Extremally T_1-spaces and Related Spaces*

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Abstract

The aim of this paper is introduce and initiate the study of extremally T_1-spaces, i.e., the spaces where all hereditarily compact C_2-subspaces are closed. A C_2-space is a space whose nowhere dense sets are finite.

1 Introduction

In this paper, we consider a new class of topological spaces, called extremally T_1-spaces, which is strictly placed between the classes of kc-spaces and T_1-spaces. Recall that a topological space (X, τ) is called a *kc-space* if every compact subset of X is closed. Such spaces have been considered by Hewitt [16], Ramanathan [19] and Vaidyanathaswamy [25]. Ramanathan proved that compact kc-spaces are maximally compact and minimal kc, and that every maximal compact space is a kc-space.

Kc-spaces have been also studied by Aull [1], Cullen [6], Halfar [15], Insell [17] and Wilansky [26]. It was Wilamsky's paper [26] which studied most systematically separation properties between T_1 and Hausdorff. In fact, kc-spaces were named so by Wilansky who also called the spaces with unique convergent sequences us-spaces (Aull called kc-spaces J_1'-spaces). Note that the following implications hold and none of them is reversible:

$$\text{Hausdorff space} \Rightarrow \text{kc-space} \Rightarrow \text{us-space} \Rightarrow T_1\text{-space}$$

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Cullen proved first that kc-spaces are us-spaces and he also showed that in spaces satisfying the first axiom of countability the notions of an us-space and a Hausdorff space coincide. Wilansky showed that if the first axiom of countability fails to hold, then the reverse implications need not be true. He also showed that in locally compact spaces the notions of a kc-space and a Hausdorff space coincide. Wilansky showed that the Alexandroff compactification of every kc-space is a us-space and that a space is a kc-space if and only if its Alexandroff compactification is a k-space. Also it is well-known that every (maximally hereditarily compact space and every) hereditarily compact kc-space is finite.

Just recently, in 1995 and 1996, a stronger form of hereditary compactness called sg-compactness has been introduced in three different papers. Caldas [4], Devi, Balachandran and Maki [7] and Tapi, Thakur and Sonwalkar [24] considered topological spaces in which every cover by sg-open sets has a finite subcover. Such spaces have been called *sg-compact* and their study was continued by Dontchev and Ganster in [10, 11]. (A subset of a given space is called sg-compact if it is sg-compact as a subspace.)

It turned out that sg-compactness is a much stronger property than hereditary compactness since even spaces with finite topologies need not be sg-compact. Thus the general behavior of sg-compact spaces seems to be more 'pathological' than the one of hereditarily compact spaces, especially if we consider product spaces (see [11]). Sg-compactness is stronger than semi-compactness. Recall that a topological space (X, τ) is called *semi-compact* [12] if every cover of X by semi-open sets has a finite subcover. One of the most significant characterizations of semi-compact spaces is the following: A topological space is semi-compact if and only if it is an S-closed C_2-space [13] if and only if it is a hereditarily compact C_2-space [9]. A space (X, τ) is called a *C_2-space* [13] (originally to satisfy condition C2) if every nowhere dense subset is finite. We also observe that every semi-compact space is hereditarily semi-compact.

We also want to mention two separation axioms between us and kc which were introduced by Aull in [2]. A topological space (X, τ) is called an *S_1-space* (resp. *S_2-space*) if X is an us-space and every convergent sequence has a subsequence without side points (resp. no convergent sequence has a side point). Recall that a point p is called a *side point* of a sequence $\{x_n\}$ if p is an accumulation point of $\{x_n\}$ but no subsequence of $\{x_n\}$ converges to p.

Further definitions of unknown concepts may be found in [7, 10, 11].

2 Extremally T_1-spaces

Definition 1 A topological space (X, τ) is called an *extremally T_1-space* if every sg-compact subspace is closed.

Remark 2.1 Every extremally T_1-space is T_1.

Theorem 2.2 *For a topological space (X, τ) the following conditions are equivalent:*

(0) *X is an extremally T_1-space.*

(1) *Every semi-compact subspace of X is closed.*

(2) *Every hereditarily compact C_2 subspace of X is closed.*

(3) *Every hereditarily sg-compact subspace of X is closed.*

Proof. (0) $\Rightarrow$ (1) Follows from the fact that in T_1-spaces semi-open sets and sg-open sets coincide.

(1) $\Rightarrow$ (0) and (1) $\Rightarrow$ (3) are obvious, since every sg-compact space is semi-compact.

(1) $\Leftrightarrow$ (2) This is a consequence from the fact that a space is semi-compact if and only if it is a hereditarily compact C_2-space.

(3) $\Rightarrow$ (1) By (3), X is T_1 and so sg-open sets are semi-open. Hence every semi-compact subspace is hereditarily sg-compact and thus closed in X. $\square$

Remark 2.3 Perhaps the reader wants some explanation why we have called the spaces in Definition 1 extremally T_1. If in that definition, we replace 'sg-compact' with any stronger known form of compactness we seem to get an equivalent definition of T_1-spaces. For example, if sg-compactness is replaced with β-compactness (i.e., every cover by β-open sets has a finite subcover, where a set is β-open if and only if it is dense in some regular closed subspace), we get nothing but the separation axiom T_1. Note that it was observed by Ganster [14] that every β-compact space is finite. At present, we are not aware of any nontrivial form of compactness strictly stronger than sg-compactness.

Remark 2.4 Since every semi-compact space is hereditarily semi-compact, in an extremally T_1-space every semi-compact subspace is closed and discrete, and hence has to be finite. Let us call a space (X, τ) an *scf-space* if its semi-compact subspaces are finite. The following result is now obvious.

Proposition 2.5 *A space (X, τ) is extremally T_1 if and only if it is a T_1 scf-space.*

Clearly the following diagram holds and none of the implications is reversible:

$$\begin{tikzcd}
\text{Hausdorff space} \arrow[r] & \text{kc-space} \arrow[r] \arrow[d] & \text{extremally } T_1\text{-space} \arrow[r] & T_1\text{-space} \\
 & S_2\text{-space} \arrow[r] & S_1\text{-space} \arrow[r] & \text{us-space} \arrow[u]
\end{tikzcd}$$

Example 2.6 A T_1-space need not be extremally T_1. Consider the real line $\mathbb{R}$ with the cofinite topology τ. Then $(\mathbb{R}, \tau)$ is T_1 and semi-compact, hence cannot be extremally T_1 by Proposition 2.5.

Question. Is there an example of an S_2-space which is not extremally T_1?

Example 2.7 An extremally T_1-space need not be a kc-space, not even an us-space. Let $\mathbb{R}$ be the real line with the following topology τ. Each point $x \notin \{1, 2\}$ is isolated. If $x \in \{1, 2\}$ then a basic open neighbourhood of x is a cofinite subset of $\mathbb{R}$ containing x. If A is an sg-compact subspace then $A \setminus \{1, 2\}$ must be clearly finite. Hence $(\mathbb{R}, \tau)$ is extremally T_1. Note, however, that any sequence in $\mathbb{R} \setminus \{1, 2\}$ converges to both 1 and 2, so the space is not us and hence not kc.

Question. When is an extremally T_1-space a kc-space (or Hausdorff)?

Proposition 2.8 *Every subspace of an extremally T_1-space is an extremally T_1-space.*

Proof. Follows immediately from Proposition 2.5. □

Proposition 2.9 *For a topological space (X, τ) the following conditions are equivalent:*

(0) *X is an extremally T_1-space.*

(4) *X is the topological sum of finitely many extremally T_1-spaces.*

(5) *X is the finite union of closed extremally T_1-spaces.*

Proof. (0) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (2) Assume that $X = \cup_{i=1}^{n} A_i$, where A_i is closed and extremally T_1 for $1 \leq i \leq n$. Then X is clearly T_1. Let S be a semi-compact subspace of X. For $1 \leq i \leq n$, $S \cap A_i$ is a semi-compact subspace of A_i and hence finite by Proposition 2.5. Thus S is finite. Again by Proposition 2.5, we conclude that X is extremally T_1. $\square$

Recall that a space (X, τ) is called *ckc* [20] if every countable, compact set is closed. Clearly, we have the following implications:

$$\text{kc-space} \Rightarrow \text{ckc-space} \Rightarrow \text{us-space}$$

Singal [20] left open the question whether those implications are reversible. We now provide two examples showing that the implications are strict.

Example 2.10 (i) Let X be the Arens-Fort space (see [22], page 54). Then X is a Hausdorff space whose compact subsets are finite. If $Y = X \cup \{a\}$ denotes the Alexandroff compactification of X then it is easily checked that Y is an us-space. Since the neighbourhoods of $\{a\}$ are the union of $\{a\}$ and a cofinite subset of X, it follows readily that $Y \setminus \{(0,0)\}$ is a countable and compact subset which is not closed. Hence Y is not a ckc-space.

(ii) Let Y be an uncountable set with the co-countable topology, let p be a point not in Y and let $X = Y \cup \{p\}$. A topology on X is defined in the following way: the co-countable subsets of Y are open in X, a basic neighbourhood of p consists of $\{p\}$ and cofinite subset of Y. Now, every subset of X containing $\{p\}$, and hence X itself, is compact. If X_1 is an uncountable subset of X containing p such that the complement of X_1 is also uncountable, then X_1 is compact but not closed. So X is not kc. Let C be a denumerable compact subset of X. Then p must be an element of C (since the only compact subsets of Y are the finite ones). If $x \notin C$, then $x \in Y$. Now, $Y \setminus C$ is clearly a neighbourhood of x which is disjoint from C. So C is closed, and hence X is a ckc-space.

A space (X, τ) is called *semi-pre-$T_{\frac{1}{2}}$-space* [8] if every generalized semi-preclosed set is semi-preclosed, or equivalently, if every nowhere dense singleton is closed.

Theorem 2.11 *Let (X, τ) be a second countable T_0-space. If X is either (a) ckc or (b) a semi-pre-$T_{\frac{1}{2}}$, C_2-space whose countable, compact sets are nowhere dense, then X is extremally T_1.*

Proof. Let $S \subseteq X$ be hereditarily compact and C_2. Since by assumption, $(S, \tau|S)$ is second countable and T_0, it follows from a result of Stone [23] that S is countable. If X is ckc, then S is closed and thus X is extremally T_1. If X is a semi-pre-$T_{\frac{1}{2}}$, C_2-space whose countable, compact sets are nowhere dense, then S is nowhere dense in X. Since (X, τ) is C_2, it follows that S is finite. Moreover, every nowhere dense singleton of X is closed, and so S is closed. Thus X is extremally T_1. $\Box$

Recall that a space (X, τ) is called *$\mathbb{Z}$-pseudocompact* [18] if there exists no continuous function from X onto $\mathbb{Z}$. Furthermore, (X, τ) is called *mildly Lindelöf* [21] if every clopen cover of X has a countable subcover.

Theorem 2.12 *If a mildly Lindelöf space (X, τ) is homeomorphic to the topological sum of finite family of connected spaces (or equivalently if a mildly Lindelöf space X is weakly locally connected and $\mathbb{Z}$-pseudocompact), then X is extremally T_1 if and only if every quasi-component of X is extremally T_1.*

Proof. If X is extremally T_1, then every quasi-component is extremally T_1 by Proposition 2.8. Assume next that every quasi-component is extremally T_1. Since X is weakly locally connected, then every quasi-component is clopen. Thus, $X = \cup\{Q_x : x \in X$ and Q_x is the quasi-component of $x\}$ and since X is mildly Lindelöf, X is countable union of clopen extremally T_1-spaces. Now, the $\mathbb{Z}$-pseudocompactness of X implies that X is finite union of clopen extremally T_1-spaces. By Proposition 2.9, X is extremally T_1. $\Box$

Question. Under what kind of mappings are extremally T_1-spaces preserved? What is their behaviour under forming products?

In 1979, Bankston [3] introduced the *anti* operator on a topological space. A space (X, τ) is called *anti-compact* if the only compact subsets of X are the finite ones. Anti-compact spaces are also known under the names *pseudo-finite spaces* or *cf-spaces*. We shall say that a topological space (X, τ) is *weakly anti-compact* if every compact subspace of X is a C_2-space.

Example 2.13 (i) Every C_2-space is weakly anti-compact but not vice versa. Let us consider the density topology on the real line. A measurable set $E \subseteq \mathbb{R}$ has density d at $x \in \mathbb{R}$ if

$$\lim_{h \to 0} \frac{m(E \cap [x-h, x+h])}{2h}$$

exists and is equal to d. Set $\phi(E) = \{x \in \mathbb{R} \colon d(x, E) = 1\}$. The open sets of the density topology $\mathcal{T}$ are those measurable sets E that satisfy $E \subseteq \phi(E)$. Clearly, the density topology $\mathcal{T}$ is finer than the usual topology on the real line. Note that the density topology is (weakly) anti-compact but not C_2, since the nowhere dense subsets of the density topology are precisely the Lebesgue null sets (hence the nowhere dense sets are not necessarily finite).

(ii) The cofinite topology on the real line shows that a weakly anti-compact space need not be anti-compact. This example also shows that a T_1, hereditarily compact C_2-space need not be submaximal. Recall that a topological space is called *submaximal* if every dense subset is open or, equivalently, if every subset is locally closed. Next we consider what happens if 'T_1' is replaced with 'extremally T_1'.

Theorem 2.14 *Every locally hereditarily compact subspace S of a weakly anti-compact, extremally T_1-space (X, τ) is locally closed.*

Proof. Let $x \in S$. Since $(S, \tau|S)$ is locally hereditarily compact, there exists $U \in \tau$ such that $U \cap S$ is hereditarily compact and hence C_2, since X is weakly anti-compact. Thus $U \cap S$ is closed in X, since by assumption X is extremally T_1. Clearly, $U \cap S$ is also closed in U. This shows that S is locally closed. $\Box$

Corollary 2.15 *Every locally hereditarily compact, weakly anti-compact, extremally T_1-space (X, τ) is submaximal.*

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