ON P_i-METACOMPACT SPACES

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Abstract

In a recent paper Al Ghour [1] considered two new variations of metacompactness by utilizing preopen sets. The aim of our paper is to continue the study of these notions and to answer the open questions posed in [1].

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1 Introduction and Preliminaries

Recently Al Ghour [1] introduced P_1-metacompact spaces and P_2-metacompact spaces as variations of metacompact spaces. A topological space (X, τ) is said to be P_1-metacompact (resp. P_2-metacompact) if every preopen cover has a point-finite open (resp. a point-finite preopen) refinement. The following open questions have been posed in [1] :

Question 1: Are P_2-metacompact spaces metacompact?

Question 2: Are P_2-metacompact T_1 spaces P_1-metacompact?

In our paper we investigate the notions above and also introduce the new notion of P_3-metacompact spaces. As a result, we are able to answer both Question 1 and Question 2 in the negative.

Let (X,τ) be a topological space. The closure and the interior of a subset A of (X,τ) will be denoted by int$A$ and cl$A$, respectively. A subset $S \subseteq X$ is called preopen [3] if $S \subseteq \text{int}(\text{cl}S)$. We denote the family of all preopen subsets of (X,τ) by $PO(X,\tau)$. It has been observed in [2] that $S \subseteq X$ is preopen if and only if $S = U \cap D$ where U is open and D is dense. Recall that a space (X,τ) is called submaximal if every dense subset is open. Clearly (X,τ) is submaximal if and only if $\tau = PO(X,\tau)$.

The cardinality of a set A is denoted by $|A|$. No separation axioms are assumed unless stated otherwise.

2 P_i-metacompact spaces

Definition 1 A space (X,τ) is called

i) P_1-metacompact [1] if every preopen cover has a point-finite open refinement,

ii) P_2-metacompact [1] if every preopen cover has a point-finite preopen refinement,

iii) P_3-metacompact if every open cover has a point-finite preopen refinement.

Remark 2.1 If (X, τ) is submaximal then all these notions coincide and are equivalent to metacompactness.

Moreover we have the following obvious implications:

$$\begin{array}{ccc} P_1\text{-metacompact} & \Rightarrow & P_2\text{-metacompact} \\ \Downarrow & & \Downarrow \\ \text{metacompact} & \Rightarrow & P_3\text{-metacompact} \end{array}$$

In the sequel we will show that in general none of these implications can be reversed. But first we provide a useful characterization of submaximality.

Proposition 2.2 *For a space (X, τ) the following are equivalent:*

1) (X, τ) is submaximal.

2) Every preopen cover has an open refinement.

3) Every cover by dense subsets has an open refinement.

Proof. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) are obvious. We now show that 3) $\Rightarrow$ 1) . Suppose that D is dense in (X, τ) and that $A = D \setminus \text{int} D$ is nonempty. Since $\text{int} A = \emptyset$, $X \setminus A$ is dense and $\{D, X \setminus A\}$ is a cover of X by dense subsets. By assumption we have an open refinement $\{U, V\}$ where $U \subseteq D$ and $V \subseteq (X \setminus A)$. Now pick $x \in A$. Then $x \in U \subseteq D$, hence $x \in \text{int} D$, a contradiction. Thus A is empty, i.e. D is open.

Corollary 2.3 *[1] A space (X, τ) is P_1-metacompact if and only if (X, τ) is submaximal and metacompact.*

Corollary 2.4 *The usual space of reals is metacompact but not P_1-metacompact.*

We now consider the following (known) set-theoretic lemma. For the sake of completeness we include a short proof.

Lemma 2.5 *Let X be a set of cardinality μ where μ is infinite. Then every cover of X by sets of cardinality μ has a disjoint refinement consisting of sets of cardinality μ .*

Proof. Let $\mathcal{A}$ be a cover of X by subsets of cardinality μ . Without loss of generality we may assume that $\mathcal{A} = \{A_\alpha : \alpha < \mu^*\}$ where μ^* denotes the first ordinal of cardinality μ . By induction of length μ^* we construct elements $b_{\alpha,\beta}$ where $\alpha, \beta < \mu^*$. First pick $b_{0,0} \in A_0$. Suppose at stage $\gamma < \mu^*$ we have distinct elements $b_{\alpha,\beta}$ with $\alpha, \beta < \gamma$ such that $b_{\alpha,\beta} \in A_\alpha$ for each $\alpha, \beta < \gamma$. Since the set of already chosen elements has cardinality $< \mu^*$ we can find distinct elements $b_{\gamma,\beta} \in A_\gamma$, $\beta \leq \gamma$ and distinct elements $b_{\alpha,\gamma} \in A_\alpha$, $\alpha < \gamma$ such that no two elements that we choose ever coincide. This completes the induction.

Now let $B_\alpha = \{b_{\alpha,\beta} : \beta < \mu^*\}$ for each $\alpha < \mu^*$. By construction, $\{B_\alpha : \alpha < \mu^*\}$ forms a disjoint refinement of $\mathcal{A}$ consisting of sets of cardinality μ .

Our next two results follow immediately from Lemma 2.5.

Theorem 2.6 *Let τ be the cofinite topology on a set X with $|X| = \aleph_0$. Then (X, τ) is T_1, compact and P_2-metacompact, but fails to be P_1-metacompact. Hence Question 2 is answered in the negative.*

Proof. Observe that the preopen sets of (X, τ) are the infinite subsets of X and that (X, τ) clearly fails to be submaximal.

Theorem 2.7 *Let τ be the co-countable topology on a set X with $|X| = \aleph_1$. Then (X, τ) is T_1 and P_2-metacompact hence P_3-metacompact but neither metacompact nor P_1-metacompact. This answers both Question 1 and Question 2 in the negative.*

Proof. Observe that the preopen sets of (X, τ) are the uncountable subsets of (X, τ). It is known that (X, τ) fails to be metacompact (see e.g. [5]). Clearly (X, τ) is not submaximal hence cannot be P_1-metacompact.

We now consider another set-theoretic construction.

Lemma 2.8 *Let X be an infinite set with $|X| = \mu$, and let $X = A \cup B$ where $|A| = \mu$, $|B| = \upsilon < \mu$ and $A \cap B = \emptyset$. For each $x \in A$ let $C_x = B \cup \{x\}$. Then $\mathcal{C} = \{C_x : x \in A\}$ is a cover of X . Suppose that $\mathcal{D}$ is a refinement of $\mathcal{C}$. For each $x \in A$ there exists $D_x \in \mathcal{D}$ such that $x \in D_x \subseteq C_x$. Observe that $D_x \neq D_y$ for distinct $x, y \in A$. We assume that $|D_x| > 1$ for each $x \in A$.*
For each $z \in B$ let $A_z = \{x \in A : z \in D_x\}$. If $|A_z| < \mu$ for each $z \in B$, then $|\bigcup_{z \in B} A_z| < \mu$ and so there must be some $x \in A \setminus \bigcup_{z \in B} A_z$. Since we assume that $|D_x| > 1$ there exists some $z \in B \cap D_x$ and so $x \in A_z$, a contradiction. Hence there exists $z \in B$ such that $|A_z| = \mu$, i.e. z lies in at least μ elements of the refinement $\mathcal{D}$.

Theorem 2.9 *Let (X, τ) be an infinite space which possesses a dense subset D with $|D| < |X|$, and $\{x\}$ is not preopen for each $x \notin D$. Then (X, τ) fails to be P_2-metacompact.*

Proof. Using Lemma 2.8, let $A = X \setminus D$ and $B = D$. Then $\mathcal{C}$ is a preopen cover. It follows readily that any preopen refinement cannot be point-finite.

Corollary 2.10 *i) Let (X, τ) be an uncountable, separable and dense-in-itself T_1 space. Then (X, τ) fails to be P_2-metacompact.*
ii) The usual space of reals fails to be P_2-metacompact.
iii) $\beta\mathbb{N}$, the Stone-Cech compactification of $\mathbb{N}$, fails to be P_2-metacompact.

Corollary 2.11 *i) Let $|X| > \aleph_0$ and let τ be the cofinite topology on X . Then (X, τ) fails to be P_2-metacompact.*
ii) Let $|X| > \aleph_1$ and let τ be the co-countable topology on X . Then (X, τ) fails to be P_2-metacompact.

Proof. We will show ii) . Pick $B \subseteq X$ with $|B| = \aleph_1$ and let $A = X \setminus B$. Observe that the preopen sets in (X, τ) are the uncountable subsets and apply Lemma 2.8 . The proof of i) is similar.

Let τ be the cofinite (resp. the co-countable topology) on a set X . It follows from Theorem 2.6, Theorem 2.7 and Corollary 2.11 that (X, τ) is P_2-metacompact if and only if $|X| \leq \aleph_0$ (resp. $|X| \leq \aleph_1$). Hence, if X is the set of reals and τ the co-countable topology on X then the P_2-metacompactness of (X, τ) depends on whether the continuum hypothesis holds or not. We conclude with the following result.

Theorem 2.12 *Let τ be the co-countable topology on a set X . Then (X, τ) is P_3-metacompact.*

Proof. We may assume that $|X| = \mu$ and μ is uncountable. Let $\mathcal{U}$ be an open cover of (X, τ) . Then every member of $\mathcal{U}$ has cardinality μ . By Lemma 2.5 there exists a disjoint refinement $\mathcal{V}$ of $\mathcal{U}$ by sets of cardinality μ . Clearly $\mathcal{V}$ is a preopen refinement of $\mathcal{U}$. Hence (X, τ) is P_3-metacompact.

In particular, if $|X| > \aleph_1$ then (X, τ) is P_3-metacompact but neither P_2-metacompact nor metacompact.

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