Recent Progress in the Theory of Generalized Closed Sets *

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Abstract

In this paper we present an overview of our research in the field of generalized closed sets (in the sense of N. Levine). We will demonstrate that certain key concepts play a decisive role in the study of the various generalizations of closed sets.

1 Introduction and Preliminaries

In recent years there has been considerable interest in the study of generalized closed sets in the sense of N. Levine, and their relationships to other classes of sets such as α-open sets, semi-open sets and preopen sets. This investigation has led to significant contributions to the theories of separation axioms, covering properties and generalizations of continuity. In this paper we shall give an overview of our approach to these topics, thereby demonstrating that certain key notions seem to play a fundamental role in the overall discussion.

For the convenience of the reader we first review some basic concepts, although most of them are very well known from the literature. A subset S of a topological space (X, τ) is called *α–open* (*semi–open, preopen, semi–preopen*) if $S \subseteq int(cl(intS))$ ($S \subseteq cl(intS)$, $S \subseteq int(clS)$, $S \subseteq cl(int(clS))$). Moreover, S is said to be *α–closed* (*semiclosed, preclosed, semi–preclosed*) if $X \setminus S$ is α–open (semi–open, preopen, semi–preopen) or, equivalently, if $cl(int(clS)) \subseteq S$ ($int(clS) \subseteq S$, $cl(intS) \subseteq S$, $int(cl(intS)) \subseteq S$). The *$\alpha$-closure* (*semi–closure, preclosure, semi–preclosure*) of $S \subseteq X$ is the smallest α–closed (semiclosed, preclosed, semi–preclosed) set containing S. It is well known that α–$clS = S \cup cl(int(clS))$ and

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$sclS = S \cup int(clS)$, $pclS = S \cup cl(intS)$ and $spclS = S \cup int(cl(intS))$. Njastad [25] has shown that the collection of α–open sets of a space (X, τ) is a topology τ^α on X. Moreover, if $SO(X, \tau)$ denotes the collection of all semi-open sets of (X, τ), then $SO(X, \tau)$ is a topology if and only if (X, τ) is extremally disconnected, i.e. the closure of every open set is open. In this case, $SO(X, \tau) = \tau^\alpha$ (see [25]).

Recall that a space (X, τ) is called *resolvable* if there exists a pair of disjoint dense subsets. Otherwise it is called *irresolvable*. (X, τ) is said to be *strongly irresolvable* if every open subspace is irresolvable. Hewitt [17] has shown that every space (X, τ) has a decomposition $X = F \cup G$, where F is closed and resolvable and G is open and hereditarily irresolvable. We shall call this decomposition the *Hewitt decomposition* of (X, τ). There is another important decomposition of a space which we shall call the *Jankovic–Reilly decomposition*. Since every singleton $\{x\}$ of a space (X, τ) is either nowhere dense or preopen (see [18]), we clearly have $X = X_1 \cup X_2$, where $X_1 = \{x \in X : \{x\}$ is nowhere dense $\}$ and $X_2 = \{x \in X : \{x\}$ is preopen $\}$.

Remark 1.1. Throughout this paper, F resp. G will always refer to the Hewitt decomposition, and X_1 resp. X_2 always to the Jankovic-Reilly decomposition.

In 1970, N. Levine [19] called a subset A of a space (X, τ) *generalized closed*, shortly *g–closed*, if $clA \subseteq O$ whenever $A \subseteq O$ and O is open. Complements of g–closed sets are called *g–open*. It is obvious that every closed set is g–closed but not vice versa. A space (X, τ) is called $T_{1/2}$ [19] if every g–closed set is closed, or equivalently, if every singleton is either open or closed [15].

Definition 1. A subset A of a space (X, τ) is called

(1) *α–generalized closed* (briefly, *αg–closed*) [20] if α–$clA \subseteq U$ whenever $A \subseteq U$ and U is open,

(2) *generalized α–closed* (briefly, *$g\alpha$–closed*) [21], if α–$clA \subseteq U$ whenever $A \subseteq U$ and U is α–open,

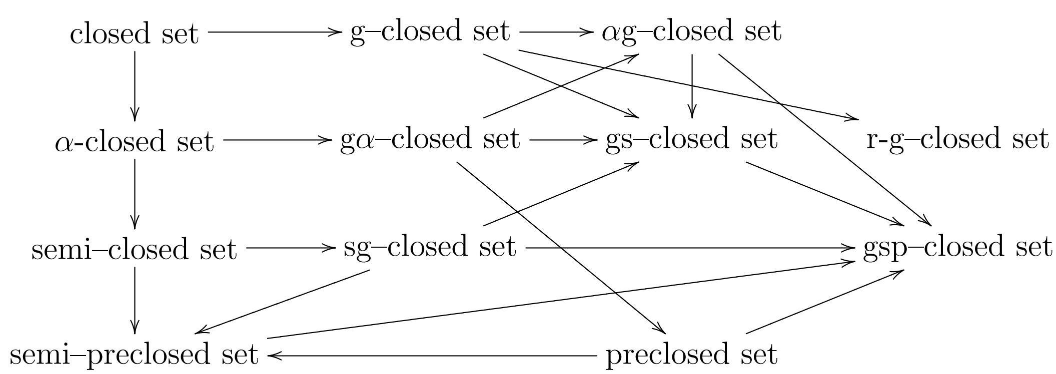
(3) *generalized semiclosed* (briefly, *gs–closed*) [1] if $sclA \subseteq U$ whenever $A \subseteq U$ and U is open,

(4) *semi–generalized closed* (briefly, *sg–closed*) [2], if $sclA \subseteq U$ whenever $A \subseteq U$ and U is semi–open,

(5) *generalized semi–preclosed* (briefly, *gsp–closed*) [13] if $spclA \subseteq U$ whenever $A \subseteq U$ and U is open.

(6) *regular generalized closed* (briefly, *r-g–closed*) [26] if $clA \subseteq U$ whenever $A \subseteq U$ and U is regular open.

In [14], J. Dontchev summarized the relationships between these notions in a beautiful diagram. He also pointed out that none of the implications can be reversed.



2 Results

Our starting point in the investigation of generalized closed sets were two open questions that Dontchev posed in [14], namely :

Characterize those spaces where

(A) Every semi–preclosed set is sg–closed, and

(B) Every preclosed set is gα–closed.

These questions have been solved by Cao, Ganster and Reilly in [4]. To our surprise, both decompositions mentioned before, i.e. the Hewitt decomposition and the Jankovic–Reilly decomposition, played a key role in our solution to these questions. Further studies have shown that these decompositons are important in many more questions concerning generalized closed sets.

Recall that a space (X, τ) is called *submaximal* (resp. *g–submaximal*) if every dense subset is open (resp. g–open). (X, τ) is said to be *locally indiscrete* if every open subset is closed.

Theorem 2.1. *[4]*

For a space (X, τ) the following are equivalent:

(1) (X, τ) satisfies (A),

(2) $X_1 \cap sclA \subseteq spclA$ for each $A \subseteq X$,

(3) $X_1 \subseteq int(clG)$,

(4) (X, τ) is the topological sum of a locally indiscrete space and a strongly irresolvable space,

(5) (X, τ) satisfies (B),

(6) (X, τ^α) is g-submaximal.

This result motivated us to look for other possible converses in Dontchev's diagram. Out of the many results we obtained we shall present here two key results.

Theorem 2.2. *[5]*

For a space (X, τ) the following are equivalent:

(1) every semi–preclosed set is gα–closed,

(2) (X, τ^α) is extremally disconnected and g–submaximal.

Theorem 2.3. *[5]*

For a space (X, τ) the following are equivalent:

(1) $X_1 \subseteq clG$,

(2) every preclosed subset is sg–closed,

(3) (X, τ) is sg–submaximal,

(4) (X, τ^α) is sg–submaximal,

Corollary 2.4. If (X, τ^α) is g–submaximal then (X, τ^α) is also sg–submaximal. The converse, however, is false (see [5]).

3 Lower Separation Axioms

We already mentioned that the closer investigation of generalized closed sets had great impact on the theory of separation axioms. If we again have a look at Dontchev's diagram, the search for converses of other implications leads to the consideration of certain lower separation axioms.

Recall that Maki et al. [22] have called a space (X, τ) a T_{gs} space if every gs–closed subset is sg–closed. We have been able to characterize T_{gs} spaces in the following way.

Theorem 3.1. *[6]*

For a space (X, τ), the following are equivalent:

(1) (X, τ) is a T_{gs} space,

(2) every nowhere dense subset of (X, τ) is a union of closed subsets, i.e. (X, τ) is T_1^* [16],

(3) every gsp–closed set is semi–preclosed, i.e. (X, τ) is semi–pre–$T_{1/2}$ [13],

(4) every singleton of (X, τ) is either preopen or closed.

A space (X, τ) is called *semi-*T_1 [23] if each singleton is semi–closed, it is called *semi-*$T_{1/2}$ [2] if every singleton is either semi-closed or semi-open. Let τ_s denote the semi-regularization topology of a space (X, τ). The closure of a subset $A \subseteq X$ with respect to τ_s will be denoted by $\delta - clA$. A subset A of X is called *δ–generalized closed* if $\delta - clA \subseteq U$ when $A \subseteq U$ and U is open in (X, τ). Moreover, (X, τ) is called a $T_{3/4}$*–space* [12] if every δ-generalized closed subset of (X, τ) is closed in (X, τ_s). The well-known digital line, also called the Khalimsky line, is a $T_{3/4}$-space which fails to be T_1.

We now have the following result.

Theorem 3.2. For every space (X, τ),

(1) $T_{3/4} = T_{gs}$ + semi-T_1 [12],

(2) $T_{1/2} = T_{gs}$ + semi-$T_{1/2}$ [22],

(3) T_{gs} = every αg–closed set is gα–closed,

(4) T_{gs} + extremally disconnected = every gs–closed set is preclosed.

Corollary 3.3. In a T_{gs} space, every g–closed set is gα–closed.

This has led to the natural question of characterizing those spaces where every gα–closed set is g–closed.

Theorem 3.4. *[6]*

For a space (X, τ) the following are equivalent:

(1) Every gα–closed set is g–closed,

(2) every nowhere dense subset is locally indiscrete as a subspace,

(3) every nowhere dense subset is g–closed,

(4) every α–closed set is g–closed.

Observe, however, that there exist spaces in which every nowhere dense subset is g–closed but there exists a nowhere dense set which is not closed (see [6]).

4 Gp–closed Sets

Definition 2. A subset A of a space (X, τ) is called *generalized preclosed*, briefly *gp–closed*, [24] if $pclA \subseteq U$ whenever $A \subseteq U$ and U is open.

Our study of generalized preclosed sets has been carried out to a great detail in [7]. As one might expect, here also the Hewitt decomposition, the Jankovic–Reilly decomposition, submaximality and extremal disconnectedness play a significant role. Out of the many results that we obtained we mention here two important characterizations.

Theorem 4.1. *[7]*

For a space (X, τ) the following are equivalent :

(1) (X, τ) is a T_{gs}-space,

(2) Every gp–closed subset of (X, τ) is preclosed,

(3) Every gsp–closed subset of (X, τ) is semi–preclosed,

(4) Every gp–closed subset of (X, τ) is semi–preclosed.

Theorem 4.2. *[7]*

For a space (X, τ) the following are equivalent :

(1) Every gsp–closed subset of (X, τ) is gp–closed,

(2) Every semi–preclosed subset of (X, τ) is gp–closed,

(3) (X, τ) is extremally disconnected.

5 Sg–compact Spaces

Definition 3. A topological space (X, τ) is called *sg–compact* if every cover by sg–open sets has a finite subcover.

The class of sg–compact spaces has been introduced by Caldas [3], Devi, Balachandran and Maki [9] and Tapi, Thakur and Sonwalkar [27]. Sg–compact spaces are quite interesting because sg–openness seems to be the weakest form of generalized openness for which there exists a nontrivial corresponding notion of compactness. For example, the cofinite topology on any infinite set yields a sg–compact space. Clearly, every sg–compact space is semi–compact and thus hereditarily compact.

Dontchev and Ganster [10] called a subset A of a space (X, τ) *hereditarily sg–closed*, briefly *hsg–closed*, if every subset is A is sg–closed. A space (X, τ) is said to be a C_3*space* [10] if every hsg–closed set is finite. It is easily observed that every nowhere dense set is hsg–closed. Moreover, $A \subseteq X$ is hsg–closed if and only if $X_1 \cap int(clA) = \emptyset$ [10].

Theorem 5.1. *[10]*

For a space (X, τ) the following are equivalent :

(1)(X, τ) is sg–compact,

(2) (X, τ) is a C_3 space.

The question concerning products of sg–compact spaces is rather tricky. It has been shown in [11] that there exists a space (X, τ) which is sg–compact but $X \times X$ fails to be sg–compact. In addition, the following result holds.

Theorem 5.2. *[11]*

(1) If $X = \prod\{X_i : i \in I\}$ is sg–compact then only finitely many X_i are not indiscrete,

(2) Suppose that $X = \prod\{X_i : i \in I\}$ is sg–compact. Then : either all X_i are finite, or exactly one of them is infinite and sg–compact and the rest are finite and locally indiscrete.

6 Concluding Remark

We want to draw the attention of the reader to a forthcoming paper of Cao, Greenwood and Reilly [8] where all the various notions of generalized closedness considered in the literature so far have been brought under a common framework.

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