On Pre-Λ-Sets and Pre-V-sets *

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Abstract

In this paper we introduce the notions of a pre-Λ-set and a pre-V-set in a topological space. We study the fundamental properties of pre-Λ-sets and pre-V-sets and investigate the topologies defined by these families of sets.

1 Introduction

In 1986, Maki [11] continued the work of Levine [9] and Dunham [5] on generalized closed sets and closure operators by introducing the notion of a generalized Λ-set in a topological space (X, τ) and by defining an associated closure operator, i.e. the Λ-closure operator. He studied the relationship between the given topology τ and the topology τ^{Λ} generated by the family of generalized Λ-sets. Caldas and Dontchev [3] built on Maki's work by introducing and studying so-called Λ_s-sets and V_s-sets, and also other forms called g.Λ_s-sets and g.V_s-sets. They were able to use these notions to provide new characterizations of semi-T_1 spaces, semi-R_0 spaces and semi-$T_{1/2}$ spaces.

The purpose of our paper is to continue research along these directions but this time by utilizing preopen sets. We introduce pre-Λ-sets and pre-V-sets in a given topological space and thus obtain new topologies defined by these families of sets. We also consider some of the fundamental properties of these new topologies.

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2 Pre-Λ-Sets and Pre-V-Sets

A subset S of a topological space (X, τ) is said to be *preopen* [12] (resp. *semi-open* [10], *β-open* [1]) if $S \subseteq \mathrm{int}(\mathrm{cl}\ S)$ (resp. $S \subseteq \mathrm{cl}(\mathrm{int}\ S)$, $S \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}\ S))$), where int S and cl S denote the interior and the closure of S. The complement of a preopen set is called *preclosed*. The intersection of all preclosed supersets of a subset S is called the *preclosure* of S and is denoted by pcl S. It is well known that a subset S is preclosed if and only if $\mathrm{cl}(\mathrm{int}\ S) \subseteq S$, and that $\mathrm{pcl}\ S = S \cup \mathrm{cl}(\mathrm{int}\ S)$ for any subset S. We shall denote the families of all preopen sets (resp. preclosed sets) in a space (X, τ) by $\mathrm{PO}(X, \tau)$ (resp. $\mathrm{PC}(X, \tau)$).

In the following X and Y (or (X, τ) and (Y, σ)) will always denote topological spaces. No separation axioms are assumed unless stated explicitly.

Definition 1 Let S be a subset of a space (X, τ). We define subsets $\Lambda_p(\mathrm{S})$ and $V_p(\mathrm{S})$ as follows:

$\Lambda_p(\mathrm{S}) = \bigcap\{\ \mathrm{G} : \mathrm{S} \subseteq \mathrm{G}\ ,\ \mathrm{G} \in \mathrm{PO}(X, \tau)\}$ and

$V_p(\mathrm{S}) = \bigcup\{\ \mathrm{D} : \mathrm{D} \subseteq \mathrm{S}\ ,\ \mathrm{D} \in \mathrm{PC}(X, \tau)\}$.

Observe that in [8] $\Lambda_p(\mathrm{S})$ is called the *pre-kernel* of S. In our first result we summarize the fundamental properties of the sets $\Lambda_p(\mathrm{S})$ and $V_p(\mathrm{S})$.

Lemma 2.1 For subsets S, Q and S_i , $i \in I$, of a space (X, τ) the following properties hold:

(1) $\mathrm{S} \subseteq \Lambda_p(\mathrm{S})$,

(2) $\mathrm{Q} \subseteq \mathrm{S}$ implies that $\Lambda_p(\mathrm{Q}) \subseteq \Lambda_p(\mathrm{S})$,

(3) $\Lambda_p(\ \Lambda_p(\mathrm{S})\ ) = \Lambda_p(\mathrm{S})$,

(4) If $\mathrm{S} \in \mathrm{PO}(X, \tau)$ then $\mathrm{S} = \Lambda_p(\mathrm{S})$,

(5) $\Lambda_p(\ \bigcup\{\ S_i : i \in I\}\ ) = \bigcup\{\ \Lambda_p(S_i) : i \in I\}$,

(6) $\Lambda_p(\ \bigcap\{\ S_i : i \in I\}\ ) \subseteq \bigcap\{\ \Lambda_p(S_i) : i \in I\}$,

(7) $\Lambda_p(\mathrm{X} \setminus \mathrm{S}) = \mathrm{X} \setminus V_p(\mathrm{S})$.

Proof. (1), (2), (4), (6) and (7) are immediate consequences of Definition 1. To prove (3), first observe that by (1) and (2), we have $\Lambda_p(\mathrm{S}) \subseteq \Lambda_p(\ \Lambda_p(\mathrm{S})\ )$. If $x \notin \Lambda_p(\mathrm{S})$, then there exists $\mathrm{G} \in \mathrm{PO}(X, \tau)$ such that $\mathrm{S} \subseteq \mathrm{G}$ and $x \notin \mathrm{G}$. Hence $\Lambda_p(\mathrm{S}) \subseteq \mathrm{G}$,and so we have $x \notin \Lambda_p(\ \Lambda_p(\mathrm{S})\ )$. Thus $\Lambda_p(\ \Lambda_p(\mathrm{S})\ ) = \Lambda_p(\mathrm{S})$.

To prove (5), let $\mathrm{S} = \bigcup\{\ S_i : i \in I\}$. By (2), we have that $\bigcup\{\ \Lambda_p(S_i) : i \in I\} \subseteq \Lambda_p(\mathrm{S})$. If $x \notin \bigcup\{\ \Lambda_p(S_i) : i \in I\}$, then, for each $i \in I$, there exists $G_i \in \mathrm{PO}(X, \tau)$ such that $S_i \subseteq G_i$ and $x \notin G_i$. If $\mathrm{G} = \bigcup\{\ G_i : i \in I\}$ then $\mathrm{G} \in \mathrm{PO}(X, \tau)$ with $S \subseteq G$ and $x \notin G$. Hence $x \notin \Lambda_p(\mathrm{S})$, and so (5) holds. □

By using Lemma 2.1 (7), one can easily verify our next result.

Lemma 2.2 For subsets S, Q and S_i , $i \in I$, of a space (X, τ) the following properties hold:

(1) $V_p(\mathrm{S}) \subseteq \mathrm{S}$,

(2) $\mathrm{Q} \subseteq \mathrm{S}$ implies that $V_p(\mathrm{Q}) \subseteq V_p(\mathrm{S})$,

(3) $V_p(\ V_p(\mathrm{S})\ ) = V_p(\mathrm{S})$,

(4) If $\mathrm{S} \in \mathrm{PC}(X, \tau)$ then $\mathrm{S} = V_p(\mathrm{S})$,

(5) $V_p(\ \bigcap\{\ S_i : i \in I\}\ ) = \bigcap\{\ V_p(S_i) : i \in I\}$,

(6) $\bigcup\{\ V_p(S_i) : i \in I\} \subseteq V_p(\ \bigcup\{\ S_i : i \in I\}\ )$.

Note that in general we have $\Lambda_p(S \cap Q) \neq \Lambda_p(\mathrm{S}) \cap \Lambda_p(\mathrm{Q})$ as the following example shows.

Example 2.3 Let $X = \{a, b, c\}$ and let $\tau = \{X, \emptyset, \{a\}\}$. If $S = \{b\}$ and $Q = \{c\}$, then $\Lambda_p(S \cap Q) = \emptyset$ but $\Lambda_p(\mathrm{S}) \cap \Lambda_p(\mathrm{Q}) = \{a\}$.

Definition 2 A subset S of a space (X, τ) is called

(1) a *pre-Λ-set* (resp. a *pre-V-set*) if $S = \Lambda_p(S)$ (resp. $V = V_p(S)$).

(2) a *Λ-set* (resp. a *V-set*) if $S = S^\Lambda$ (resp. $S = S^V$) [11], where $S^\Lambda = \bigcap\{\ \mathrm{O} : \mathrm{S} \subseteq \mathrm{O}\ ,\ \mathrm{O} \in \tau\}$ and $S^V = \bigcup\{\ \mathrm{F} : \mathrm{F} \subseteq \mathrm{S}\ ,\ X \setminus F \in \tau\}$.

(3) a *Λ_s-set* (resp. a *V_s-set*) if $S = S^{\Lambda_s}$ (resp. $S = S^{V_s}$) [3], where $S^{\Lambda_s} = \bigcap\{\ \mathrm{O} : \mathrm{S} \subseteq \mathrm{O}\ ,\ \mathrm{O} \in SO(X, \tau)\}$ and $S^V = \bigcup\{\ \mathrm{F} : \mathrm{F} \subseteq \mathrm{S}\ ,\ X \setminus F \in SO(X, \tau)\}$.

Clearly, a subset S is a pre-Λ-set (resp. a pre-V-set) if and only if it is an intersection (resp. a union) of preopen (resp. preclosed) sets. A subset S is a Λ-set (resp. a V-set) if and only if it is an intersection (resp. a union) of open (resp. closed) sets. A subset S is a Λ_s-set (resp. a V_s-set) if and only if it is an intersection (resp. a union) of semi-open (resp. semi-closed) sets. Hence Λ-sets and preopen sets are pre-Λ-sets, and V-sets and preclosed sets are pre-V-sets.

Observe also, that a subset S is a pre-Λ-set if and only if $X \setminus S$ is a a pre-V-set.

Proposition 2.4 For a space (X, τ) the following statements hold:

(1) $\emptyset$ and X are pre-Λ-sets and pre-V-sets.

(2) Every union of pre-Λ-sets (resp. pre-V-sets) is a pre-Λ-set (resp. pre-V-set).

(3) Every intersection of pre-Λ-sets (resp. pre-V-sets) is a pre-Λ-set (resp. pre-V-set).

Proof. We shall only consider the case of pre-Λ-sets. (1) and (3) are obvious. Let $\{S_i : i \in I\}$ be a family of pre-Λ-sets in (X, τ). If $S = \bigcup\{ S_i : i \in I\}$, then by Lemma 2.1 we have $S = \bigcup\{ \Lambda_p(S_i) : i \in I\} = \Lambda_p(S)$. $\square$

Remark 2.5 Let τ^{Λ_p} (resp. τ^{V_p}) denote the family of all pre-Λ-sets (resp. pre-V-sets) in (X, τ). Then τ^{Λ_p} (resp. τ^{V_p}) is a topology on X containing all preopen (resp. preclosed) sets. Clearly, (X, τ^{Λ_p}) and (X, τ^{V_p}) are Alexandroff spaces [2], i.e. arbitrary intersections of open sets are open.

Recall that a space (X, τ) is said to be *pre-T_1* [14] if for each pair of distinct points x and y of X there exists a preopen set containing x but not y . Clearly a space (X, τ) is pre-T_1 if and only if singletons are preclosed. We now offer additional characterizations of pre-T_1 spaces.

Theorem 2.6 For a space (X, τ) the following are equivalent:

(1) (X, τ) is pre-T_1,

(2) Every subset of X is a pre-Λ-set,

(3) Every subset of X is a pre-V-set,

(4) Every semi-open subset of X is a pre-V-set.

Proof. Clearly (2) $\Leftrightarrow$ (3) .

(1) $\Rightarrow$ (3): Let $A \subseteq X$. Since $A = \bigcup\{\{x\} : x \in A\}$, A is a union of preclosed sets, hence a pre-V-set.

(3) $\Rightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (1): First observe that every singleton is open or preclosed. Let $x \in X$. If $\{x\}$ is open, then by assumption, $\{x\}$ is a pre-V-set and so preclosed. Hence each singleton is preclosed, i.e. (X, τ) is pre-T_1. □

3 Generalized pre-Λ-sets

Following the lines of investigation of Maki in [11] one could now define generalized pre-Λ-sets and generalized pre-V-sets in the following way.

Definition 3 A subset S of a space (X, τ) is called

(i) a *generalized pre-Λ-set*, briefly g-Λ_p-set, if Λ_p(S) $\subseteq$ P whenever $S \subseteq P$ and P $\in$ PC(X, τ) ,

(ii) a *generalized pre-V-set*, briefly g-V_p-set, if V $\subseteq$ V_p(S) whenever $V \subseteq S$ and V $\in$ PO(X, τ).

We shall see, however, that we obtain nothing new.

Proposition 3.1 Let S be a subset of a space (X, τ).

(i) S is a generalized pre-Λ-set if and only if S is a pre-Λ-set,

(ii) S is a generalized pre-V-set if and only if S is a pre-V-set.

Proof. (i) Clearly, every pre-Λ-set is a generalized pre-Λ-set. Now let S be a generalized pre-Λ-set. Suppose there exists $x \in \Lambda_p(S) \setminus S$. Observe that $\{x\}$ is open or preclosed, and that $S \subseteq X \setminus \{x\}$. If $\{x\}$ is open, then $X \setminus \{x\}$ is closed, hence preclosed, and so $\Lambda_p(S) \subseteq X \setminus \{x\}$, a contradiction. If $\{x\}$ is preclosed, then $X \setminus \{x\}$ is preopen and so $\Lambda_p(S) \subseteq X \setminus \{x\}$, a contradiction. Hence S is a pre-Λ-set.

(ii) This is proved in a similar way. □

4 Properties of pre-Λ-Sets and pre-V-Sets

Recall that a subset A of a space (X, τ) is said to be *generalized closed*, briefly g-closed [9], if cl $A \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$. A space (X, τ) is said to be a $T_{1/2}$ *space* if every g-closed subset of X is closed. Dunham [6] pointed out that a space (X, τ) is $T_{1/2}$ if and only if each singleton is open or closed.

Proposition 4.1 Let (X, τ) be a space.

(1) (X, τ^{Λ_p}) and (X, τ^{V_p}) are always $T_{1/2}$ spaces,

(2) If (X, τ) is pre-T_1, then both (X, τ^{Λ_p}) and (X, τ^{V_p}) are discrete spaces,

(3) The identity function $id : (X, \tau^{\Lambda_p}) \to (X, \tau)$ is continuous,

(4) The identity function $id : (X, \tau^{V_p}) \to (X, \tau)$ is contra-continuous [4], i.e. inverse images of open sets are closed.

Proof. (1): Let $x \in X$. Then $\{x\}$ is open or preclosed in (X, τ). If $\{x\}$ is open, thus preopen, then $\{x\} \in \tau^{\Lambda_p}$. If $\{x\}$ is preclosed in (X, τ), then $X \setminus \{x\}$ is preopen and so $X \setminus \{x\} \in \tau^{\Lambda_p}$, i.e. $\{x\}$ is closed in (X, τ^{Λ_p}). Hence (X, τ^{Λ_p}) and (X, τ^{V_p}) are $T_{1/2}$ spaces.

(2): This follows from Theorem 2.6.

(3) and (4) are obvious. $\square$

Recall that a space (X, τ) is called *resolvable* if it has two disjoint dense subsets.

Corollary 4.2 If (X, τ) is resolvable, then (X, τ^{Λ_p}) and (X, τ^{V_p}) are discrete.

Proof. We will show that (X, τ) is pre-T_1. Let D and E be disjoint dense subsets of (X, τ), and let $x \in X$, wlog $x \in D$. Then $X \setminus \{x\} = E \cup (D \setminus \{x\})$ is dense, hence preopen, and so $\{x\}$ is preclosed. $\square$

Proposition 4.3 If (X, τ^{Λ_p}) is connected, then (X, τ) is preconnected, i.e. X cannot be represented as a disjoint union of nonempty preopen subsets of (X, τ).

Proof. Suppose that (X, τ) is not preconnected. Hence there exist nonempty disjoint preopen sets S, T in (X, τ) such that $S \cup T = X$. Since S and T are open in (X, τ^{Λ_p}), we have a contradiction.

Observe also that (X, τ^{Λ_p}) is connected if and only if (X, τ^{V_p}) is connected. $\square$

Proposition 4.1 points out that for many spaces (X, τ), τ^{Λ_p} is the discrete topology. In our final result we provide an example of an infinite space (X, τ) such that (X, τ^{V_p}) (and thus (X, τ^{Λ_p})) is not discrete.

Let (X, τ) be a space having a point x_0 such that $\{x_0\}$ is open and dense. Let $S = \{x_0\}$. Clearly, the only preclosed set containing S is X. Hence, if $x_0 \in O$ with $O \in \tau^{V_p}$, then $O = X$. This shows that (X, τ^{V_p}) is a compact and connected space, and thus cannot be discrete.

Example 4.4 Let $X = \mathbf{N}$, i.e. the set of natural numbers, and let $\tau = \{\emptyset\} \cup \{\{1, 2, ..., n\} : n \in \mathbf{N}\}$. Then $\{1\}$ is open and dense in (X, τ), hence (X, τ^{V_p}) is an infinite, compact and connected space. Observe also that for $n \neq 1$, $\{n\}$ is preclosed in (X, τ) and so $\{n\} \in \tau^{V_p}$. Now let $V \in PO(X, \tau^{V_p})$ with $1 \in V$. By a result of Ganster [7], $V = O \cap D$ where $O \in \tau^{V_p}$ and D is dense in (X, τ^{V_p}). From our previous observations it follows that $O = X$. Since $\{n\} \in \tau^{V_p}$ for $n \neq 1$, we also have $D = X$ and thus $V = X$. This shows that (X, τ^{Λ_p}) is also strongly compact [13], i.e. every preopen cover has a finite subcover.

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