# On $S_i$-metacompact Spaces *

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**Abstract**

The primary purpose of this paper is to introduce and study new variations of metacompactness by utilizing semi-open sets.

## 1 Introduction and Preliminaries

Let $(X, \tau)$ be a topological space. We shall denote the closure and the interior of a subset $A$ of $(X, \tau)$ by cl$A$ and int$A$, respectively. A subset $A$ of $(X, \tau)$ is called *$\alpha$-open* (or an *$\alpha$-set* [6]), if $A \subseteq \text{int}(\text{cl}(\text{int}A))$, *semi-open* [4] if $A \subseteq \text{cl}(\text{int}A)$, *preopen* [5] if $A \subseteq \text{int}(\text{cl}A)$ and *regular closed* if $A = \text{cl}(\text{int}A)$. Clearly, $\alpha$-sets and regular closed sets are semi-open. Njastad [6] pointed out that the family of all $\alpha$-sets of $(X, \tau)$, denoted by $\tau^\alpha$, is a topology on $X$ finer than $\tau$. Reilly and Vamanamurthy observed in [7] that a subset A of $(X, \tau)$ is an $\alpha$-set if and only if it is semi-open and preopen. We denote the family of all semi-open (resp. preopen, regular closed) sets in $(X, \tau)$ by $SO(X, \tau)$ (resp. $PO(X, \tau)$, $RC(X, \tau)$).

**Remark 1.1** [3] Let $(X, \tau)$ be a space. Then $(\tau^\alpha)^\alpha = \tau^\alpha$, $SO(X, \tau^\alpha) = SO(X, \tau)$, $PO(X, \tau^\alpha) = PO(X, \tau)$ and $RC(X, \tau^\alpha) = RC(X, \tau)$.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively. The Stone-Cech compactification of $\mathbb{N}$ is denoted by $\beta\mathbb{N}$. The cardinality of a set $A$ will be denoted by $|A|$ . No separation axioms are assumed unless stated explicitly. For concepts not defined in our paper we refer the reader to [2].

*2000 Math. Subject Classification — 54D20.
Key words and phrases — $S_1$-metacompact, $S_2$-metacompact, $S_3$-metacompact, $\alpha$-open, semi-open.

## 2 $S_i$-metacompact spaces

**Definition 1** A space $(X, \tau)$ is called
(i) $S_1$-*metacompact* if every semi-open cover has a point-finite open refinement,
(ii) $S_2$-*metacompact* if every semi-open cover has a point-finite semi-open refinement,
(iii) $S_3$-*metacompact* if every open cover has a point-finite semi-open refinement.

**Definition 2** A space $(X, \tau)$ is called
(i) *globally disconnected* [1] if every semi-open set is open,
(ii) *nodec* [10] (or an $\alpha$-*space*) if every nowhere dense subset is closed, or equivalently (see [6]), if $\tau = \tau^\alpha$ .

Clearly, if $(X, \tau)$ is globally disconnected then all the notions in Definition 1 coincide and are equivalent to metacompactness. In general, we have the following obvious implications:

$$\begin{array}{ccc} S_1\text{-metacompact} & \Rightarrow & S_2\text{-metacompact} \\ \Downarrow & & \Downarrow \\ \text{metacompact} & \Rightarrow & S_3\text{-metacompact} \end{array}$$

It will be seen, however, that none of the implications above can be reversed, and that $S_2$-metacompactness and metacompactness are independent of each other. We shall first characterize globally disconnected spaces and $S_1$-metacompact spaces.

**Proposition 2.1** For a space $(X, \tau)$ the following are equivalent:
1) $(X, \tau)$ is globally disconnected,
2) every semi-open cover of $(X, \tau)$ has an open refinement.

**Proof.** 1) $\Rightarrow$ 2) is clear. To show that 2) $\Rightarrow$ 1) let $A \in SO(X, \tau)$ and let $U$=int$A$ . Then $\mathcal{U} = \{U \cup \{x\} \ : \ x \in \text{cl}U\} \cup \{X \setminus \text{cl}U\}$ is a semi-open cover of $(X, \tau)$. By assumption, there exists an open refinement of $\mathcal{U}$ , say $\mathcal{V}$ . Suppose that $A$ is not open, i.e. there is an $x \in A \setminus U$ . Pick $V \in \mathcal{V}$ such that $x \in V$ . Then $V \subseteq U \cup \{x\} \subseteq A$ and thus $x \in U$, a contradiction. Hence $A$ is open.

**Corollary 2.2** A space $(X,\tau)$ is $S_1$-metacompact if and only if it is metacompact and globally disconnected.

Recall that a space $(X,\tau)$ is said to be *extremally disconnected* if the closure of each open set is open, or equivalently, if each regular closed set is clopen. Jankovic [3] observed that $(X,\tau)$ is extremally disconnected if and only if $SO(X,\tau) \subseteq PO(X,\tau)$ . The next (known) result follows easily from the fact that $\tau^\alpha = SO(X,\tau) \cap PO(X,\tau)$ [7] and from Remark 1.1.

**Remark 2.3** For a space $(X,\tau)$ the following are equivalent:
(i) $(X,\tau)$ is extremally disconnected,
(ii) $(X,\tau^\alpha)$ is extremally disconnected,
(iii) $SO(X,\tau) \subseteq \tau^\alpha$ ,
(iv) $(X,\tau^\alpha)$ is globally disconnected.

**Remark 2.4** (i) A space $(X,\tau)$ is globally disconnected if and only if it is extremally disconnected and nodec.
(ii) If $(X,\tau)$ is extremally disconnected then the concepts of $(X,\tau^\alpha)$ being metacompact, $S_1$-metacompact, $S_2$-metacompact and $S_3$-metacompact coincide.

**Remark 2.5** The notions of $(X,\tau)$ being metacompact and of $(X,\tau^\alpha)$ being metacompact are, in general, independent of each other. In Corollary 2.10 we provide an example of a compact, thus metacompact, space $(X,\tau)$ such that $(X,\tau^\alpha)$ fails to be metacompact. On the other hand, if $(X,\tau)$ denotes the space in Theorem 2.14(2) then $(X,\tau)$ fails to be metacompact while $(X,\tau^\alpha)$ is metacompact.

Clearly, every discrete space is $S_1$-metacompact. To provide an example of a non-discrete $S_1$-metacompact space, we consider $X = \mathbb{N} \cup \{p\}$ where $p$ denotes a free ultrafilter on $\mathbb{N}$. We take as a base for the topology $\tau$ all sets of the form $U \cup \{p\}$ where $U \in p$ , together with all singletons $\{n\}, n \in \mathbb{N}$ . It is very well known (see e.g. [9], Ex. 114) that $(X,\tau)$ is perfectly normal, extremally disconnected and metacompact. Clearly every nowhere dense set is closed, so $(X,\tau)$ is nodec and thus globally disconnected. Hence $(X,\tau)$ is $S_1$-metacompact by Corollary 2.2.

**Remark 2.6** Spaces which fail to be $S_1$-metacompact exist in profusion. The usual space $\mathbb{R}$ of reals is metacompact but not $S_1$-metacompact. $\beta\mathbb{N}$ is a compact $T_2$ space which is not $S_1$-metacompact as $\mathbb{N} \cup \{p\}$ with $p \in \beta\mathbb{N} \setminus \mathbb{N}$ is a semi-open subset of $\beta\mathbb{N}$ that is not open.

We shall now consider $S_2$-metacompact spaces and $S_3$-metacompact spaces. First of all, by Remark 1.1 we have that $(X, \tau)$ is $S_2$-metacompact if and only if $(X, \tau^\alpha)$ is $S_2$-metacompact.

**Example 2.7** Let $X^* = X \cup \{p\}$ denote the 1-point-compactification of an infinite discrete space $X$, where $p \notin X$. Let $\mathcal{U}$ be a semi-open cover of $X^*$. Pick $S \in \mathcal{U}$ such that $p \in S$. Then $S = \{p\} \cup A$ where $A$ is an infinite subset of $X$. Clearly, $\mathcal{V} = \{S\} \cup \{\{x\} : x \in X \setminus A\}$ is a disjoint semi-open refinement of $\mathcal{U}$. Thus $X^*$ is a compact Hausdorff $S_2$-metacompact space. In addition, it is easily seen that $X^*$ fails to be globally disconnected, hence cannot be $S_1$-metacompact.

Next we recall the following set-theoretic construction that has been observed in [8].

**Lemma 2.8** [8] Let $X$ be an infinite set with $|X| = \mu$ and let $X = A \cup B$ where $|A| = \mu$, $|B| = \nu < \mu$ and $A \cap B = \emptyset$ . For each $x \in A$ let $C_x = B \cup \{x\}$ . Then $\mathcal{C} = \{C_x : x \in A\}$ is a cover of $X$ . Suppose that $\mathcal{D}$ is a refinement of $\mathcal{C}$ . For each $x \in A$ there exists $D_x \in \mathcal{D}$ such that $x \in D_x \subseteq C_x$ . If $|D_x| > 1$ for each $x \in A$ then there exists an element $z \in B$ such that $z$ lies in at least $\mu$ elements of the refinement $\mathcal{D}$ .

**Theorem 2.9** Let $(X, \tau)$ be an infinite space possessing a dense open subset $D$ such that $|D| < |X|$ . Then $(X, \tau)$ fails to be $S_2$-metacompact.

**Proof.** Let $A = X \setminus D$ and $B = D$ . Utilizing Lemma 2.8 we obtain a semi-open cover $\mathcal{C}$ that does not have a semi-open point finite refinement. Thus $(X, \tau)$ fails to be $S_2$-metacompact.

**Corollary 2.10** (i) $\beta\mathbb{N}$ fails to be $S_2$-metacompact.
(ii) Let $\sigma$ denote the topology of $\kappa\mathbb{N}$, the Katetov extension of $\mathbb{N}$, and let $\tau$ be the topology of $\beta\mathbb{N}$. It is well known that $\sigma = \tau^\alpha$ . Clearly $\kappa\mathbb{N}$ fails to be $S_2$-metacompact. Since $\beta\mathbb{N}$ is known to be extremally disconnected, by Remark 2.4, $\kappa\mathbb{N}$ even fails to be $S_3$-metacompact.

**Theorem 2.11** Let $(X, \tau)$ be an infinite space possessing a closed nowhere dense subset $N$ and a dense subset $B$ such that $|B| < |N| \geq \aleph_0$ . Then $(X, \tau)$ fails to be $S_2$-metacompact.

**Proof.** For each $x \in N$ let $S_x = (X \setminus N) \cup \{x\}$ . Then $\mathcal{U} = \{S_x \ : \ x \in N\}$ is a semi-open cover of $X$ . Suppose that $\mathcal{V}$ is a semi-open refinement of $\mathcal{U}$. For each $x \in N$ there exists $T_x \in \mathcal{V}$ such that $x \in T_x$. Clearly $T_x \subseteq S_x$ for $x \in N$, and $T_x \neq T_y$ for distinct points $x, y \in N$ . Now, $\{\text{int}T_x \ : \ x \in N\}$ is a family of $|N|$ nonempty open subsets of $(X, \tau)$. Since $B$ is dense and $|B| < |N| \geq \aleph_0$, there must be a point $z \in B$ which is contained in infinitely many sets $T_x$. Thus $\mathcal{V}$ cannot be point-finite and $(X, \tau)$ fails to be $S_2$-metacompact.

**Corollary 2.12** The usual space $\mathbb{R}$ fails to be $S_2$-metacompact (let $N$ be the Cantor set, $B$ the set of rationals and apply Theorem 2.11).

In the following we exhibit a useful method for generating new topological spaces from given spaces. Let $(Y, \sigma)$ be a topological space, $Z$ be an infinite set disjoint from $Y$ and let $X = Y \cup Z$ . We define a topology $\tau$ on $X$ in the following way: every set $S \subseteq Y$ with $S \in \sigma$ is open in $(X, \tau)$ and a basic neighbourhood of $z \in Z$ is of the form $\{z\} \cup Y$ . Clearly, $(Y, \sigma)$ is a dense and open subspace of $(X, \tau)$. It is easily checked that for $S \neq \emptyset$ we have that $S \in SO(X, \tau)$ if and only if $S \cap Y$ is a nonempty semi-open subset of $(Y, \sigma)$.

**Proposition 2.13** Let $(Y, \sigma)$ be a topological space, $Z$ be an infinite set and let $(X, \tau)$ be as defined above. Then

(a) $(X, \tau)$ fails to be metacompact,

(b) $(X, \tau)$ is extremally disconnected if and only if $(Y, \sigma)$ is hyperconnected (i.e. every nonempty open set in $(Y, \sigma)$ is dense in $(Y, \sigma)$),

(c) if $(Y, \sigma)$ has a nonempty finite open set, then $(X, \tau)$ is not $S_2$-metacompact,

(d) if $Z$ is countable and $(Y, \sigma)$ is countably infinite, $T_1$ and dense-in-itself then $(X, \tau)$ is $S_2$-metacompact,

(e) Let $Z$ be countable. Then $(X, \tau)$ is $S_3$-metacompact if and only if $(Y, \sigma)$ has a countably infinite point-finite family of nonempty semi-open sets.

**Proof.** (a) First observe that the open cover $\mathcal{U} = \{Y \cup \{z\} : z \in Z\}$ cannot have a point-finite open refinement. Thus $(X, \tau)$ fails to be metacompact.
(b) Suppose that $(X, \tau)$ is extremally disconneced and let $\emptyset \neq U \subseteq Y$ be open in $(Y, \sigma)$. Then $U \in \tau$ and $Z \subseteq \mathrm{cl}U$. By assumption, $\mathrm{cl}U$ is open in $(X, \tau)$ and thus $\mathrm{cl}U = X$. Hence $U$ is dense in $(Y, \sigma)$, i.e. $(Y, \sigma)$ is hyperconnected. To prove the converse, suppose that $(Y, \sigma)$ is hyperconnected and let $U \neq \emptyset$ be open in $(X, \tau)$. Then $V = U \cap Y$ is nonempty and open in $(Y, \sigma)$. Since $\mathrm{cl}V = \mathrm{cl}_Y V \cup Z$, we have, by assumption, that $\mathrm{cl}V = X$ and thus $\mathrm{cl}U = X$, i.e. $(X, \tau)$ is extremally disconnected. Observe that we have also shown that $(X, \tau)$ is hyperconnected if and only if $(Y, \sigma)$ is hyperconnected.
(c) Now let $U \neq \emptyset$ be a finite open subset of $(Y, \sigma)$. We may assume that $U$ is minimal open, i.e. $U$ does not properly contain a nonempty open set. Let $\mathcal{V}$ be a semi-open refinement of the semi-open cover $\mathcal{U} = \{Y\} \cup \{U \cup \{z\} : z \in Z\}$. For each $z \in Z$ there is $T_z \in \mathcal{V}$ such that $z \in T_z$, and $T_z \neq T_{z'}$ whenever $z \neq z'$. Since $U$ is minimal open we have $T_z = U \cup \{z\}$. This shows that $\mathcal{V}$ cannot be point-finite, i.e. (c) holds.
(d) Let $Z = \{z_n : n \in \mathbb{N}\}$, $Y = \{y_n : n \in \mathbb{N}\}$ and suppose that $(Y, \sigma)$ is $T_1$ and dense-in-itself. Let $C_1 = Y$ and $C_n = Y \setminus \{y_1, y_2, ..., y_{n-1}\}$ for each $n > 1$. By assumption, each $C_n$ is open and dense in $(Y, \sigma)$. Suppose that $\mathcal{U}$ is a semi-open cover of $(X, \tau)$. For each $n \in \mathbb{N}$ we pick $U_n, V_n \in \mathcal{U}$ such that $y_n \in U_n$ and $z_n \in V_n$. Let $S_n = U_n \cap C_n$ and $T_n = \{z_n\} \cup (V_n \cap C_n)$ for each $n \in \mathbb{N}$. Observe that $S_n$ and $V_n \cap C_n$ are always nonempty semi-open subsets of $(Y, \sigma)$. Clearly we have $y_n \in S_n \subseteq U_n$ and $z_n \in T_n \subseteq V_n$ for each $n \in \mathbb{N}$, hence $\mathcal{V} = \{S_n : n \in \mathbb{N}\} \cup \{T_n : n \in \mathbb{N}\}$ is a semi-open refinement of $\mathcal{U}$. Moreover, it is easily checked that $\mathcal{V}$ is point-finite, thus (d) holds.
(e) Let $Z = \{z_n : n \in \mathbb{N}\}$. First suppose that $(X, \tau)$ is $S_3$-metacompact. Then $\mathcal{U} = \{\{z_n\} \cup Y : n \in \mathbb{N}\}$ is an open cover of $(X, \tau)$. By assumption, there exists a point-finite semi-open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $n \in \mathbb{N}$ pick $V_n \in \mathcal{V}$ such that $z_n \in V_n$. Then $\{V_n \cap Y : n \in \mathbb{N}\}$ is a countably infinite point-finite family of nonempty semi-open subsets of $(Y, \sigma)$. To prove the converse, let $\{S_n : n \in \mathbb{N}\}$ be a point-finite family of nonempty semi-open subsets of $(Y, \sigma)$. Let $\mathcal{U}$ be an open cover of $(X, \tau)$. For each $n \in \mathbb{N}$ there exists $U_n \in \mathcal{U}$ such that $z_n \in \{z_n\} \cup Y \subseteq U_n$. If $T_n = \{z_n\} \cup S_n$ for each $n \in \mathbb{N}$, then

$\mathcal{V} = \{T_n : n \in \mathbb{N}\} \cup \{Y\}$ is a point-finite semi-open refinement of $\mathcal{U}$. Hence $(X, \tau)$ is $S_3$-metacompact.

As an application of Proposition 2.13 we have the following result which concludes our discussion of $S_i$-metacompact spaces.

**Theorem 2.14** (1) There is an $S_3$-metacompact space that is neither $S_2$-metacompact nor metacompact.
(2) There is an $S_2$-metacompact space that is not metacompact.

**Proof.** (1) Let $Z$ be a countably infinite set and let $(Y, \sigma)$ be an infinite discrete space such that $Y \cap Z = \emptyset$. By applying Proposition 2.13(e),(c) and (a), the resulting space $(X, \tau)$ is $S_3$-metacompact but neither $S_2$-metacompact nor metacompact.
(2) Let $Y$ and $Z$ be two disjoint countably infinite sets and let $\sigma$ be the cofinite topology on $Y$. By applying Proposition 2.13(d) and (a), the resulting space $(X, \tau)$ is $S_2$-metacompact but not metacompact. In addition, $(Y, \sigma)$ is hyperconnected so $(X, \tau)$ is extremally disconnected by Proposition 2.13 (b). Since $(X, \tau^\alpha)$ is also $S_2$-metacompact, $(X, \tau^\alpha)$ is metacompact by Remark 2.4.

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