On compactness with respect to countable extensions of ideals and the generalized Banach Category theorem*

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Abstract

We extend a theorem of Hamlett and Janković by proving that if a topological space (X, τ) is compact with respect to the countable extension of $\mathcal{I}$, then the local function $A^*(\mathcal{I})$ of every subset A of X with respect to τ and $\mathcal{I}$ is a compact subspace with respect to the extension $\tilde{\mathcal{I}}$ in $A^*(\mathcal{I})$. We also give a generalized version of the Banach Category theorem.

1 Introduction

In 1990, Hamlett and Janković [4] improved a theorem of Rančin from 1972 [9] by showing that if a space is $\mathcal{M}$-compact (that is every open cover has a finite subfamily covering the space but a meager set), then X^* (this is the local function of X with respect to the ideal $\mathcal{M}$) is QHC (= quasi-H-closed) as a subspace. A topological space (X, τ) is said to be *quasi-H-closed* (= QHC) if for every open cover $\{V_\alpha : \alpha \in A\}$ of X, there exists a finite subset A_0 of A such that $X = \cup\{\mathrm{Cl}(V_\alpha) : \alpha \in A_0\}$.

A nonempty family $\mathcal{I}$ of subsets on a topological space (X, τ) is called an *ideal* on X if it satisfies the following two conditions:

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(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (heredity).

(2) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

A *σ-ideal* on a topological space (X, τ) is an ideal which satisfies:

(3) If $\{A_i : i = 1, 2, 3, \ldots\} \subseteq \mathcal{I}$, then $\bigcup\{A_i : i = 1, 2, 3, \ldots\} \in \mathcal{I}$ (countable additivity).

If $X \notin \mathcal{I}$, then $\mathcal{I}$ is called a *proper* ideal. The collection of the complements of all elements of a proper ideal is a filter, hence proper ideals are sometimes called *dual filters*.

The following collections form important ideals on a topological space (X, τ): the ideal of all finite sets $\mathcal{F}$, the ideal of all countable sets $\mathcal{C}$, the ideal of all closed and discrete sets $\mathcal{CD}$, the ideal of all nowhere dense sets $\mathcal{N}$, the ideal of all first category (= meager) sets $\mathcal{M}$, the ideal of all scattered sets $\mathcal{S}$ (here X must be T_0) and the ideal of all Lebesgue null sets $\mathcal{L}$ (here X stands for the real line).

By an *ideal topological space*, we mean a topological space (X, τ) with an ideal $\mathcal{I}$ on X and we denote it by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}) = \{x \in X :$ for every $U \in \tau(x), U \cap A \notin \mathcal{I}\}$ is called the *local function* of A with respect to $\mathcal{I}$ and τ [5, 6]. We simply write A^* instead of $A^*(\mathcal{I})$ in case there is no chance for confusion.

Recall that a subset A of a space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-compact* [4, 7, 9] if for each τ-open cover $(U_i)_{i \in I}$ of A there exists a finite subcollection $(U_1, \ldots, U_n)$ such that $A \setminus (U_1 \cup \ldots \cup U_n) \in \mathcal{I}$. An ideal topological space $(X, \tau, \mathcal{I})$ is *$\mathcal{I}$-compact* if X is an $\mathcal{I}$-compact set. Clearly, classical compactness coincides with compactness with respect to the minimal ideal $\{\emptyset\}$ (or with respect to $\mathcal{F}$ if you prefer).

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on (X, τ), then the *extension of $\mathcal{I}$ via $\mathcal{J}$* [5], denoted by $\mathcal{I} * \mathcal{J}$ is the collection $\{A \subseteq X : A^*(\mathcal{I}) \in \mathcal{J}\}$. The extension of $\mathcal{I}$ over the ideal of nowhere dense sets $\mathcal{N}$ is usually denoted by $\tilde{\mathcal{I}}$ [5]. Clearly, $\tilde{\mathcal{I}} = \{A \subseteq X : \mathrm{Int} A^*(\mathcal{I}) = \emptyset\}$. The *countable extension of $\mathcal{I}$* [5], denoted by $\tilde{\mathcal{I}}_\sigma$, is the family $\{A \subseteq X : A = \cup_{n \in \mathbb{N}} A_n,\ A_n \in \tilde{\mathcal{I}}$ for each $n \in \mathbb{N}\}$.

2 The generalized Banach Category theorem

An important property of ideals was first proved for the ideal of meager sets by Banach in 1930 [1]. Recall that the topology τ of an ideal topological space $(X, \tau, \mathcal{I})$ is *compatible* with the ideal $\mathcal{I}$ [8], denoted $\tau \sim \mathcal{I}$, if the following condition holds for every subset A of X: if for every $x \in A$ there exists a $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, i.e., if $A \cap A^*(\mathcal{I}) = \emptyset$, then $A \in \mathcal{I}$. If this condition holds, then the ideal $\mathcal{I}$ is also sometimes said to be *local relative to the topology* [10] or to have the *strong Banach's localization property* [11]. For example, the σ-ideal of meager sets is always local and every topology is compatible with the ideal of meager subsets—this result is known as the *Banach category theorem* and was first proven by Banach [1] for metric spaces and extended to general topological spaces by Kuratowski [6].

In 1992, Janković and Hamlett [5] extended the results of Banach and Kuratowski by proving the following:

Theorem 2.1 [5] (Generalized Banach Category theorem) *For every ideal topological space* $(X, \tau, \mathcal{I})$*, we have* $\tilde{\mathcal{I}}_\sigma \sim \tau$*.*

The generalized Banach Category theorem plays a central role in the proof of our next result which in turn extends the already mentioned theorem of Hamlett and Janković from [4].

Theorem 2.2 *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If* (X, τ) *is compact with respect to the countable extension* $\tilde{\mathcal{I}}_\sigma$ *of* $\mathcal{I}$*, then the local function of every subset* A *of* X *with respect to* τ *and* $\tilde{\mathcal{I}}_\sigma$ *is a compact subspace with respect to the extension* $\tilde{\mathcal{I}}$ *in the subspace* $(A^*, \tau|A^*)$*.*

Proof. Throughout this proof, A^* will stand for $A^*(\tilde{\mathcal{I}}_\sigma)$. Let $\{U_i : i \in I\}$ be a cover of A^* consisting of open subsets of $(A^*, \tau|A^*)$. For each $i \in I$, let V_i be open in (X, τ) such that $V_i \cap A^* = U_i$. Since A^* is closed in (X, τ), the family $\{V_i : i \in I\} \cup \{X \setminus A^*\}$ is an open cover of (X, τ). Due to the $\tilde{\mathcal{I}}_\sigma$-compactness of (X, τ), there exists a finite subset $F \subseteq I$ such that $X \setminus ((\cup_{i \in F} V_i) \cup (X \setminus A^*)) = S \in \tilde{\mathcal{I}}_\sigma$. Hence, $(\cup_{i \in F} U_i) \cup S = A^*$ and $(\cup_{i \in F} V_i) \cap S = \emptyset$, in particular, S is closed in $(A^*, \tau|A^*)$. We now claim that $S \in \tilde{\mathcal{I}}|A^*$. Assuming the contrary, suppose there exists nonempty set $W \in \tau|A^*$ such that $W \subseteq S^*(\mathcal{I}|A^*) \subseteq \mathrm{Cl}_{A^*} S = S$. Let $W' \in \tau$ such that $W' \cap A^* = W$. Clearly, if $x \in W$ then $x \in A^*$ and so $W' \cap A \notin \tilde{\mathcal{I}}_\sigma$. On the other hand, $W' \cap A = (W' \cap (A^* \cap A)) \cup (W' \cap (A \setminus A^*))$. Set $F = W' \cap (A^* \cap A)$ and $G = W' \cap (A \setminus A^*)$. Since $F \subseteq S$ and $S \in \tilde{\mathcal{I}}_\sigma$, we have $F \in \tilde{\mathcal{I}}_\sigma$. If $x \in G$, then $x \notin A^*$

and so there exists $H \in \tau$ such that $H \cap A \in \tilde{\mathcal{I}}_\sigma$. Since $G \subseteq A$, it follows immediately from the Generalized Banach Category theorem that $G \in \tilde{\mathcal{I}}_\sigma$. Due to the finite additivity of $\tilde{\mathcal{I}}_\sigma$, we have $F \cup G = W' \cap A \in \tilde{\mathcal{I}}_\sigma$, a contradiction. This shows that $(A^*, \tau|A^*)$ is $\tilde{\mathcal{I}}|A^*$-compact. □

Note that, if $\mathcal{I} = \{\emptyset\}$ then $\tilde{\mathcal{I}} = \mathcal{N}$, so its countable extension is $\mathcal{M}$. Clearly, a space X is QHC if and only if X is $\mathcal{N}$-compact. An immediate consequence of the theorem above is the following result of Hamlett and Janković.

Corollary 2.3 [4] *If a topological space (X, τ) is $\mathcal{M}$-compact, then X^* is QHC as a subspace.*

Proof. Set $\mathcal{I} = \{\emptyset\}$, $A = X$ and apply Theorem 2.2. □

Example 2.4 Let $\mathbb{Q}$ be the space of all rationals with the usual topology (inherited from the real line). Clearly, $\mathbb{Q}$ is $\mathcal{M}$-compact but not QHC. Thus, if a space $(X, \tau, \mathcal{I})$ is compact with respect to the countable extension $\tilde{\mathcal{I}}_\sigma$ of an ideal $\mathcal{I}$, then not every subset A of X need to be a compact subspace with respect to the extension $\tilde{\mathcal{I}}$ in A.

A topological space (X, τ) is called *sporadic* [2] if the Cantor-Bendixson derivative of X is meager. Spaces without isolated points are usually called *dense-in-themselves* or *crowded*. Recall also that a space X is called a $T_{\frac{1}{2}}$*-space* if each singleton is open or closed.

Corollary 2.5 *If (X, τ) is a crowded sporadic $T_{\frac{1}{2}}$-space. Then the derived set A^d of every subset A of X is QHC.*

Proof. We will apply Theorem 2.2 for $\mathcal{I} = \mathcal{F}$. First we note that X is T_1 (crowded + $T_{\frac{1}{2}} \Rightarrow T_1$). Thus $A^*(\mathcal{F}) = A^d$ for every subset A of X. Since $\tilde{\mathcal{F}} = \mathcal{N}$ by [5, Theorem 5.2], we have that A^d is QHC $\Leftrightarrow$ A^d is $\mathcal{N}$-compact $\Leftrightarrow$ A^d is $\tilde{\mathcal{F}}$-compact. Now, $\tilde{\mathcal{F}}_\sigma = \mathcal{N}_\sigma = \mathcal{M}$ and since X^d is meager, $X^d = X$ is trivially $\tilde{\mathcal{F}}_\sigma$-compact. By Theorem 2.2, A^d is QHC. □

Recall that an ideal $\mathcal{I}$ is called *codense* if each of its members is codense. Note that an ideal $\mathcal{I}$ is codense if and only if $\tau \cap \mathcal{I} = \{\emptyset\}$. Observe that a topological space (X, τ) is a Baire space if and only if $\mathcal{M}$ is codense. If $\mathcal{I} \cap PO(X, \tau) = \{\emptyset\}$ (resp. $\mathcal{I} \cap RO(X, \tau) = \{\emptyset\}$), then $\mathcal{I}$ is called *completely codense* [3] (resp. *regular*). Here, $PO(X,\tau)$ (resp. $RO(X,\tau)$) denotes the collection of all preopen (resp. regular open) subsets of (X,τ), where a set A is *preopen* (resp. *regular open*) if $A \subseteq \text{Int}(\text{Cl}(A))$ (resp. $A = \text{Int}(\text{Cl}(A))$). We will say that $\mathcal{I}$ is *normal* if $\mathcal{I}$ is regular and $\mathcal{N} \subseteq \mathcal{I}$.

Note that every codense ideal is regular, in particular, $\mathcal{N}$ is regular. Most of the well-known ideals defined on the real line with the usual topology are regular, for example $\mathcal{F}, \mathcal{C}, \mathcal{CD}, \mathcal{N}, \mathcal{L}, \mathcal{S}$. In hyperconnected (= irreducible) spaces every dual filter is regular. One can also show that $\mathcal{I}$ is completely codense and normal if and only if $\mathcal{I} = \mathcal{N}$.

Example 2.6 Consider the density topology on the real line. Recall that a measurable set $E \subseteq \mathbb{R}$ has density d at $x \in \mathbb{R}$ if

$$\lim_{h\to 0} \frac{m(E \cap [x-h, x+h])}{2h}$$

exists and is equal to d. Set $\phi(E) = \{x \in \mathbb{R} : d(x,E) = 1\}$. The open sets of the density topology $\mathcal{T}$ are those measurable sets E which satisfy the condition $E \subseteq \phi(E)$. Note that the density topology $\mathcal{T}$ is finer than the usual topology on the real line. Now, it can be easily seen that $\mathcal{N}, \mathcal{M}, \mathcal{CD}, \mathcal{L}$ are all normal ideal in the density topology. Note, however, that (for example) $\mathcal{CD}$ and $\mathcal{L}$ are not normal in the usual topology.

Our next example will show that $\tilde{\mathcal{I}}$ does not necessarily coincide with the extension of $\mathcal{I}$ over an arbitrary codense (and hence regular) ideal. However, we will show that $\tilde{\mathcal{I}}$ is precisely the extension of $\mathcal{I}$ over an arbitrary normal ideal.

Example 2.7 Let $X = \{a,b,c\}$, $\tau = \{\emptyset, \{a,b\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}\}$. Note that $\mathcal{N} = \{\emptyset, \{c\}\}$. It is easily checked that $\tilde{\mathcal{I}} = \mathcal{I} * \mathcal{N} = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}$ and $\mathcal{I} * \mathcal{J} = \{\emptyset, \{a\}\}$.

Proposition 2.8 *Let $(X,\tau,\mathcal{I})$ be an ideal topological space. Then $\tilde{\mathcal{I}}$ coincides with the extension of $\mathcal{I}$ via an arbitrary normal ideal $\mathcal{J}$.*

Proof. Let $\mathcal{J}$ be an arbitrary normal ideal. We claim that $\tilde{\mathcal{I}} = \mathcal{I} * \mathcal{J}$. Assume first that $A \in \tilde{\mathcal{I}}$. Then $A^*(\mathcal{I}) \in \mathcal{N}$ and since $\mathcal{J}$ is normal, we have $A^*(\mathcal{I}) \in \mathcal{J}$. This shows that $A \in \mathcal{I} * \mathcal{J}$. For the converse, assume that $A \in \mathcal{I} * \mathcal{J}$, i.e. $A^*(\mathcal{I}) \in \mathcal{J}$. Due to the heredity of $\mathcal{J}$, we have $\text{Int}A^*(\mathcal{I}) \in \mathcal{J}$. On the other hand, $\text{Int}(\text{Cl}(A^*(\mathcal{I}))) = \text{Int}A^*(\mathcal{I})$ is regular open in (X, τ). Since $\mathcal{J}$ is normal and since $\text{Int}A^*(\mathcal{I}) \in RO(X, \tau) \cap \mathcal{J}$, we have $\text{Int}A^*(\mathcal{I}) = \emptyset$. Thus $A^*(\mathcal{I})$ is codense and hence nowhere dense (being closed). This shows that $A^*(\mathcal{I}) \in \mathcal{N}$. Consequently, $A \in \tilde{\mathcal{I}}$. □

Now, in the notion of Proposition 2.8, we can generalize the Banach Category theorem as follows:

Theorem 2.9 *Let (X, τ) be a topological space. Then, the countable extension of every ideal $\mathcal{I}$ via an arbitrary normal ideal $\mathcal{J}$ is always compatible with the topology τ.*

Proof. Follows directly from Theorem 2.1 and Proposition 2.8. □

Question. Does Theorem 2.9 remain true if 'normal' is replaced with 'regular'?

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