}\alpha\text{-closed set} \arrow[r] \arrow[ur] \arrow[ddr] & \text{gs-closed set} \arrow[dr] & \text{r-g-closed set} \\
\text{semi-closed set} \arrow[r] \arrow[d] & \text{sg-closed set} \arrow[ur] \arrow[rr] \arrow[dl] & & \text{gsp-closed set} \\
\text{semi-preclosed set} \arrow[urrr] & & \text{preclosed set} \arrow[ll] \arrow[ur] &
\end{tikzcd}
$$

Concerning possible converses of some implications in the above diagram, Dontchev [7] posed two questions asking for the class of spaces in which every semi-preclosed subset is sg-closed, and for the class of spaces in which every preclosed subset is $g\alpha$-closed. These two questions have been considered and answered by Cao, Ganster and Reilly in [3]. Other possible converses of implications were investigated in [4]. This lead to some new characterizations of T_{gs}-spaces, extremally disconnected spaces and sg-submaximal spaces.

The main purpose of this paper is to summarise recent work in this direction, and also to present some new results. Throughout this paper, no separation axioms are assumed unless stated explicitly.

2 Dontchev's questions

In a recent paper [7], Dontchev posed the following two open questions concerning generalized closed sets:

Question 2.1. [7] *Characterize the spaces in which*
1) Every semi-preclosed set is sg-closed.
2) Every preclosed set is $g\alpha$-closed.

In order to answer these questions we need some preparation. Let S be a subset of a space X. A *resolution* of S is a pair $\langle E_1, E_2 \rangle$ of disjoint dense subsets of S. The subset S is said to be *resolvable* if it possesses a resolution, otherwise S is called *irresolvable*. In addition, S is called *strongly irresolvable*, if every open subspace of S is irresolvable. Observe that if $\langle E_1, E_2 \rangle$ is a resolution of S then E_1 and E_2 are codense in X, i.e. have empty interior.We also note that every submaximal space is hereditarily irresolvable.

Lemma 2.2. [14, 12] *Every space X has a unique decomposition $X = F \bigcup G$, where F is closed and resolvable and G is open and hereditarily irresolvable.*

In this paper, the representation $X = F \bigcup G$, where F and G are as in Lemma 2.2, will be called the *Hewitt decomposition* of X.

Lemma 2.3. [15] *Let X be a space. Then every singleton of X is either nowhere dense or preopen.*

For a space X, we now define $X_1 = \{x \in X : \{x\}$ is nowhere dense $\}$, and $X_2 = \{x \in X : \{x\}$ is preopen $\}$. Then $X = X_1 \bigcup X_2$ is a decomposition of X, which will be called the *Janković-Reilly decomposition*. Recall that a space X is said to be *locally indiscrete* if every open subset is closed. We are now ready to state the following theorem which was proved in [3] answering Question 2.1.

Theorem 2.4. [3] *For a space X with Hewitt decomposition $X = F \bigcup G$ the following are equivalent:*
(1) every semi-preclosed subset of X is sg-closed,
(2) $X_1 \bigcap \mathrm{scl}\ A \subseteq \mathrm{spcl}\ A$ for each $A \subseteq X$,
(3) $X_1 \subseteq \mathrm{int}(\mathrm{cl}\ G)$,
(4) $X \approx Y \oplus Z$, where Y is locally indiscrete and Z is strongly irresolvable,
(5) every preclosed subset of X is $g\alpha$-closed,
(6) X is g-submaximal with respect to $\alpha(X)$.

3 Lower separation axioms

A space X is called $T_{1/2}$ [16] if every g-closed subset of X is closed. Dunham [11] proved that a space X is $T_{1/2}$ if and only if singletons of X are either open or closed. For further results concerning this class of spaces we refer the reader to [8] and [11].

Theorem 3.1. [5] *Every submaximal space is $T_{1/2}$.*

It is obvious that in any topological space X, every sg-closed subset of X is gs-closed. In [20], the class of T_{gs}-spaces was introduced where a space X is called T_{gs} if every gs-closed subset of X is sg-closed. The following result exhibits the relationship between T_{gs}-spaces and $T_{1/2}$-spaces.

Theorem 3.2. [4, 8] *For a space X, the following statements are equivalent:*
(1) X is a T_{gs}-space,
(2) every nowhere dense subset of X is a union of closed subsets of X (i.e. X is T_1^* [13]),
(3) every generalized semi-preclosed subset of X is semi-preclosed (i.e. X is semi-pre-$T_{1/2}$ [6]),
(4) every singleton of X is either preopen or closed.

Corollary 3.3. *Every $T_{1/2}$-space is T_{gs}.*

A space X is called *semi-T_1* [17] if each singleton is semi-closed, it is called *semi-$T_{1/2}$* [2] if every singleton is either semi-closed or semi-open. Let $s(X)$ be the semi-regularization of a space X. The closure of a subset A of X with respect to $s(X)$ will be denoted by δ-cl A. A subset A of X is called *δ-generalized closed* if δ-cl $A \subseteq U$ when $A \subseteq U$ and U is open in X. Moreover, X is called a *$T_{3/4}$-space* [8] if every δ-generalized closed subset of X is closed in $s(X)$. The well-known digital line, also called the Khalimsky line, is a $T_{3/4}$-space which fails to be T_1.

Theorem 3.4. *For any space X,*
(1) [8] $T_{3/4} = T_{gs}$ + semi-T_1.
(2) [20] $T_{1/2} = T_{gs}$ + semi-$T_{1/2}$.

The results above clarify some connections between the T_{gs} property and other lower separation axioms. Note, however, that not every T_{gs}-space is T_0. For example, a three point space $X = \{a, b, c\}$ in which the only proper open subset is $\{a, b\}$, is a T_{gs}-space, but not a T_0-space. To obtain more characterizations of T_{gs}-spaces, we need the following lemma.

Lemma 3.5. *A subset A of a space X is $g\alpha$-closed if and only if* $X_1 \bigcap \alpha\text{-cl } A \subseteq A$.

Proof. Suppose that A is $g\alpha$-closed, and let $x \in X_1 \bigcap \alpha\text{-cl } A$. If $x \notin A$, then $X \setminus \{x\}$ is an α-open set containing A and so $\alpha\text{-cl } A \subseteq X \setminus \{x\}$, which is impossible.

Conversely, suppose that $X_1 \bigcap \alpha\text{-cl } A \subseteq A$. Let U be an α-open set containing A, and let $x \in \alpha\text{-cl } A$. If $x \in X_1$, then $x \in A \subseteq U$. Now let $x \in X_2$ and assume that $x \notin U$. Then $X \setminus U$ is an α-closed set containing x, and thus, $\alpha\text{-cl}(\{x\}) = \{x\} \bigcup \text{cl}(\text{int}(\text{cl}(\{x\}))) \subseteq X \setminus U$. Since $\{x\}$ is preopen, we have $\text{int}(\text{cl}(\{x\})) \bigcap A \neq \emptyset$. Pick a point $y \in \text{int}(\text{cl}(\{x\})) \bigcap A$. Then $y \in A \bigcap (X \setminus U) \subseteq U \bigcap (X \setminus U)$, which is a contradiction. □

Theorem 3.6. *A space X is T_{gs} if and only if every αg-closed subset of X is $g\alpha$-closed.*

Proof. Suppose that X is T_{gs}. Let A be αg-closed and let $x \in X_1 \bigcap \alpha\text{-cl } A$. Then $\{x\}$ is closed by Theorem 3.2. Assume that $x \notin A$, i.e. $A \subseteq X \setminus \{x\}$. Since A is αg-closed and $X \setminus \{x\}$ is open we have $x \in \alpha\text{-cl}A \subseteq X \setminus \{x\}$, which is a contradiction. Therefore, $X_1 \bigcap \alpha\text{-cl } A \subseteq A$. By Lemma 3.5, A is $g\alpha$-closed.

Conversely, assume that every αg-closed subset of X is $g\alpha$-closed. Let $x \in X_1$ and suppose that $\{x\}$ is not closed. Then $X \setminus \{x\}$ is dense and αg-closed, thus $g\alpha$-closed. It follows from Lemma 3.5 that

$$X_1 \bigcap \alpha\text{-cl}(X \setminus \{x\}) = X_1 \bigcap X = X_1 \subseteq X \setminus \{x\}.$$

So we obtain $x \in X \setminus \{x\}$, a contradiction. By Theorem 3.2 , X is T_{gs}. □

Remark 3.7. A space whose αg-closed subsets are $g\alpha$-closed may be called a *$T_{\alpha g}$-space*. Theorem 3.6 shows, however, that the class of $T_{\alpha g}$-spaces is precisely the class of T_{gs}-spaces.

Proposition 3.8. *A space X is T_{gs} and extremally disconnected if and only if every gs-closed subset of X is preclosed.*

Proof. Suppose that X is T_{gs} and extremally disconnected. Let A be a gs-closed subset. Then A is sg-closed. By Theorem 2.3 (3) of [4], A is preclosed.

Now assume that every gs-closed subset is preclosed. Let $x \in X$ and suppose that $\{x\}$ is not closed. Then $X \setminus \{x\}$ is not open, hence it is gs-closed. By assumption, $X \setminus \{x\}$ is preclosed, and so $\{x\}$ is preopen. By

Theorem 3.2, X is a T_{gs}-space. Moreover, since every sg-closed subset is preclosed, again by Theorem 2.3(3) of [4], X is extremally disconnected. $\Box$

By Theorem 3.6, in a T_{gs}-space, every g-closed subset is $g\alpha$-closed. This suggests a natural question: Characterize those spaces whose $g\alpha$-closed subsets are g-closed. Clearly, if X is nodec, i.e. every nowhere dense subset of X is closed [10], then every $g\alpha$-closed subset of X is g-closed since in that case $\alpha(X)$ coincides with the given topology on X. Also observe that X is nodec if and only if every nowhere dense subset is discrete as a subspace.

Theorem 3.9. *For a space X the following are equivalent:*
(1) Every $g\alpha$-closed set is g-closed,
(2) every nowhere dense subset is locally indiscrete as a subspace,
(3) every nowhere dense subset is g-closed,
(4) every α-closed set is g-closed.

Proof. (1) $\to$ (2) : Let $N \subseteq X$ be nowhere dense, let U be open and let $N_1 = U \bigcap N$. We have to show that N_1 is closed in N. Since N_1 is nowhere dense it is α-closed, hence $g\alpha$-closed and so g-closed. Since $N_1 \subseteq U$, we have $\mathrm{cl}N_1 \subseteq U$ and hence $\mathrm{cl}N_1 \bigcap N = N_1$, i.e. N_1 is closed in N.

(2) $\to$ (3) : Let $N \subseteq U$ where N is nowhere dense and U is open. Let $x \in \mathrm{cl}N$. Then $\mathrm{cl}\{x\} \subseteq \mathrm{cl}N$. Since $\mathrm{cl}N$ is nowhere dense, by (2) we have that $\mathrm{cl}\{x\}$ is also open in $\mathrm{cl}N$, i.e. there exists an open set W such that $\mathrm{cl}\{x\} = W \bigcap \mathrm{cl}N$. Suppose that $x \notin U$. Then $\mathrm{cl}\{x\} \subseteq X \setminus U$ and so $W \bigcap N \subseteq W \bigcap \mathrm{cl}N \bigcap U = \emptyset$, a contradiction. Therefore $\mathrm{cl}N \subseteq U$.

(3) $\to$ (4) : Let $F \subseteq X$ be α-closed. Then $F = A \bigcup N$ where A is closed and N is nowhere dense. If $F \subseteq U$ where U is open then, by our assumption, $\mathrm{cl}N \subseteq U$ and so $\mathrm{cl}F \subseteq U$, i.e. F is g-closed.

(4) $\to$ (1) : Let A be $g\alpha$-closed with $A \subseteq U$ where U is open. By assumption $\alpha\text{-cl}A = A \bigcup \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \subseteq U$. It is easily checked that $N = A \setminus \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$ is nowhere dense, hence α-closed and so g-closed by assumption. Since $N \subseteq U$ we have $\mathrm{cl}A \bigcap (X \setminus \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \subseteq \mathrm{cl}N \subseteq U$. It follows readily that $\mathrm{cl}A \setminus \alpha\text{-cl}A \subseteq U$ and so $\mathrm{cl}A \subseteq U$, i.e. A is g-closed. $\Box$

In our next example we will show that there exist spaces whose nowhere dense subsets are g-closed but which are not nodec.

Example 3.10. Let X be the real line and let $X_1 = \{x \in X : x > 0\}$ and $X_2 = \{x \in X : x < 0\}$. We now define a topology on X in the following way. Let $\{0\}$ be open. If $x \in X_1$, a basic (minimal) open neighbourhood of x is $X_1 \bigcup \{0\}$. If $x \in X_2$, a basic (minimal) open neighbourhood of x is $X_2 \bigcup \{0\}$. Clearly, if $x \in X_1$ then $\{x\}$ is nowhere dense but not closed, so X fails to be nodec. Now let $N \subseteq U$ where N is nowhere dense and U is open. Then $0 \notin N$. Let $N_1 = N \bigcap X_1$ and $N_2 = N \bigcap X_2$. If $x \in N_1$ then $x \in U$ and so $X_1 \subseteq U$. Hence $\mathrm{cl}N_1 \subseteq X_1 \subseteq U$. In the same manner, $\mathrm{cl}N_2 \subseteq U$ and so $\mathrm{cl}N \subseteq U$, i.e. N is g-closed.

4 More Characterizations

We now return to the diagram in Section 1 to consider other possible converses of some of the implications in that diagram. The following result about the class of extremally disconnected spaces was proved in [4].

Theorem 4.1. [4] *For a space X, the following statements are equivalent:*
(1) (X, τ) is extremally disconnected,
(2) $\mathrm{scl}(A \bigcup B) = \mathrm{scl}A \bigcup \mathrm{scl}B$ for all $A, B \subseteq X$,
(3) the union of two semi-closed subsets of X is semi-closed,
(4) the union of two sg-closed subsets of X is sg-closed,
(5) every semi-preclosed subset of X is preclosed,
(6) every sg-closed subset of X is preclosed,
(7) every semi-closed subset of X is preclosed,
(8) every semi-closed subset of X is α-closed,
(9) every semi-closed subset of X is $g\alpha$-closed.

Next we present one more characterization of extremal disconnectedness using generalized closed subsets.

Theorem 4.2. *A space X is extremally disconnected if and only if every gs-closed subset of X is αg-closed.*

Proof. Suppose that X is extremally disconnected. Let A be gs-closed and let U be an open set containing A. Then $\mathrm{scl}A \subseteq U$, i.e. $\mathrm{int}(\mathrm{cl}A) \subseteq U$. Since $\mathrm{int}(\mathrm{cl}A)$ is closed, we have $\alpha\text{-}\mathrm{cl}A = A \bigcup \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \subseteq A \bigcup \mathrm{int}(\mathrm{cl}A) \subseteq U$. Hence A is αg-closed.

To prove the converse, let every gs-closed subset of X be αg-closed. Let $A \subseteq X$ be regular open. Then A is gs-closed and so αg-closed. It follows that $\mathrm{cl}A = \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) = \alpha\text{-}\mathrm{cl}A \subseteq A$. Therefore A is closed and X is extremally disconnected. □

We now consider the property of sg-submaximality. First we give some elementary characterizations of sg-submaximal spaces. Since the proof of the following result is straightforward, we will omit it.

Theorem 4.3. *For a space X, the following are equivalent:*

(1) X is sg-submaximal,

(2) every subset of X is an intersection of a closed subset and an sg-open subset of X,

(3) every subset of X is a union of an open subset and an sg-closed subset of X,

(4) every codense subset A of X is sg-closed,

(5) $\mathrm{cl}A \setminus A$ is sg-closed for every subset A of X.

A more advanced result about sg-submaximality was obtained in [4].

Theorem 4.4. [4] *For a space X with Hewitt decomposition $X = F \bigcup G$, the following are equivalent:*

(1) $X_1 \subseteq \mathrm{cl}G$,

(2) every preclosed subset of X is sg-closed,

(3) X is sg-submaximal,

(4) X is sg-submaximal with respect to $\alpha(X)$.

We shall now improve the equivalence of (2) and (3) in Theorem 4.4 thereby providing a new characterization of sg-submaximal spaces.

Theorem 4.5. *A space X is sg-submaximal if and only if every preclosed subset of X is gs-closed.*

Proof. The necessity is trivial by Theorem 4.4 (2). For the sufficiency, suppose that every preclosed subset is gs-closed. Let $X = F \bigcup G$ be the Hewitt decomposition of X, and let $\langle E_1, E_2 \rangle$ be a resolution of $\mathrm{int}F$.

We first claim that every open set $V \subseteq \mathrm{int}\ F$ is regular open. In fact, $V \bigcap E_1$ is codense and contained in V. Since codense sets are preclosed, by assumption, they are gs-closed. Thus $\mathrm{int}(\mathrm{cl}(V \bigcap E_1)) \subseteq V$. On the other hand, E_1 is dense in $\mathrm{int}F$, hence we have $\mathrm{int}(\mathrm{cl}(V \bigcap E_1)) = \mathrm{int}(\mathrm{cl}V)$. It follows that $V = \mathrm{int}(\mathrm{cl}V)$.

Now let $x \in \mathrm{int}F$ and let $V = \mathrm{int}F \bigcap (X \setminus \mathrm{cl}(\{x\}))$. Suppose that $\{x\}$ is nowhere dense. Then $X \setminus \mathrm{cl}\{x\}$ is dense and $\mathrm{int}(\mathrm{cl}V) = \mathrm{int}(\mathrm{cl}(\mathrm{int}F)) = \mathrm{int}F$. By our claim, $\mathrm{int}F = V$. Hence $\mathrm{int}F \subseteq X \setminus \{x\}$, a contradiction. Therefore $\{x\}$ has to be preopen. We have thus proved that $\mathrm{int}F \subseteq X_2$, i.e. $X_1 \subseteq \mathrm{cl}G$. By Theorem 4.4, X is sg-submaximal. □

Remark 4.6. One may define a space X to be *gs-submaximal* if each dense subset of X is gs-open. Similar to the proof of Theorem 4.4, one checks easily that a space X is gs-submaximal if and only if each preclosed subset of X is gs-closed. In the light of Theorem 4.5, the notion of gs-submaximality coincides with that of sg-submaximality.

Acknowledgement

The authors wish to thank Dr. Julian Dontchev for providing the latex version of the diagram in Section 1.

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