On Preclosed Sets and Their Generalizations

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Abstract

This paper continues the study of preclosed sets and of generalized preclosed sets in a topological space. Our main objective is to establish results about the relationships between the various types of generalized closed sets. As a by-product, we are able to provide characterizations of certain known classes of topological spaces by using preclosed sets and their generalizations.

1 Introduction

Let X be a topological space. Recall that a subset A of X is said to be *preclosed* if $\mathrm{cl}(\mathrm{int}A) \subseteq A$. The *preclosure* of A, denoted by $\mathrm{pcl}A$, is the smallest preclosed set in X containing A. It is easy to check that $\mathrm{pcl}A = A \bigcup \mathrm{cl}(\mathrm{int}A)$. Complements of preclosed sets are called *preopen* (= nearly open or locally dense [8]) sets. The notion of preclosed sets plays an important role in questions concerning generalized continuity. It leads to the notion of precontinuity. Recall that a function $f : X \to Y$ is *precontinuous* if the inverse image of each closed set of Y is preclosed in X. Blumberg [4] proved that any function $f : \mathbf{R} \to \mathbf{R}$ is precontinuous at each point of a certain dense set of the real line $\mathbf{R}$. Naimpally [18] showed that every linear function between Banach spaces is precontinuous. In Functional Analysis, precontinuous functions are important in the study of various versions of Closed Graph Theorems and Open Mapping Theorems.

In 1970, Levine [13] initiated the investigation of so-called generalized closed sets. By definition, a subset A of a topological space X is called

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generalized closed, briefly *g-closed*, if $\mathrm{cl}A \subseteq U$ whenever $A \subseteq U$ and U is open. Moreover, A is called *generalized open*, or *g-open*, if $X \setminus A$ is g-closed. This concept has been studied extensively in recent years by a large number of topologists (see e.g. References of this paper). In [17], Maki et al introduced the concepts of *pg*-closed sets and *gp*-closed sets in an analogous manner. These notions are generalizations of preclosed sets which were further studied by Dontchev and Maki [10], leading to a new decomposition of precontinuity.

Definition 1. Let X be a topological space. A subset A of X is called

(1) *pre-generalized closed* (briefly, *pg-closed*) [17], if $\mathrm{pcl}A \subseteq U$ whenever $A \subseteq U$ and U is preopen;

(2) *generalized preclosed* (briefly, *gp-closed*) [17] if $\mathrm{pcl}A \subseteq U$ whenever $A \subseteq U$ and U is open.

A space X has been called a *pre-*$T_{\frac{1}{2}}$*-space* [17] if every *pg*-closed set of X is preclosed. It was proved in [17] that a space X is a pre-$T_{\frac{1}{2}}$-space if and only if every singleton of X is either preopen or preclosed. However, it is easily observed that in any topological space, a singleton is either open or preclosed. Therefore, every *pg*-closed set is preclosed, or equivalently, every space is *pre-*$T_{\frac{1}{2}}$. In this paper, we shall therefore study *gp*-closed sets and other generalizations of preclosed sets. Let us recall some basic concepts first, although most of these concepts are well known. A subset A of a topological space X is called α*-open* (resp. *semi-open, semi-preopen*) if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$ (resp. $A \subseteq \mathrm{cl}(\mathrm{int}A)$, $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$). Moreover, A is said to be α*-closed* (resp. *semi-closed, semi-preclosed*) if $X \setminus A$ is α-open (resp. semi-open, semi-preopen) or, equivalently, if $\mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \subseteq A$ (resp. $\mathrm{int}(\mathrm{cl}A) \subseteq A$, $\mathrm{int}(\mathrm{cl}(\mathrm{int}A)) \subseteq A$). The α*-closure* (resp. *semi-closure, semi-preclosure*) of $A \subseteq X$ is the smallest α-closed (resp. semi-closed, semi-preclosed) set containing A. It is well-known that $\alpha\text{-cl}A = A \bigcup \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$, $\mathrm{scl}A = A \bigcup \mathrm{int}(\mathrm{cl}A)$ and $\mathrm{spcl}A = A \bigcup \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$.

Definition 2. Let X be a topological space. A subset A of X is called

(1) *semi-generalized closed* (briefly, *sg-closed*) [3], if $\mathrm{scl}A \subseteq U$ whenever $A \subseteq U$ and U is semi-open;

(2) *generalized semiclosed* (briefly, *gs-closed*) [2] if $\mathrm{scl}A \subseteq U$ whenever $A \subseteq U$ and U is open;

(3) *generalized* α*-closed* (briefly, $g\alpha$*-closed*) [14], if $\alpha\text{-cl}A \subseteq U$ whenever $A \subseteq U$ and U is α-open;

(4) α*-generalized closed* (briefly, αg*-closed*) [15] if $\alpha\text{-cl}A \subseteq U$ whenever $A \subseteq U$ and U is open;

(5) *generalized semi-preclosed* (briefly, *gsp-closed*) [9] if $\mathrm{spcl}A \subseteq U$ whenever $A \subseteq U$ and U is open.

The fundamental relationships between the various types of generalized closed sets considered in Definition 1 and Definition 2 can be summarized in the following diagram.

$$\begin{tikzcd}
 & g\alpha\text{-closed} \arrow[dl] \arrow[dr] & \\
\text{preclosed} \arrow[dr] \arrow[dd] & & \alpha g\text{-closed} \arrow[dl] \\
 & gp\text{-closed} \arrow[dr] & \\
\text{semi-preclosed} \arrow[rr] & & gsp\text{-closed}
\end{tikzcd}$$

Diagram 1

We observe that none of the implications in the above diagram can be reversed in general. The main aim of our paper is to characterize the classes of spaces where the converses of the implications in our diagram hold.

2 T_{gs}-spaces and generalized preclosed sets

In this section, we start with establishing relations between various generalized preclosed sets. This will lead us to characterize the class of T_{gs}-spaces and the class of extremally disconnected spaces. Recall that a space X is called a *T_{gs}-space* [16] if every gs-closed set of X is sg-closed. The following characterization of T_{gs}-spaces has been obtained in [5] and [7].

Lemma 2.1. *For a space X the following are equivalent:*

(1) X is a T_{gs}-space.

(2) Every singleton is either preopen or closed [5].

(3) Every αg-closed subset of X is $g\alpha$-closed [7].

In our next result we offer additional characterizations of T_{gs}-spaces thereby answering several possible questions about our diagram.

Theorem 2.2. *For a space X the following are equivalent:*

(1) X is a T_{gs}-space.

(2) Every gp-closed subset of X is preclosed.

(3) Every gsp-closed subset of X is semi-preclosed.

(4) Every gp-closed subset of X is semi-preclosed.

Proof. The route of our proof is: (1) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (1) $\rightarrow$ (2) $\rightarrow$ (4). Observe that the implications (3) $\rightarrow$ (4) and (2) $\rightarrow$ (4) are obvious.

(1) $\rightarrow$ (3). Suppose that X is a T_{gs}-space and let A be a *gsp*-closed subset of X. We want to show that $\mathrm{spcl}A \subseteq A$. So let $x \in \mathrm{spcl}A$ and suppose that $x \notin A$, i.e. $A \subseteq X \setminus \{x\}$. If $\{x\}$ is preopen, then $\mathrm{spcl}A \subseteq \mathrm{pcl}A \subseteq X \setminus \{x\}$, a contradiction. If $\{x\}$ is closed, then $\mathrm{spcl}A \subseteq X \setminus \{x\}$ since A is *gsp*-closed, also a contradiction. Thus A is semi-preclosed.

(4) $\rightarrow$ (1). First observe that every singleton in any topological space is either preopen or nowhere dense (see e.g. [12]). Now suppose that every *gp*-closed subset of X is semi-preclosed. Let $x \in X$. If $\{x\}$ is preopen we are done. Suppose that $\{x\}$ is nowhere dense, i.e. $\mathrm{int}(\mathrm{cl}\{x\}) = \emptyset$, and not closed. Then $X \setminus \{x\}$ is *gp*-closed, since the only open set containing $X \setminus \{x\}$ is the whole space X itself. By assumption, $X \setminus \{x\}$ is semi-preclosed, i.e. $\{x\}$ is semi-preopen, and so $\{x\} \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}\{x\}))$, a contradiction to the fact that $\{x\}$ is nowhere dense. Thus X is a T_{gs}-space.

The proof of (1) $\rightarrow$ (2) is very similar to the proof of (1) $\rightarrow$ (3) and hence omitted. $\square$

Recall that a space X is said to be *extremally disconnected* if the closure of each open subset of X is open, or equivalently, if every regular closed subset of X is open.

Theorem 2.3. *For a space X the following are equivalent:*

(1) Every gsp-closed subset of X is gp-closed.

(2) Every semi-preclosed subset of X is gp-closed.

(3) The space X is extremally disconnected.

Proof. (1) $\rightarrow$ (2) is obvious. Therefore, we have to show that (2) $\rightarrow$ (3) and (3) $\rightarrow$ (1).

(2) $\rightarrow$ (3). Let A be a regular open subset of X. Then A is semi-preclosed. By hypothesis, A is *gp*-closed and so $\mathrm{pcl}A \subseteq A$ which implies $A = \mathrm{cl}(\mathrm{int}A)$. Therefore, A is closed and hence X is extremally disconnected.

(3) $\rightarrow$ (1). Let A be a *gsp*-closed subset of X, and let $U \subseteq X$ be open with $A \subseteq U$. If $B = \mathrm{spcl}A$ then, by assumption, $A \subseteq B \subseteq U$. Since B is semi-preclosed, by Theorem 2.3 in [5] we have that B is preclosed. Therefore $\mathrm{pcl}A \subseteq B \subseteq U$, i.e. A is *gp*-closed. $\square$

As a consequence of Theorem 2.2 and Theorem 2.3 we now have

Corollary 2.4. *For a space X the following are equivalent:*

(1) Every gsp-closed subset of X is preclosed.

(2) X is T_{gs} and extremally disconnected.

3 When is every gp-closed set αg-closed?

In order to answer this question we need some preparation. A subspace A of a space X is called *resolvable* if it has two disjoint dense subsets, otherwise it is called *irresolvable.* In addition, A is said to be *strongly irresolvable* if every open subspace of A is irresolvable. It has been shown by Hewitt [11] that every topological space X has a decomposition $X = F \cup G$, where F is closed and resolvable, and G is open and hereditarily irresolvable. We will call this decomposition the *Hewitt decomposition* of X. Recall also that a space X is said to be *locally indiscrete* if every open subset of X is closed. The following crucial result has been obtained in [6].

Theorem 3.1. [6] *Let $X = F \cup G$ be the Hewitt decomposition of a space (X, τ) and let $X_1 = \{x \in X : \{x\}$ is nowhere dense $\}$. Then the following are equivalent:*

(1) Every semi-preclosed set is sg-closed.

(2) $X_1 \cap \mathrm{scl}A \subseteq \mathrm{spcl}A$ for each $A \subseteq X$.

(3) $X_1 \subseteq \mathrm{int}(\mathrm{cl}G)$.

(4) (X, τ) is the topological sum of a locally indiscrete space and a strongly irresolvable space.

(5) Every preclosed set is $g\alpha$-closed.

(6) (X, τ^α) is g-submaximal, i.e. every dense set is g-open in (X, τ^α).

Proposition 3.2. *Let $X = F \cup G$ be the Hewitt decomposition of a space X. If every preclosed set is αg-closed then* $\mathrm{int}F$ *is a clopen locally indiscrete subspace.*

Proof. Let U be an open subset of $\mathrm{int}F$. Since $\mathrm{int}F$ is resolvable, it can be decomposed into two disjoint dense subsets, say E_1 and E_2. If $R = U \cap E_1$ then R has empty interior, hence is preclosed in X. By assumption, $R \subseteq U$

is αg-closed and so α-cl$R \subseteq U$. Since cl$R =$ cl$U$, we have cl(int(cl$U$)) $\subseteq U$ and so cl(int(cl$U$)) $= U$, i.e. U is closed in X. It follows that int$F$ is a clopen locally indiscrete subspace. $\square$

Remark 3.3. *If $X = F \cup G$ is the Hewitt decomposition of a space X, then $X \setminus \text{int}F = \text{cl}G$ is always strongly irresolvable. Hence, under the hypothesis of Proposition 3.2, (4) in Theorem 3.1 is fulfilled.*

We are now able to answer the question posed in the title of this section.

Theorem 3.4. *For a space X the following are equivalent:*

(1) Every gp-closed set is αg-closed.

(2) Every preclosed set is αg-closed.

(3) Every preclosed set is $g\alpha$-closed.

Proof. (1) $\rightarrow$ (2) and (3) $\rightarrow$ (2) are obvious, and (2) $\rightarrow$ (3) follows from Remark 3.3 and Theorem 3.1.

(2) $\rightarrow$ (1). Let A be *gp*-closed and $A \subseteq U$ where U is open. If $B =$ pcl$A$ then $B \subseteq U$. By assumption, B is αg-closed and so α-cl$A \subseteq \alpha$-cl$B \subseteq U$, i.e. A is αg-closed. $\square$

Corollary 3.5. *For a space X the following are equivalent:*

(1) Every gp-closed set is $g\alpha$-closed.

(2) X is T_{gs} and every gp-closed set is αg-closed.

Proof. (1) $\rightarrow$ (2). We have to show that X is T_{gs}. Let $x \in X$ and suppose that $\{x\}$ is nowhere dense and not closed. Then $X \setminus \{x\}$ is α-open and *gp*-closed, and so α-cl$(X \setminus \{x\}) \subseteq X \setminus \{x\}$. Thus $X \setminus \{x\}$ is α-closed and $\{x\}$ is α-open, a contradiction. This proves that X is T_{gs}.

(2) $\rightarrow$ (1). This follows from Lemma 2.1. $\square$

In concluding this section we provide an example of a space where every *gp*-closed set is αg-closed but which fails to be T_{gs}, hence must have a *gp*-closed subset which is not $g\alpha$-closed.

Example 3.6. Let X be the set of natural numbers with $\emptyset, X$ and sets of the form $\{1, 2, ..., n\}, n \in \mathbf{N}$, as open sets. Since $\{1\} \subseteq U$ for every open set U, X is strongly irresolvable and so, by Theorem 3.1, every preclosed set is $g\alpha$-closed. By Theorem 3.4, every *gp*-closed set is αg-closed. If $m > 1$ then cl$\{m\} = \{m, m+1, ...\}$. Hence $\{m\}$ is nowhere dense but not closed, so X is not T_{gs}.

4 gp-closed sets and sg-closed sets

In this section we shall consider the relationships between gp-closed sets and sg-closed sets (resp. gs-closed sets). First observe that every sg-closed set is obviously gs-closed. The relationships between sg-closed sets (gs-closed) sets and other generalized preclosed sets can be illustrated in the following diagram.

$$\begin{tikzcd}
sg\text{-closed} \arrow[r] \arrow[d] & gs\text{-closed} \arrow[d] \\
\text{semi-preclosed} \arrow[r] & gsp\text{-closed}
\end{tikzcd}$$

Diagram 2

In general, the notions of gp-closed sets and sg-closed (gs-closed) sets are independent of each other. Recall also that a space X is said to be *sg-submaximal* [5] if every dense subset is sg-open.

Theorem 4.1. *For a space X the following are equivalent:*
(1) Every gs-closed subset of X is gp-closed.
(2) Every sg-closed subset of X is gp-closed.
(3) Every semi-closed subset of X is gp-closed.
(4) The space X is extremally disconnected.

Proof. (1) $\rightarrow$ (2) and (2) $\rightarrow$ (3) are obvious. We shall show implications (3) $\rightarrow$ (4) and (4) $\rightarrow$ (1).

(3) $\rightarrow$ (4). Let A be a regular open subset of X. Then A is semi-closed. By assumption, A is gp-closed and $A \subseteq A$. So $\mathrm{pcl}A = \mathrm{cl}A \subseteq A$, i.e. A is closed and thus X is extremally disconnected.

(4) $\rightarrow$ (1). Let A be gs-closed with $A \subseteq U$ where U is open. Then $\mathrm{scl}A = A \cup \mathrm{int}(\mathrm{cl}A) \subseteq U$. By assumption, $\mathrm{int}(\mathrm{cl}A)$ is closed and so clearly $\mathrm{pcl}A = A \cup \mathrm{cl}(\mathrm{int}A) \subseteq U$, i.e. A is gp-closed. $\square$

Theorem 4.2. *For a space X the following are equivalent:*
(1) Every gp-closed set is gs-closed.
(2) Every preclosed set is gs-closed.
(3) X is sg-submaximal.

Proof. (1) $\rightarrow$ (2) is obvious and (2) $\leftrightarrow$ (3) is Theorem 4.5 in [7].

(2) $\rightarrow$ (1). Let A be gp-closed with $A \subseteq U$ where U is open. If $B = \mathrm{pcl}A$ then B is preclosed and $B \subseteq U$. By assumption, B is gs-closed and so $\mathrm{scl}A \subseteq \mathrm{scl}B \subseteq U$, i.e. A is gs-closed. □

The proof of the following result is similar to that of Theorem 4.2, thus is omitted.

Theorem 4.3. *For a space X the following are equivalent:*
(1) Every gsp-closed set of X is gs-closed.
(2) Every semi-preclosed set of X is gs-closed.

Proposition 4.4. *If every gp-closed subset of a space X is sg-closed, then X is T_{gs}.*

Proof. Suppose that $\{x\}$ is nowhere dense but not closed. Then $X \setminus \{x\}$ is semi-open and gp-closed. By assumption, $X \setminus \{x\}$ is sg-closed and thus semi-closed. So $\{x\}$ is semi-open, contradicting the fact that $\{x\}$ is nowhere dense. □

As corollaries to Theorem 3.1, Theorem 4.2, Theorem 4.3 and Proposition 4.4 we now have the following results.

Corollary 4.5. *For a space X the following are equivalent:*
(1) Every gp-closed set is sg-closed.
(2) X is both T_{gs} and sg-submaximal.

Recall that a space is $T_{\frac{1}{2}}$ (resp. *semi-*$T_{\frac{1}{2}}$) if every g-closed set is closed (resp. every sg-closed set is semi-closed). It is known that a space X is $T_{\frac{1}{2}}$ if and only if every singleton is either open or closed. Moreover, X is *semi-*$T_{\frac{1}{2}}$ if every singleton is either semi-open or semi-closed [3]. Every $T_{\frac{1}{2}}$ space is semi-$T_{\frac{1}{2}}$. By Theorem 3.1 and Theorem 3.3 in [5], strong irresolvability implies sg-submaximality. But Example 3.5 in [5] shows that these two notions are distinct in general. In the following, we shall prove that strong irresolvability is equivalent to sg-submaximality in the class of $T_{\frac{1}{2}}$ spaces.

Corollary 4.6. *For a space X the following are equivalent:*
(1) X is both $T_{\frac{1}{2}}$ and sg-submaximal.
(2) Every gp-closed set is semi-closed.
(3) X is both $T_{\frac{1}{2}}$ and strongly irresolvable.

Proof. (1) → (2). Follows from Lemma 2.1, Corollary 4.5 and the definition of semi-$T_{\frac{1}{2}}$ spaces.

(2) → (3). Obviously, under the hypothesis, every preclosed set of X is semi-closed. By Theorem 3.2 in [1], X is strongly irresolvable. Thus, it suffices to show that X is $T_{\frac{1}{2}}$. By Proposition 4.4, X is T_{gs}, that is, each singleton $\{x\}$ is either preopen or closed. Suppose that $\{x\}$ is preopen. Then, by the hypothesis, $\{x\}$ is semi-open, which implies that $\{x\}$ is open.

(3) → (1). It follows from Theorem 4.2 directly. □

Corollary 4.7. *For a space X the following are equivalent:*
(1) Every gsp-closed set is sg-closed.
(2) X is T_{gs} and every gsp-closed set is gs-closed.
(3) X is T_{gs} and (X, τ^α) is g-submaximal.

Remark 4.8. One might ask whether every *sg*-submaximal space has to be T_{gs}. This is, however, not the case. The space in our Example 3.6 is not T_{gs} and has the property that every preclosed set is $g\alpha$-closed and thus *gs*-closed. Hence, by Theorem 4.2, it is *sg*-submaximal.

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