ACCELERATION OF LAMPLIGHTER RANDOM WALKS

Lorenz A. Gilch

Graz University of Technology, Graz, Austria

Abstract. Suppose we are given an infinite, finitely generated group G and a transient random walk on the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr G$, such that its projection on G is transient and has finite first moment. This random walk can be interpreted as a lamplighter random walk on G. Our aim is to show that the random walk on the wreath product escapes to infinity with respect to a suitable (pseudo-)metric faster than its projection onto G. We also address the case where the pseudo-metric is the length of a shortest "travelling salesman tour". In this context, and excluding some degenerate cases if $G = \mathbb{Z}$, the linear rate of escape is strictly bigger than the rate of escape of the lamplighter random walk's projection on G.

1. Introduction

Let G be an infinite group generated by a finite symmetric set S, and imagine a lamp sitting at each group element. These lamps have two states: 0 ("off") or 1 ("on"), and initially all lamps are off. We think of a lamplighter walking randomly on G and switching lamps on or off as he walks. We investigate the following model: at each step the lamplighter may walk to some random neighbour vertex, and may flip some lamps in a bounded neighbourhood of his position. This model can be interpreted as a random walk on the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr G$ governed by a probability measure μ. The random walk is described by a Markov chain $(Z_n)_{n\in\mathbb{N}_0}$, which represents the position X_n of the lamplighter and the lamp configuration $\eta_n : G \to \mathbb{Z}/2\mathbb{Z}$ at time n. We assume that the lamplighter random walk's projection on G has finite first moment and is also transient.

For better visualization, we identify G with its Cayley graph with respect to the generating set S. Suppose we are given "lengths" of the elements of S such that $s \in S$ and $s^{-1} \in S$ have the same length. The length of a path in G is the sum of the lengths of its edges, and we denote by $d(\cdot,\cdot)$ the metric on G induced by the lengths of the edges. We denote by $d_{\mathrm{TS}}(\eta, x)$ the length of an optimal "travelling salesman tour" from the identity e to $x \in G$ that visits each point in $\mathrm{supp}(\eta)$ (where $\eta : G \to \mathbb{Z}/2\mathbb{Z}$ has finite support). Then a natural length function $\ell(\eta, x)$ is given by

Date: October 2, 2008.

2000 *Mathematics Subject Classification.* Primary 60G50; Secondary 20E22, 60B15.

Key words and phrases. Random Walks, Lamplighter Groups, Rate of Escape.

$d_{\mathrm{TS}}(\eta, x) + c_{\mathcal{L}} \cdot |\mathrm{supp}(\eta)|$ for an arbitrary, but fixed constant $c_{\mathcal{L}} \geq 0$. By transience, our random walk escapes to infinity with respect to this length function.

The (new) topic that we address in this paper is the comparison of the limits $\ell = \lim_{n\to\infty} \ell(Z_n)/n$ and $\ell_0 = \lim_{n\to\infty} d(e, X_n)/n$, which exist almost surely. They describe the speed of the lamplighter random walk and its projection on G, respectively. The number ℓ_0 is called the *rate of escape*, or the *drift*, of $(X_n)_{n\in\mathbb{N}_0}$ and ℓ is the *rate of escape of the lamplighter random walk* $(Z_n)_{n\in\mathbb{N}_0}$. It is well-known that the rate of escape exists for random walks with finite first moment on transitive graphs. This follows from *Kingman's subadditive ergodic theorem*; see Kingman [15], Derriennic [6] and Guivarc'h [10]. We will prove that, under some weak assumptions on G, we have $\ell > \ell_0$, that is, the lamplighter random walk escapes strictly faster to infinity than its projection onto G, on which we have the metric $d(\cdot,\cdot)$. If the lamplighter random walk's projection on G is transient and has zero drift, then the acceleration of the lamplighter random walk follows from results of Kaimanovich and Vershik [12] and of Varopoulos [18]. Thus, we may restrict ourselves to the case $\ell_0 > 0$. More explicitly, we will prove that $\lim_{n\to\infty} |\mathrm{supp}(\eta_n)|/n > 0$, where η_n is the lamp configuration at time n. From this it follows directly that $\ell > \ell_0$. We also prove $\lim_{n\to\infty} d_{\mathrm{TS}}(Z_n)/n > \ell_0$ (except for some degenerate cases), providing $\ell > \ell_0$.

Let us briefly review a few selected results regarding the rate of escape. The classical case is that of random walks on the k-dimensional grid $\mathbb{Z}^k$, where $k \geq 1$, which can be described by the sum of n i.i.d. random variables, the increments of n steps. By the law of large numbers the limit $\lim_{n\to\infty} \|Z_n\|/n$, where $\|\cdot\|$ is the distance on the grid to the starting point of the random walk, exists almost surely. Furthermore, this limit is positive if the increments have non-zero mean vector. There is an important link between drift and the Liouville property: the entropy (introduced by Avez [1]) of any random walk on a group is non-zero if and only if non-constant harmonic functions exist; see Kaimanovich and Vershik [12] and Derriennic [6]. Moreover, if the rate of escape is zero, then the entropy is zero (first observed by Guivarc'h [10]). Varopoulos [18] has shown the converse for symmetric finite range random walks on groups. The recent work of Karlsson and Ledrappier [14] generalizes this result to symmetric random walks with finite first moment of the step lengths.

In this paper we deal with random walks on wreath products, for which there are many detailed results: Lyons, Pemantle and Peres [16] gave a lower bound for the rate of escape of inward-biased random walks on lamplighter groups. Revelle [17] examined the rate of escape of random walks on wreath products. He proved laws of the iterated logarithm for the inner and outer radius of escape. For a finitely generated group A, Dyubina [7] proved that the drift w.r.t. the word metric of a random walk on the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr A$ is zero if and only if the random walk's projection onto A is recurrent.

It is not obvious that lamplighter random walks are in general faster than their projections onto G: e.g., consider the Switch-Walk lamplighter random walk on $\mathbb{Z}$ with drift: in each step switch the lamp at the actual position with probability $p \in (0,1)$ and then walk to a random neighbour vertex. Then the rate of escape of the lamplighter random walk is equal to the random walk's projection onto $\mathbb{Z}$ whenever $c_{\mathcal{L}} = 0$; compare with Bertacchi [3]. However, this example is more or less the only counterexample. The author of this article has investigated the rate of escape of lamplighter random walks arising from a simple random walk on homogeneous trees providing tight lower and upper bounds for the rate of escape; see Gilch [9]. In particular, the lamplighter random walk is significantly faster than its projection onto the tree. This was the starting point for the investigation of the relation between ℓ and ℓ_0 on more general classes of graphs.

The structure of this article is as follows: in Section 2 we give an introduction to lamplighter random walks on groups and some basic properties. In Section 3 we prove that $\lim_{n\to\infty} |\mathrm{supp}(\eta_n)|/n > 0$ if and only if the lamplighter random walk's projection on G is transient. In Section 4 we prove $\lim_{n\to\infty} d_{\mathrm{TS}}(Z_n)/n > \ell_0$ under some weak assumptions on S, which exclude some degenerate cases. Finally, in Section 5 we give some additional remarks regarding extensions of the presented results.

2. Lamplighter Groups

2.1. **Groups and Random Walks.** Consider an infinite, finitely generated group G with identity e and a finite, symmetric set of generators $S \subseteq G \setminus \{e\}$, which generates G as a semigroup. We assign to each $s \in S$ a *length* $l(s) = l(s^{-1}) > 0$. We write $r_1 := \min_{s\in S} l(s)$. These lengths induce a metric on G: the distance between $x, y \in G$ is given by

$$d(x,y) := \min\left\{ \sum_{i=1}^{n} l(s_i) \,\middle|\, s_1,\dots,s_n \in S \text{ such that } y = x s_1 s_2 \cdots s_n \right\}.$$

We identify G with its Cayley graph with respect to S. A *path* in G is a finite sequence of group elements $[x_0, x_1, \dots, x_n]$ such that $x_{i-1}^{-1}x_i \in S$. The *length* of this path is $\sum_{i=1}^{n} l(x_{i-1}^{-1}x_i)$. The *ball* $B(x,r)$ centered at $x \in G$ with radius $r \geq 0$ is given by the set of all elements $y \in G$ with $d(x,y) \leq r$.

2.2. **Lamplighter Random Walks.** Imagine a *lamp* sitting at each vertex of G, which can be switched off or on, encoded by "0" and "1". We think of a lamplighter walking on G and switching lamps on and off. The lamp configurations are encoded by functions $\eta : G \to \mathbb{Z}/2\mathbb{Z}$. Writing $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, the *set of finitely supported configurations* of lamps is

$$\mathcal{N} := \big\{ \eta : G \to \mathbb{Z}_2 \;\big|\; |\mathrm{supp}(\eta)| < \infty \big\}.$$

Here, $|A|$ denotes the cardinality of a set A. Denote by $\mathbf{0}$ the zero configuration, which will be the initial lamp configuration of the random walk, and by $\mathbb{1}_x$ the configuration where only the lamp at $x \in G$ is on and all other lamps are off. The *wreath product* of $\mathbb{Z}_2$ with G is

$$\mathcal{L} := \Big(\sum_{x\in G} \mathbb{Z}_2\Big) \rtimes G = \mathbb{Z}_2 \wr G.$$

The elements of $\mathcal{L}$ are pairs of the form $(\eta, x) \in \mathcal{N} \times G$, where η represents a configuration of the lamps and x the position of the lamplighter. For $x, w \in G$ and $\eta \in \mathcal{N}$, define

$$(x\eta)(w) := \eta(x^{-1}w).$$

A group operation on $\mathcal{L}$ is given by

$$(\eta_1, x)(\eta_2, y) := \big(\eta_1 \oplus (x\eta_2), xy\big),$$

where $x, y \in G$, $\eta_1, \eta_2 \in \mathcal{N}$, $\oplus$ is the componentwise addition modulo 2. The group identity is $(\mathbf{0}, e)$. We call $\mathcal{L}$ together with this operation the *lamplighter group over G*.

A natural symmetric set of generators of $\mathcal{L}$ is given by

$$S_{\mathcal{L}} := \big\{(\mathbb{1}_e, e), (\mathbf{0}, s) \,\big|\, s \in S\big\}.$$

Consider the Cayley graph of $\mathcal{L}$ with respect to $S_{\mathcal{L}}$. We lift $d(\cdot,\cdot)$ to a (pseudo-)metric $d_{\mathcal{L}}(\cdot,\cdot)$ on $\mathcal{L}$ by assigning the following lengths to the elements of $S_{\mathcal{L}}$: $l\big((\mathbf{0}, s)\big) := l(s)$ for $s \in S$ and $l\big((\mathbb{1}_e, e)\big) := c_{\mathcal{L}} \geq 0$, where $c_{\mathcal{L}}$ is some arbitrary, but fixed non-negative constant. These lengths induce a (pseudo-)metric on $\mathcal{L}$. The distance between (η, x) and (η', y) is then the minimal length of all paths in the Cayley graph of $\mathcal{L}$ joining both vertices. More explicitely, we denote by $d_{\mathrm{TS}}(\eta, x)$ the minimal length of a "travelling salesman tour" on G (not on $\mathcal{L}$) from e to x, which visits each $y \in \mathrm{supp}(\eta)$. With this notation, we have

$$\ell(\eta, x) := d_{\mathcal{L}}\big((\mathbf{0}, e), (\eta, x)\big) = d_{\mathrm{TS}}(\eta, x) + c_{\mathcal{L}} \cdot |\mathrm{supp}(\eta)|.$$

The case $c_{\mathcal{L}} = 0$ can also be interpreted as the model where $S_{\mathcal{L}}$ is replaced by $\{(\mathbf{0}, s), (\mathbb{1}_e, s) \mid s \in S\}$ and where the length of $(\mathbb{1}_e, s)$ equals $l(s)$. In this case, lamp switches are not charged by the pseudo-metric.

We now consider an irreducible, transient random walk on $\mathcal{L}$ starting at the identity $(\mathbf{0}, e)$ such that the random walk's projection onto G is also transient. For this purpose, consider the sequence of i.i.d. $\mathcal{L}$-valued random variables $(\mathbf{i}_n)_{n\in\mathbb{N}}$ governed by a probability measure μ which satisfies the following conditions:

(1) $\langle \mathrm{supp}(\mu) \rangle = \mathcal{L}$.
(2) There is a non-negative real number R such that

$$\mu(\eta, x) > 0 \;\text{ implies }\; d(e, y) \leq R \text{ for all } y \in \mathrm{supp}(\eta).$$

(3) The projection of μ onto G has finite first moment, that is, $\sum_{(\eta,x)\in\mathcal{L}} d(e,x)\mu(\eta,x) < \infty$.

We write $\mu^{(n)}$ for the n-th convolution power of μ. A lamplighter random walk starting at $(\mathbf{0},e)$ is described by the sequence of $\mathcal{L}$-valued random variables $(Z_n)_{n\in\mathbb{N}_0}$ in the following natural way:

$$Z_0 := (\mathbf{0},e), \quad Z_n := Z_{n-1}\mathbf{i}_n \text{ for all } n \geq 1.$$

More precisely, we write $Z_n = (\eta_n, X_n)$, where η_n is the random configuration of the lamps at time n and X_n is the random group element at which the lamplighter stands at time n. As a *general assumption* we assume transience of $(X_n)_{n\in\mathbb{N}_0}$. We explain below what happens if this assumption fails.

The corresponding single and n-step transition probabilities of the random walk on $\mathcal{L}$ are denoted by $p(\cdot,\cdot)$ and $p^{(n)}(\cdot,\cdot)$. We write $\mathbb{P}_z[\,\cdot\,] := \mathbb{P}[\,\cdot \mid Z_0 = z]$ for any $z \in \mathcal{L}$, if we want to start the lamplighter walk at z instead of $(\mathbf{0},e)$.

Observe that by transience of $(X_n)_{n\in\mathbb{N}_0}$ each finite subset of G is visited only finitely often yielding that the sequence $(\eta_n)_{n\in\mathbb{N}_0}$ converges pointwise to a random limit configuration $\eta_\infty : G \to \mathbb{Z}_2$, which is not necessarily finitely supported. On the other hand, $(X_n)_{n\in\mathbb{N}_0}$ leaves every finite set after some finite time forever, that is, $d(e,X_n)$ goes to infinity.

As a consequence of *Kingman's subadditive ergodic theorem* there are non-negative numbers $\ell_0, \ell \in \mathbb{R}$ such that

$$\ell_0 = \lim_{n\to\infty} \frac{d(e,X_n)}{n} \quad \text{and} \quad \ell = \lim_{n\to\infty} \frac{\ell(Z_n)}{n} \quad \text{almost surely;}$$

see Derriennic [6] and Guivarc'h [10]. The number ℓ_0 is called the *rate of escape* or *drift* of the lamplighter random walk's projection onto G. Analogously, ℓ is the *rate of escape of the lamplighter random walk*. Moreover, we can write

$$\ell = \ell_{\mathrm{TS}} + c_{\mathcal{L}} \cdot \ell_{\mathrm{supp}}, \tag{2.1}$$

where

$$\begin{aligned} \ell_{\mathrm{TS}} &:= \lim_{n\to\infty} \frac{d_{\mathrm{TS}}(\eta_n, X_n)}{n} \text{ (core rate of escape) and} \\ \ell_{\mathrm{supp}} &:= \lim_{n\to\infty} \frac{|\mathrm{supp}(\eta)|}{n} \text{ (asymptotic configuration size).} \end{aligned}$$

The latter limits exist for the same reason as above. Obviously, $\ell \geq \ell_{\mathrm{TS}} \geq \ell_0$. By Kaimanovich and Vershik [12], we have $h \leq \ell \cdot g$, where

$$h = \lim_{n\to\infty} -\frac{1}{n} \sum_{(\eta,x)\in\mathcal{N}\times G} p^{(n)}\big((\mathbf{0},e),(\eta,x)\big)\, \log p^{(n)}\big((\mathbf{0},e),(\eta,x)\big)$$

is the *asymptotic entropy* and $g = \lim_{n\to\infty}(\log|B_{\mathcal{L}}(n)|)/n$ is the *growth rate* of $\mathcal{L}$, where $B_{\mathcal{L}}(n)$ denotes the ball around $(\mathbf{0}, e)$ of radius n with respect to $d_{\mathcal{L}}(\cdot,\cdot)$. Existence of h and g follows again from Kingman's subadditive ergodic theorem. We have $g < \infty$ even if one considers the balls with respect to the pseudo-metric $d_{\mathrm{TS}}(\cdot,\cdot)$, because of the relation

$$\big\{(\eta, x) \in \mathcal{L} \mid d_{\mathrm{TS}}(\eta, x) \leq n\big\} \subseteq B_{\mathcal{L}}\big(n + (\lfloor n/r_1 \rfloor + 1)c_{\mathcal{L}}\big).$$

Furthermore, we also have $h > 0$: the mapping

$$(\eta, x) \mapsto \mathbb{P}_{(\eta,x)}[\eta_\infty(e) = 0]$$

defines a non-constant bounded harmonic function. Thus, the Poisson boundary is non-trivial, that is, $h > 0$; see Kaimanovich [11] and Kaimanovich and Woess [13]. Thus, we get $\ell > 0$. As an additional remark, let us mention that Dyubina [7] proved that lamplighter random walks w.r.t. the word metric have non-zero drift if and only if the projection on G is transient. Since S is finite and the lengths of edges are bounded, we have in our situation $\ell = 0$ if G is recurrent.

Our basic aim is to show that ℓ is strictly bigger than ℓ_0, that is, the lamplighter random walk escapes faster to infinity than its projection onto G. For this purpose, we will show $\ell_{\mathrm{supp}} > 0$ in the following section, giving $\ell > \ell_0$ in the case $c_{\mathcal{L}} > 0$. In Section 4, we will prove $\ell_{\mathrm{TS}} > \ell_0$ under suitable weak assumptions on G, which exclude degenerate cases for special choices of S and $l(\cdot)$ when $G = \mathbb{Z}$.

3. The Asymptotic Configuration Size

In this section we want to show that the number of lamps which are on increases asymptotically at linear speed. We show that $\ell_{\mathrm{supp}} > 0$, giving $\ell > \ell_0$ in the case $c_{\mathcal{L}} > 0$.

Consider now the lamplighter random walk's projection on G and its range $\mathcal{R}_n \subseteq G$, which is the set of visited elements up to time n. By Derriennic [6], $|\mathcal{R}_n|/n$ converges to $\mathbb{P}[\forall n \geq 1 : X_n \neq e]$, which is strictly positive in our case by transience. For $j \in \mathbb{N}$, let

$$\mathbf{s}_j := \min\big\{n \in \mathbb{N}_0 \,\big|\, |\mathcal{R}_n| = j\big\}.$$

By transience, $\mathbf{s}_j < \infty$ almost surely. For $k \in \mathbb{N}_0$, let

$$\Delta_{k,j} := \begin{cases} 1, & \text{if } \eta_{\mathbf{s}_j+k}(X_{\mathbf{s}_j}) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

With this definition we have for all $n \in \mathbb{N}_0$

$$|\mathrm{supp}(\eta_n)| \geq \sum_{j=1}^{|\mathcal{R}_n|} \Delta_{n-\mathbf{s}_j,j}. \tag{3.1}$$

We give now a uniform lower bound for the probability $\mathbb{P}_{(\eta,x)}[\eta_\infty(x) = 1]$ with $(\eta, x) \in \mathcal{N} \times G$.

Lemma 3.1. *There are $\kappa \in \mathbb{N}$ and $C > 0$ such that for all $(\eta, x) \in \mathcal{N} \times G$*

$$\mathbb{P}_{(\eta,x)}[\forall n \geq \kappa : \eta_n(x) = 1] \geq C.$$

Proof. By transience and bounded range of the random walk $(X_n)_{n\in\mathbb{N}_0}$, there is at least one vertex $y \in G$ with $R < d(e, y)$ such that

$$\tilde{p} := \mathbb{P}_{(\eta',y)}[\forall n \geq 1 : X_n \notin B(e,R)] > 0 \quad \text{ for each } \eta' \in \mathcal{N}.$$

Moreover, there are $\kappa_0, \kappa_1 \in \mathbb{N}$ such that $C_0 := \mu^{(\kappa_0)}\big((\mathbf{0}, y)\big) > 0$ and $C_1 := \mu^{(\kappa_1)}\big((\mathbb{1}_e, y)\big) > 0$. Transitivity provides that the probability of walking from (η, x) to some (η', xy) with $x \in \operatorname{supp}(\eta')$ in at most $\kappa := \max\{\kappa_0, \kappa_1\}$ steps is at least $C' := \min\{C_0, C_1\}$. With $C := C' \cdot \tilde{p}$ follows the claim of the lemma. □

The next lemma gives a non-trivial uniform lower bound for $\mathbb{E}[\Delta_{n,j}]$ with $n \geq \kappa$:

Lemma 3.2. *For all $j, n \in \mathbb{N}$ with $n \geq \kappa$ we have $\mathbb{E}[\Delta_{n,j}] \geq C > 0$.*

Proof. In order to bound $\mathbb{P}[\Delta_{n,j} = 1]$ uniformly from below, we decompose according to all possible states of $X_{\mathbf{s}_j}$, followed by walking steps to some $X_{\mathbf{s}_j} y \in G \setminus B(X_{\mathbf{s}_j}, R)$, such that the lamp at $X_{\mathbf{s}_j}$ – if necessary – will be switched on and after reaching $X_{\mathbf{s}_j} y$ the random walk does not return to $B(X_{\mathbf{s}_j}, R)$.

By vertex-transitivity and Lemma 3.1, the probability of starting in $X_{\mathbf{s}_j}$ and walking in at most κ steps to some vertex $X_{\mathbf{s}_j} y \in G \setminus B(X_{\mathbf{s}_j}, R)$, such that the lamp at $X_{\mathbf{s}_j}$ rests on and no further visit in $B(X_{\mathbf{s}_j}, R)$ after reaching $X_{\mathbf{s}_j} y$ is at least $C > 0$.

Observe that, by transience we have for each $j \in \mathbb{N}$

$$\sum_{x\in G} \sum_{m\geq 0} \mathbb{P}[\mathbf{s}_j = m, X_m = x] = \mathbb{P}[\mathbf{s}_j < \infty] = 1.$$

We get for $n \geq \kappa$:

$$\begin{aligned} \mathbb{E}[\Delta_{n,j}] &= \sum_{(\eta,x)\in\mathcal{L}} \sum_{m\geq 0} \mathbb{P}\big[\mathbf{s}_j = m, Z_m = (\eta, x), \eta_{m+n}(x) = 1\big] \\ &\geq \sum_{(\eta,x)\in\mathcal{L}} \sum_{m\geq 0} \mathbb{P}\big[\mathbf{s}_j = m, Z_m = (\eta, x)\big] \cdot \mathbb{P}_{(\eta,x)}\big[\eta_n(x) = 1\big] \\ &\geq \mathbb{P}[\mathbf{s}_j < \infty] \cdot C = C > 0. \end{aligned}$$

□

Now we can conclude:

Theorem 3.3. *For the lamplighter random walk with respect to the infinite, finitely generated group G,*

$$\ell_{\mathrm{supp}} > 0.$$

Moreover, if $c_{\mathfrak{L}} > 0$, then $\ell > \ell_0$.

Proof. Recall the definition of κ from Lemma 3.1. We have:

$$\ell_{\mathrm{supp}} = \lim_{n\to\infty} \frac{|\mathrm{supp}(\eta_{n+\kappa})|}{|\mathcal{R}_n|} \frac{|\mathcal{R}_n|}{n} \frac{n}{n+\kappa} = \ell_1 \cdot \bar{F},$$

where $\ell_1 := \lim_{n\to\infty} |\mathrm{supp}(\eta_{n+\kappa})|/|\mathcal{R}_n|$ and $\bar{F} := \mathbb{P}[\forall n \geq 1 : X_n \neq e] > 0$. As ℓ_{supp} exists, the limit ℓ_1 also exists. If we set $D_n := \sum_{j=1}^{|\mathcal{R}_n|} \Delta_{n+\kappa-\mathbf{s}_j,j}/|\mathcal{R}_n|$, then (3.1) yields the inequality $\ell_1 \geq \limsup_{n\in\mathbb{N}} D_n$. Since the D_n's are bounded, Fatou's lemma yields

$$\ell_1 \geq \mathbb{E}\Big[\limsup_{n\in\mathbb{N}} D_n\Big] \geq \limsup_{n\in\mathbb{N}} \mathbb{E}[D_n].$$

With the help of Lemma 3.2 we obtain:

$$\begin{aligned}
\mathbb{E}\Bigg[\frac{1}{|\mathcal{R}_n|}\sum_{j=1}^{|\mathcal{R}_n|} \Delta_{n+\kappa-\mathbf{s}_j,j}\Bigg] &= \sum_{m=1}^{n+1} \mathbb{P}\big[|\mathcal{R}_n| = m\big] \cdot \mathbb{E}\Bigg[\frac{1}{|\mathcal{R}_n|}\sum_{j=1}^{|\mathcal{R}_n|} \Delta_{n+\kappa-\mathbf{s}_j,j} \,\Bigg|\, |\mathcal{R}_n| = m\Bigg] \\
&\geq \sum_{m=1}^{n+1} \mathbb{P}\big[|\mathcal{R}_n| = m\big] \cdot C = C > 0.
\end{aligned}$$

This yields

$$\ell_{\mathrm{supp}} \geq \bar{F} \cdot \limsup_{n\in\mathbb{N}} \mathbb{E}[D_n] \geq \bar{F}\cdot C > 0.$$

The rest follows by (2.1) and $\ell \geq \ell_{\mathrm{TS}} \geq \ell_0$. □

We can generalize the last theorem, if we do not necessarily assume transience of the projection $(X_n)_{n\in\mathbb{N}_0}$:

Theorem 3.4. *For the lamplighter random walk with respect to the infinite, finitely generated group G, we have $\ell_{\mathrm{supp}} > 0$ if and only if $(X_n)_{n\in\mathbb{N}_0}$ is transient.*

Proof. By Theorem 3.3, transience implies $\ell_{\mathrm{supp}} > 0$. For the proof of the inverse direction assume now $\ell_{\mathrm{supp}} > 0$. We have $|\mathrm{supp}(\eta_n)| \leq |B(e,R)| \cdot |\mathcal{R}_n|$. This yields

$$\begin{aligned}
0 < \ell_{\mathrm{supp}} &= \lim_{n\to\infty} \frac{|\mathrm{supp}(\eta_n)|}{n} \\
&\leq \lim_{n\to\infty} \frac{|B(e,R)|\cdot|\mathcal{R}_n|}{n} = |B(e,R)| \cdot \mathbb{P}[\forall n \geq 1 : X_n \neq e].
\end{aligned}$$

Thus, $\mathbb{P}[\exists n \geq 1 : X_n = e] < 1$, that is, $(X_n)_{n\in\mathbb{N}_0}$ is transient. □

We get also as a consequence of Theorem 3.3:

Corollary 3.5. *The core rate of escape satisfies* $\ell_{\mathrm{TS}} > 0$.

Proof. We have $d_{\mathrm{TS}}(\eta_n, X_n) \geq \big(|\mathrm{supp}(\eta_n)| - 1\big) \cdot r_1$. Dividing both sides of the inequality by n and taking limits yields the proposed claim. □

Finally, we give an explicit formula for ℓ_{supp} for the special case of a "Walk-Switch" lamplighter random walk.

Example 3.6: Suppose we are given a probability measure μ_0 on G with finite support and $\langle \mathrm{supp}(\mu_0)\rangle = G$. Then a "Walk-Switch" lamplighter random walk over G is given by the transition probabilities

$$p\big((\eta_1, x_1), (\eta_2, x_2)\big) := \frac{1}{2}\mu_0(x_1^{-1}x_2)$$

for $x_1, x_2 \in G$ and $\eta_1, \eta_2 \in \mathcal{N}$ with $\eta_1 = \eta_2$ or $\eta_1 \oplus \mathbb{1}_{x_2} = \eta_2$. It is easy to see that $\mathbb{E}[|\mathrm{supp}(\eta_n)|] = \mathbb{E}[|\mathcal{R}_n|]/2$, providing

$$\ell_{\mathrm{supp}} = \frac{1}{2}\big(1 - \mathbb{P}[\exists n \geq 1 : X_n = e]\big).$$

4. The Core Rate of Escape

In this section we want to prove $\ell_{\mathrm{TS}} > \ell_0$ whenever G is generated as a semigroup by a symmetric set S with at least three elements. If, however, $G = \mathbb{Z}$ we have to make some weak assumption on the lengths of the elements of S to show $\ell_{\mathrm{TS}} > \ell_0$; otherwise we can construct counterexamples where $\ell_{\mathrm{TS}} = \ell_0$. In this section we may again assume $\ell_0 > 0$, since $\ell_{\mathrm{TS}} > 0$.

4.1. **Groups generated by at least three elements.** In this section we assume that $S = \{s_1, \ldots, s_r\}$ is symmetric with $r \geq 3$ such that there is no symmetric set $\{s, s'\} \subseteq S$ with $G = \langle s, s'\rangle$. If $G = \langle s, s'\rangle$, then G is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = e\rangle$, and we treat these cases in Section 4.2. We want to prove that $\ell_{\mathrm{TS}} > \ell_0$ under the above assumption to S. Without loss of generality we may assume that the elements of S are ordered such that $l(s_1) \leq l(s_2) \leq \cdots \leq l(s_r)$. Our next aim is to choose three elements $\sigma_1, \sigma_2, \sigma_3 \in S$ such that $d(\sigma_k, \sigma_l) \geq \max\{d(e, \sigma_k), d(e, \sigma_l)\}$ (with one single exception). For this purpose, we have to make a case distinction:

I. If $s_1 \neq s_1^{-1}$, then we define $\sigma_1 := s_1$, $\sigma_2 := s_1^{-1}$ and we set $\sigma_3 := s_i$ with $i = \min\{k \geq 2 \mid s_k \notin \langle s_1^{-1}, s_1, s_2, \ldots, s_{k-1}\rangle\}$, that is, σ_3 is not a multiple of σ_1 or σ_2. Note that we ensured by the above assumptions on S the existence of such a σ_3.

II. If $s_1 = s_1^{-1}$ and $s_2 = s_2^{-1}$, then define $\sigma_1 := s_1$, $\sigma_2 := s_2$ and $\sigma_3 := s_i$ with $i = \min\{k \geq 3 \mid s_k \notin \langle s_1, \ldots, s_{k-1} \rangle\}$. E.g., this may happen in the case $G = \langle a, b, c \mid a^2 = b^2 = c^2 = e \rangle$.

III. If $s_1 = s_1^{-1}$ and $s_2 \neq s_2^{-1}$, then define $\sigma_1 := s_1$, $\sigma_2 := s_2$ and $\sigma_3 := s_2^{-1}$. E.g., this may happen in the case $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}^3$, where $(2,0), (1,0) \in S$.

We will see that in fact it is not relevant which one of the above cases happens. In each of the cases we get the following straightforward equalities, resp. inequalities:

$$\begin{gathered} d(e, \sigma_1) = l(\sigma_1) = r_1,\ d(e, \sigma_2) \leq l(\sigma_2),\ d(e, \sigma_3) \leq l(\sigma_3), \\ d(\sigma_1, \sigma_2) \leq l(\sigma_1) + l(\sigma_2),\ d(\sigma_1, \sigma_3) \leq l(\sigma_1) + l(\sigma_3), \\ d(\sigma_2, \sigma_3) \leq l(\sigma_2) + l(\sigma_3). \end{gathered} \tag{4.1}$$

Moreover, we get the following (uniform) lower bounds:

Lemma 4.1. *In all the cases I, II, III,*

$$(i)\ d(e, \sigma_2) = l(\sigma_2), \quad (ii)\ d(e, \sigma_3) = l(\sigma_3), \quad (iii)\ d(\sigma_1, \sigma_k) \geq l(\sigma_k) \text{ for } k \in \{2, 3\}.$$

Furthermore,

$$(iv)\ d(\sigma_2, \sigma_3) \geq \begin{cases} l(\sigma_3), & \text{in case I and II}, \\ l(\sigma_1), & \text{in case III}. \end{cases}$$

Proof. Equation (i) follows from $l(\sigma_1) = l(\sigma_1^{-1})$ in the case I. In the cases II and III, we have $s_2 \neq s_1^m \in \{e, s_1\}$ for each $m \in \mathbb{N}$. Thus (4.1) yields equation (i). In case III, equation (ii) holds, since s_2^{-1} is not a multiple of s_1. For the proof of (ii) in case I and II, assume $d(e, \sigma_3) < l(\sigma_3)$, that is, there is a path $[e, x_1, \ldots, x_m = \sigma_3]$ with $d(x_{j-1}, x_j) < l(\sigma_3)$, that is, $\sigma_3 = s_i$ can be written as a product of elements of $s_1^{-1}, s_1, \ldots, s_{i-1}$, a contradiction to the minimal choice of σ_3. The inequalities (iii) and (iv) are proved by analogous arguments. Note that the case distinction in (iv) is necessary, as the equation $s_2^2 = s_1$ in case III may hold. □

With the last lemma we can prove the following lemma:

Lemma 4.2. *Let $A = \{e, \sigma_1, \sigma_2, \sigma_3\}$ and let $\varphi : \{1, 2, 3, 4\} \to A$ be an injective function. Then in each of the cases I, II, III,*

$$d\big(\varphi(1), \varphi(4)\big) + r_1 \leq d\big(\varphi(1), \varphi(2)\big) + d\big(\varphi(2), \varphi(3)\big) + d\big(\varphi(3), \varphi(4)\big).$$

Proof. The lemma states that for each choice of φ a shortest tour from $\varphi(1)$ to $\varphi(4)$ visiting $\varphi(2)$ and $\varphi(3)$ on this trip has length at least $d\big(\varphi(1), \varphi(4)\big) + l(\sigma_1)$. Assume for the moment that $\varphi(1) = e$ and $\varphi(4) = \sigma_3$. Then before finally reaching σ_3 a shortest tour visiting all elements of A has to pass through σ_1 and σ_2 in this order (or first through σ_2 and then σ_1); it is not forbidden to visit e or σ_3 twice. This tour has a length of at least

$$d(e,\sigma_1)+d(\sigma_1,\sigma_2)+d(\sigma_2,\sigma_3)\geq \begin{cases} l(\sigma_1)+l(\sigma_2)+l(\sigma_3), & \text{in cases I and II,}\\ 2\,l(\sigma_1)+l(\sigma_2), & \text{in case III.}\end{cases}$$

But $d(e,\sigma_3)\leq l(\sigma_3)$ in case I and II, and $d(e,\sigma_3)\leq l(\sigma_2)$ in case III. Thus, the claim follows for the specific choice of φ with $\varphi(1)=e,\varphi(2)=\sigma_1,\varphi(3)=\sigma_2,\varphi(4)=\sigma_3$. For all other choices of φ the same result follows; compare with Figure 1: the first four columns build a case distinction for the choice of φ (we use symmetries!), the fifth column gives an upper bound for $d\big(\varphi(1),\varphi(4)\big)$; the sixth column gives a lower bound for the right hand side of the inequality in the lemma and the last column is a lower bound for the difference between the sixth and fifth column $\big($recall that $l(\sigma_1)\leq l(\sigma_2)\leq l(\sigma_3)\big)$. For case III, we only summarize the different possibilities, where $\varphi(i-1)=\sigma_2$, $\varphi(i)=\sigma_3$ or $\varphi(i-1)=\sigma_3$, $\varphi(i)=\sigma_2$, where we use Lemma 4.1(iv); the lower and upper bounds for all other choices for φ conincide with case I and II. Compare also with Figure 2, where the labels on the dotted lines are the lower bounds for the distances between two points. This proves the lemma. □

In other words, the lemma states that a shortest tour starting at some $a\in A$ visiting all other elements of A and finishing at some $a'\in A$ has length of at least $d(a,a')+r_1$. We will now apply this lemma independently of which of the cases I, II, III applies. For $y\in G$ let $B_y:=\{y,y\sigma_1,y\sigma_2,y\sigma_3\}$ and let be $x_y\in G\setminus B_y$. Obviously, for each choice of $w,z\in B_y$,

$$d(e,x_y)\leq d(e,w)+d(w,z)+d(z,x_y). \tag{4.2}$$

Let $\mathcal{F}$ be the set of all injective functions $\varphi:\{1,2,3,4\}\to B_y$. Then the last lemma and (4.2) yield the following inequality:

$$\begin{aligned} & d_{\mathrm{TS}}\big(\mathbb{1}_y\oplus\mathbb{1}_{y\sigma_1}\oplus\mathbb{1}_{y\sigma_2}\oplus\mathbb{1}_{y\sigma_3},x\big)\\ \geq\ & \min_{\varphi\in\mathcal{F}}\Big\{d\big(e,\varphi(1)\big)+\sum_{i=1}^{3}d\big(\varphi(i),\varphi(i+1)\big)+d\big(\varphi(4),x\big)\Big\}\\ \geq\ & d(e,x)+r_1. \end{aligned}\tag{4.3}$$

We now come back to our lamplighter random walk. Our next aim is to bound $d_{\mathrm{TS}}(Z_n)$ from below with the help of (4.3), independently of which of the cases I, II, III holds. For this purpose, define *hitting times*

$$\mathbf{t}_1:=0,\text{ and }\mathbf{t}_k:=\min\Big\{m\in\mathbb{N}\;\Big|\;m>\mathbf{t}_{k-1},X_m\notin\bigcup_{j=1}^{k-1}B\big(X_{\mathbf{t}_j},2l(\sigma_3)\big)\Big\}\text{ for }k\geq 2,$$

Cases I and II:

$\varphi(1)$	$\varphi(2)$	$\varphi(3)$	$\varphi(4)$	$d(\varphi(1), \varphi(4)) \leq$	Right Side $\geq$	Difference $\geq$
e	σ_1	σ_2	σ_3	$l(\sigma_3)$	$l(\sigma_1) + l(\sigma_2) + l(\sigma_3)$	$l(\sigma_1) + l(\sigma_2)$
e	σ_1	σ_3	σ_2	$l(\sigma_2)$	$l(\sigma_1) + 2\,l(\sigma_3)$	$l(\sigma_1) + l(\sigma_3)$
e	σ_2	σ_1	σ_3	$l(\sigma_3)$	$2\,l(\sigma_2) + l(\sigma_3)$	$2\,l(\sigma_2)$
e	σ_3	σ_1	σ_2	$l(\sigma_2)$	$l(\sigma_2) + 2\,l(\sigma_3)$	$2\,l(\sigma_3)$
e	σ_2	σ_3	σ_1	$l(\sigma_1)$	$l(\sigma_2) + 2\,l(\sigma_3)$	$2\,l(\sigma_3)$
e	σ_3	σ_2	σ_1	$l(\sigma_1)$	$l(\sigma_2) + 2\,l(\sigma_3)$	$2\,l(\sigma_3)$
σ_1	e	σ_2	σ_3	$l(\sigma_1) + l(\sigma_3)$	$l(\sigma_1) + l(\sigma_2) + l(\sigma_3)$	$l(\sigma_2)$
σ_1	e	σ_3	σ_2	$l(\sigma_1) + l(\sigma_2)$	$l(\sigma_1) + 2\,l(\sigma_3)$	$l(\sigma_3)$
σ_1	σ_2	e	σ_3	$l(\sigma_1) + l(\sigma_3)$	$2\,l(\sigma_2) + l(\sigma_3)$	$l(\sigma_2)$
σ_1	σ_3	e	σ_2	$l(\sigma_1) + l(\sigma_2)$	$l(\sigma_2) + 2\,l(\sigma_3)$	$l(\sigma_3)$
σ_2	e	σ_1	σ_3	$l(\sigma_2) + l(\sigma_3)$	$l(\sigma_1) + l(\sigma_2) + l(\sigma_3)$	$l(\sigma_1)$
σ_2	σ_1	e	σ_3	$l(\sigma_2) + l(\sigma_3)$	$l(\sigma_1) + l(\sigma_2) + l(\sigma_3)$	$l(\sigma_1)$

Case III:

$\varphi(1)$	$\varphi(2)$	$\varphi(3)$	$\varphi(4)$	$d(\varphi(1), \varphi(4)) \leq$	Right Side $\geq$	Difference $\geq$
e	σ_1	σ_2	σ_3	$l(\sigma_2)$	$2\,l(\sigma_1) + l(\sigma_2)$	$2\,l(\sigma_1)$
e	σ_1	σ_3	σ_2	$l(\sigma_2)$	$2\,l(\sigma_1) + l(\sigma_2)$	$2\,l(\sigma_1)$
e	σ_2	σ_3	σ_1	$l(\sigma_1)$	$l(\sigma_1) + 2\,l(\sigma_2)$	$2\,l(\sigma_2)$
e	σ_3	σ_2	σ_1	$l(\sigma_1)$	$l(\sigma_1) + 2\,l(\sigma_2)$	$2\,l(\sigma_2)$
σ_1	e	σ_2	σ_3	$l(\sigma_1) + l(\sigma_2)$	$2\,l(\sigma_1) + l(\sigma_2)$	$l(\sigma_1)$
σ_1	e	σ_3	σ_2	$l(\sigma_1) + l(\sigma_2)$	$2\,l(\sigma_1) + l(\sigma_2)$	$l(\sigma_1)$

FIGURE 1. Comparision for the choice of φ

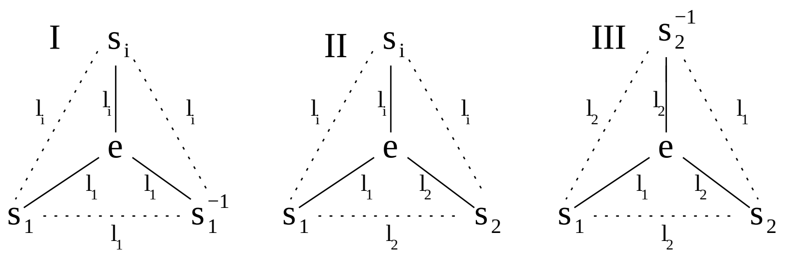

FIGURE 2. Distances between e, σ_1, σ_2, σ_3 with $l_1 = l(s_1)$, $l_2 = l(s_2)$ and $l_i = l(s_i)$.

that is, $\mathbf{t}_k$ is the first instant of time after time $\mathbf{t}_{k-1}$ for which the random walk leaves the finite set $\bigcup_{j=0}^{k-1} B\big(X_{\mathbf{t}_j}, 2l(\sigma_3)\big)$. By transience, $\mathbf{t}_k < \infty$ almost surely. Furthermore, we write $\mathbf{H}_k := X_{\mathbf{t}_k}$ and $\mathcal{R}'_n := \{X_{\mathbf{t}_j} \,|\, j \in \mathbb{N} \text{ with } \mathbf{t}_j \leq n\}$. Observe that

$$\{\mathbf{H}_k, \mathbf{H}_k\sigma_1, \mathbf{H}_k\sigma_2, \mathbf{H}_k\sigma_3\} \cap \{\mathbf{H}_l, \mathbf{H}_l\sigma_1, \mathbf{H}_l\sigma_2, \mathbf{H}_l\sigma_3\} = \varnothing \tag{4.4}$$

for $k \neq l$. The idea is to investigate, if enough lamps are on in $B\big(X_{\mathbf{t}_j}, l(\sigma_3)\big)$ such that we have $d_{\mathrm{TS}}(\eta, y) > d(e, y)$, where $y \in G$ and $\mathrm{supp}(\eta)$ is a subset of this ball. Our aim is to construct deviations to establish such a situation for each of these balls with a strict positive probability independently of k. See Figure 3.

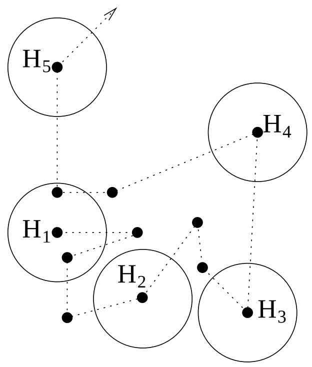

FIGURE 3. Hitting points

For $n \in \mathbb{N}_0$, $k \in \mathbb{N}$, define

$$\Delta_{n,k} := \begin{cases} r_1, & \text{if } \{\mathbf{H}_k, \mathbf{H}_k\sigma_1, \mathbf{H}_k\sigma_2, \mathbf{H}_k\sigma_3\} \subseteq \mathrm{supp}(\eta_{\mathbf{t}_k+n}), \\ 0, & \text{otherwise.} \end{cases}$$

If $n \geq \mathbf{t}_k$ and $\Delta_{n-\mathbf{t}_k,k} = 1$, a shortest tour from e to X_n visiting each element of $\mathrm{supp}(\eta_n)$ has to visit in particular each element of $\big\{\mathbf{H}_k, \mathbf{H}_k\sigma_1, \mathbf{H}_k\sigma_2, \mathbf{H}_k\sigma_3\big\}$. But by Lemma 4.2 and (4.3) this means that

$$d_{\mathrm{TS}}(Z_n) \geq d(e, X_n) + r_1.$$

Due to (4.4), iterated applications of the triangular inequality and Lemma 4.2 yield

$$d_{\mathrm{TS}}(Z_n) \geq d(e, X_n) + \sum_{j=1}^{|\mathcal{R}'_n|} \Delta_{n-\mathbf{t}_j,j}. \tag{4.5}$$

Our next aim is to bound $\mathbb{P}[\Delta_{n,k} = r_1]$ for n big enough, and thus $\mathbb{E}[\Delta_{n,k}]$, uniformly from below by a non-zero constant.

Proposition 4.3. *There exist $\lambda \in \mathbb{N}$ and $D > 0$ such that $\mathbb{E}[\Delta_{n,k}] \geq D$ for all $k, n \in \mathbb{N}$ with $n \geq \lambda$.*

Proof. By transience, there is $y \in G$ with $d(e, y) \geq R + l(\sigma_3)$ such that

$$\hat{p} := \mathbb{P}_{(\eta', y)}\big[\forall n \geq 1 : X_n \notin B\big(e, R + l(\sigma_3)\big)\big] > 0 \quad \text{for all } \eta' \in \mathcal{N}.$$

For each subset $A \subseteq \{e, \sigma_1, \sigma_2, \sigma_3\}$ there is $\lambda_A \in \mathbb{N}$ such that

$$D_A := \mu^{(\lambda_A)}\Big(\Big(\sum_{w\in A} \mathbb{1}_w, y\Big)\Big) > 0.$$

We sum over all possibilities for the hitting point $\mathbf{H}_k$. Assume now for a moment that x is the hitting point. Thus, the probability of walking from x with configuration η to xy such that the lamps at $x, x\sigma_1, x\sigma_2, x\sigma_3$ are on when the lamplighter reaches xy is at least $D' := \min\big\{D_A | A \subseteq \{e, \sigma_1, \sigma_2, \sigma_3\}\big\}$. The lamplighter will not return to $B\big(x, R + l(\sigma_3)\big)$ when starting at xy with a positive probability, namely with a probability of at least $\hat{p}$. We now obtain for $n \geq \lambda := \max\big\{\lambda_A | A \subseteq \{e, \sigma_1, \sigma_2, \sigma_3\}\big\}$:

$$\begin{aligned}
\mathbb{E}[\Delta_{n,k}] &= r_1 \cdot \mathbb{P}[\Delta_{n,k} = r_1] \\
&= r_1 \cdot \sum_{(\eta,x)\in\mathcal{L}} \sum_{m\geq 0} \mathbb{P}\big[\mathbf{t}_k = m, Z_m = (\eta, x), \{x, x\sigma_1, x\sigma_2, x\sigma_3\} \subseteq \operatorname{supp}(\eta_{m+n})\big] \\
&\geq r_1 \cdot \sum_{(\eta,x)\in\mathcal{L}} \sum_{m\geq 0} \mathbb{P}\big[\mathbf{t}_k = m, Z_m = (\eta, x)\big] \cdot D' \cdot \hat{p} \\
&= r_1 \cdot D' \cdot \hat{p} =: D.
\end{aligned}$$

□

Now we can summarize:

Theorem 4.4. *For the lamplighter random walk on the infinite, finitely generated group G, generated as a semigroup by the symmetric set S such that $G \neq \langle s_k, s_l\rangle$ for $s_k, s_l \in S$,*

$$\ell \geq \ell_{\mathrm{TS}} > \ell_0.$$

Proof. In view of inequality (4.5), Fatou's lemma and Proposition 4.3 give

$$\begin{aligned}
\ell_{\mathrm{TS}} &= \int \lim_{n\to\infty} \frac{d_{\mathrm{TS}}\big(Z_{n+\lambda}\big)}{n}\, d\mathbb{P} \\
&\geq \int \limsup_{n\in\mathbb{N}} \Bigg(\frac{d(e, X_{n+\lambda})}{n} + \frac{1}{n}\sum_{j=1}^{|\mathcal{R}'_n|} \Delta_{n+\lambda-\mathbf{t}_j, j}\Bigg)\, d\mathbb{P} \\
&\geq \ell_0 + \limsup_{n\in\mathbb{N}} \int \Bigg(\frac{1}{n}\sum_{j=1}^{|\mathcal{R}'_n|} \Delta_{n+\lambda-\mathbf{t}_j, j}\Bigg)\, d\mathbb{P} \\
&= \ell_0 + \limsup_{n\in\mathbb{N}} \sum_{k=1}^{n+1} \mathbb{P}\big[|\mathcal{R}'_n| = k\big] \cdot \mathbb{E}\Bigg[\frac{1}{n}\sum_{j=1}^{|\mathcal{R}'_n|} \Delta_{n+\lambda-\mathbf{t}_j, j}\ \Bigg|\ |\mathcal{R}'_n| = k\Bigg].
\end{aligned}$$

This provides

$$\ell_{\mathrm{TS}} \geq \ell_0 + \limsup_{n\in\mathbb{N}} \sum_{k=1}^{n+1} \mathbb{P}\big[|\mathcal{R}'_n| = k\big] \cdot \frac{k}{n} \cdot D = \ell_0 + D \cdot \limsup_{n\in\mathbb{N}} \frac{\mathbb{E}[|\mathcal{R}'_n|]}{n}.$$

By Kingman's subadditive ergodic theorem $\mathbb{E}[|\mathcal{R}'_n|]/n$ converges, and due to the inequality $|\mathcal{R}_n| \leq |B\big(e, 2l(\sigma_3)\big)| \cdot |\mathcal{R}'_n|$ its limit is bounded from below by some constant $\bar{D} > 0$, completing the proof. □

4.2. **$\mathbb{Z}$-isomorphic Groups.** In this section we consider the remaining case, where G is generated as a semigroup by two elements $s_k, s_l \in S$, that is, $G \simeq \mathbb{Z}$ or $G \simeq \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = e\rangle$. For the sake of completeness we prove the following lemma:

Lemma 4.5. *Any irreducible random walk $(X_n)_{n\in\mathbb{N}_0}$ on $G = \mathbb{Z}_2 * \mathbb{Z}_2$ governed by a probability measure μ_0 with finite first moment is recurrent. In particular, $\ell_{\mathrm{TS}} = 0$.*

Proof. Observe that

$$Z := \big\{(ab)^z \mid z \in \mathbb{Z}\big\} \simeq \mathbb{Z}$$

is a subgroup of G, which has index 2. We identify from now on the elements of Z with integers and write $\overline{Z}$ for the complement $G \setminus Z$. Consider the stopping times $T_0 := 0$, $T_n := \min\{m \in \mathbb{N} \mid X_m \in Z, m > T_{m-1}\}$ for $m \in \mathbb{N}$, and define $Y_n := X_{T_n}$. Then the random walk $(X_n)_{n\in\mathbb{N}_0}$ on G is recurrent if and only if $(Y_n)_{n\in\mathbb{N}_0}$ is recurrent. We now compute the expectation of the drift $D_n := Y_n - Y_{n-1}$, which is independent of n. We write $\mu_Z(z) = \mu_0(z)/\mu_0(Z)$ for every $z \in \mathbb{Z}$ and $d_Z = \sum_{w\in Z} w\, \mu_Z(w)$. Observe that for each $w, w' \in \overline{Z}$ we have $ww' \in Z$ and $ww' = -w'w$. Now we can compute the expected drift by distinguishing how one can walk from Y_{n-1} to Y_n:

$$\begin{aligned}
\mathbb{E}[D_n] &= \sum_{w\in Z} w\, \mu_0(w) + \sum_{n\geq 1} \sum_{w_0, w_n \in \overline{Z}} \sum_{w_1,\dots,w_{n-1}\in Z} (w_0 - w_1 - \cdots - w_{n-1} + w_n) \cdot \\
&\qquad \cdot \mu_0(w_0)\mu_0(w_1)\cdots\mu_0(w_{n-1})\mu_0(w_n) \\
&= \mu_0(Z) \cdot d_Z + \sum_{n\geq 1} \mu_0(Z)^{n-1} \Big(\sum_{w_0, w_n \in \overline{Z}} (w_0 + w_n)\mu_0(w_0)\mu_0(w_n) + \\
&\qquad + \big(1 - \mu_0(Z)\big)^2 \cdot \sum_{w_1,\dots,w_{n-1}\in Z} (-w_1 - \cdots - w_{n-1})\mu_Z(w_1)\dots\mu_Z(w_{n-1}) \Big) \\
&= \mu_0(Z) \cdot d_Z - \big(1 - \mu_0(Z)\big) \cdot d_Z \sum_{n\geq 1} (n-1)\, \mu_0(Z)^{n-1} \big(1 - \mu_0(Z)\big) \\
&= \mu_0(Z) \cdot d_Z - \big(1 - \mu_0(Z)\big) \cdot d_Z \cdot \mu_0(Z)/\big(1 - \mu_0(Z)\big) = 0.
\end{aligned}$$

By Chung-Ornstein [5] it follows that $(Y_n)_{n\in\mathbb{N}_0}$ is recurrent, and thus $(X_n)_{n\in\mathbb{N}_0}$ is recurrent. From Dyubina [7] follows $\ell_{\mathrm{TS}} = 0$. □

We want to show that Theorem 4.4 holds also in the case $G = \mathbb{Z}$ under suitable assumptions on the lengths of the elements of S. Note that we may assume again that $\ell_0 > 0$ and that $|S| \geq 3$. If $|S| = 2$, then G is isomorphic to $\mathbb{Z} = \langle S \rangle$ with $S = \{-1, 1\}$, whose Cayley graph is the infinite line, on which one can easily show that the lamplighter random walk has the same speed as its projection onto G; see also Bertacchi [3]. Thus, we only have to take a closer look on $\mathbb{Z}$ generated by a symmetric set S with $-1, 1 \in S$ and $|S| \geq 3$. Observe that if $\pm 1 \notin S$, then we may apply the results of the previous section. Furthermore, assume that there is $s \in S \setminus \{\pm 1\}$ with $l(s) < |s| \cdot l(1)$; otherwise we are more or less in the situation of $S = \{\pm 1\}$, see the end of this section. Moreover, we may assume that $d(0, 1) = d(0, -1) = l(1)$; otherwise $[x_0 = 0, x_1 = 1]$ is not a shortest path from 0 to 1, that is, $S \setminus \{-1, 1\}$ provides the same metric as the metric induced by S, that is, we may apply the results of Section 4.1 in such case. Due to the same argument we may assume that the only shortest path from 0 to 1 is $[x_0 = 0, x_1 = 1]$.

We proceed similarly to Section 4.1. We make a case distinction and define:

I. If there is $s \in S \setminus \{\pm 1\}$ such that $r_1 = l(s) < l(1)$, then define $\sigma_1 = s$, $\sigma_2 = s^{-1}$ and $\sigma_3 = 1$.

II. Otherwise we set $\sigma_1 = 1$, $\sigma_2 = -1$ and $\sigma_3 = s$, where $s \in \mathbb{N} \cap (S \setminus \{1\})$ such that $l(s) < |s| \cdot l(1)$ and $l(s) \leq l(s')$ for all $s' \in S$ with $l(s') < |s'| \cdot l(1)$.

In case I we have trivially $d(0, s) = d(0, s^{-1}) = l(s) = r_1$ and $d(s, s^{-1}) \geq l(s) = r_1$, while in case II we have $d(0, 1) = d(0, -1) = l(1) = r_1$ and $d(1, -1) \geq l(1)$. Moreover:

Lemma 4.6. *We have the following equations and lower bounds:*

(1) *In case I there is some $\varepsilon_0 > 0$ such that*
  - (i) $d(0, 1) = l(1)$,
  - (ii) $d(s, 1) \geq l(1) - l(s) + \varepsilon_0$,
  - (iii) $d(s^{-1}, 1) \geq l(1) - l(s) + \varepsilon_0$.

(2) *In case II there is some $\varepsilon_0 > 0$ such that*
  - (i) $d(0, s) = l(s)$,
  - (ii) $d(1, s) \geq l(s) - l(1) + \varepsilon_0$,
  - (iii) $d(-1, s) \geq l(s) - l(1) + \varepsilon_0$.

Proof. Equation (1).(i) holds by the assumption made above. For the proof of (1).(ii) assume that $d(s, 1) \leq l(1) - l(s)$. Then there is a shortest path $[s, x_1, \ldots, x_n = 1]$ with $d(x_{i-1}, x_i) \leq l(1) - l(s)$, that is, $x_{i-1}^{-1} x_i \neq \pm 1$. But this means that there is another path from 0 to 1 of length at most $l(1)$, namely $[0, s, x_1, \ldots, x_n]$, distinct from $[0, 1]$, a contradiction to the assumptions above. As S is finite, existence of ε_0 is ensured. Inequality (1).(iii) is proved analogously.

Assume that equation (2).(i) does not hold. This implies that there is a shortest path $[0, x_1, \dots, x_n = s]$ from 0 to s with $d(x_{i-1}, x_i) < l(s)$, that is, we may assume $x_{i-1}^{-1}x_i = \pm 1$ by minimality of $l(s)$. But this implies $l(s) > d(0, s) = |s| \cdot l(1)$, a contradiction to the choice of s. To prove (2).(ii) assume $d(1, s) \le l(s) - l(1)$, that is, there is a shortest path $[1, x_1, \dots, x_n = s]$ with $d(x_{i-1}, x_i) \le l(s) - l(1)$, that is, we may assume $x_{i-1}^{-1}x_i = \pm 1$ by minimality of $l(s)$. But this provides now $l(s) \ge l(1) + d(1, s) = |s| \cdot l(1)$, a contradiction to the choice of s. Inequality (2).(iii) is proved analogously. □

We get the analogue to Lemma 4.2:

Lemma 4.7. *Let be $A = \{0, \sigma_1, \sigma_2, \sigma_3\}$ and let $\varphi : \{1, 2, 3, 4\} \to A$ be an injective function. Then in each of the cases I and II,*

$$d\big(\varphi(1), \varphi(4)\big) + \min\{\varepsilon_0, l(s), l(1)\} \le d\big(\varphi(1), \varphi(2)\big) + d\big(\varphi(2), \varphi(3)\big) + d\big(\varphi(3), \varphi(4)\big).$$

Proof. The proof works analogously to the proof of Lemma 4.2; compare with Figure 4 for the comparision of the distances in case I. The inequality for case II follows analogously by symmetry. □

$\varphi(1)$	$\varphi(2)$	$\varphi(3)$	$\varphi(4)$	$d\big(\varphi(1), \varphi(4)\big) \le$	Right Side $\ge$	Difference $\ge$
0	s	s^{-1}	1	$l(1)$	$2\,l(s) + (l(1) - l(s) + \varepsilon_0)$	$l(s) + \varepsilon_0$
0	s^{-1}	s	1	$l(1)$	$2\,l(s) + (l(1) - l(s) + \varepsilon_0)$	$l(s) + \varepsilon_0$
0	1	s^{-1}	s	$l(s)$	$l(1) + (l(1) - l(s) + \varepsilon_0) + l(s)$	$l(1) + \varepsilon_0$
0	s^{-1}	1	s	$l(s)$	$l(s) + 2(l(1) - l(s) + \varepsilon_0)$	ε_0
0	1	s	s^{-1}	$l(s)$	$l(1) + (l(1) - l(s) + \varepsilon_0) + l(s)$	$l(1) + \varepsilon_0$
0	s	1	s^{-1}	$l(s)$	$l(s) + 2(l(1) - l(s) + \varepsilon_0)$	$2\varepsilon_0$
s	0	1	s^{-1}	$2\,l(s)$	$l(s) + l(1) + (l(1) - l(s) + \varepsilon_0)$	ε_0
s	1	0	s^{-1}	$2\,l(s)$	$(l(1) - l(s) + \varepsilon_0) + l(1) + l(s)$	ε_0
s	0	s^{-1}	1	$l(s) + l(1)$	$2\,l(s) + (l(1) - l(s) + \varepsilon_0)$	ε_0
s	s^{-1}	0	1	$l(s) + l(1)$	$2l(s) + l(1)$	$l(s)$
s^{-1}	0	s	1	$l(s) + l(1)$	$2\,l(s) + (l(1) - l(s) + \varepsilon_0)$	ε_0
s^{-1}	s	0	1	$l(s) + l(1)$	$2\,l(s) + l(1)$	$l(s)$

FIGURE 4. Comparision for the choices of φ in case I

Now we can conlude:

Corollary 4.8. *For the lamplighter random walk on $G = \mathbb{Z}$, generated as a semi-group by the symmetric set S such that $-1, 1 \in S$, $|S| \ge 3$ and $l(s) < |s| \cdot l(1)$ for some $s \in S \setminus \{-1, 1\}$,*

$$\ell \ge \ell_{\mathrm{TS}} > \ell_0.$$

Proof. Due to Lemma 4.7 the proof follows analogously to the considerations of Section 4.1, where we redefine $\Delta_{n,k}$ by

$$\Delta_{n,k} := \begin{cases} \min\{\varepsilon_0, l(s), l(1)\}, & \text{if } \{\mathbf{H}_k, \mathbf{H}_k\sigma_1, \mathbf{H}_k\sigma_2, \mathbf{H}_k\sigma_3\} \subseteq \operatorname{supp}(\eta_{\mathbf{t}_k+n}), \\ 0, & \text{otherwise.} \end{cases}$$

□

We now explain the necessity of having some $s \in S \setminus \{-1, 1\}$ with $l(s) < |s| \cdot l(1)$. If this assumption is not satisfied, then the metric on $G = \mathbb{Z}$ is $d(x, y) = r_1 \cdot |x-y|$, that is, we have the natural metric on $\mathbb{Z}$ if $r_1 = 1$. In this case the lamplighter random walk has the same speed as its projection onto the group G. E.g., consider $G = \mathbb{Z}$ generated by $S = \big\{\pm 1, \pm 2, \pm 3\big\}$ with $l(\pm 1) = 1$, $l(\pm 2) = 3$ and $l(\pm 3) = 5$. Observe that $[0, 1, 2, \ldots, z]$ for $z > 0$ is a shortest path from 0 to z. Let be $p \in (1/2; 1)$. We equip $\mathbb{Z}_2 \wr \mathbb{Z}$ with a transient random walk defined by the following transition probabilities:

$$\mu(\mathbf{0}, 1) = \mu(\mathbb{1}_0, 1) = \mu(\mathbf{0}, 2) = \mu(\mathbb{1}_0, 2) = \mu(\mathbf{0}, 3) = \mu(\mathbb{1}_0, 3) = p/6,$$
$$\mu(\mathbf{0}, -1) = \mu(\mathbb{1}_0, -1) = \mu(\mathbf{0}, -2) = \mu(\mathbb{1}_0, -2) = \mu(\mathbf{0}, -3) = \mu(\mathbb{1}_0, -3) = (1-p)/6.$$

Thus, $d(e, X_n)/n$ converges almost surely to $2p - 1$. Analogously to the case $G = \mathbb{Z} = \langle \pm 1 \rangle$, it can be shown that the lamplighter does not escape faster than its projection on $\mathbb{Z}$, that is, we have $\ell_{\mathrm{TS}} = \ell_0$.

5. Remarks

5.1. **Generalization to Transitive Graphs and Markovian Distance.** The results of Sections 3 and 4 can be generalized to transient lamplighter random walks on transitive, connected, locally finite graphs, which are not necessarily Cayley graphs of finitely generated groups. Again, it is assumed that the lamplighter random walk's projection onto the base graph is transient. The results of the previous sections also apply in this case, if graph automorphisms leave the lamplighter random walk operator invariant; compare with Gilch [8].

One can also investigate the rate of escape with respect to the *Markovian distance* $d_{\mathbb{P}}\big((\eta, x), (\eta', x')\big)$ on $\mathbb{Z}_2 \wr G$, which is given by

$$\min \left\{ \sum_{i=1}^{n} d(x_{i-1}, x_i) \;\middle|\; \begin{array}{c} n \in \mathbb{N}_0, \text{there are } x_0, x_1, \ldots x_n \in G \text{ with } x_0 = x, x_n = x' \\ \text{such that } \mathbb{P}_{(\eta,x)}\big[X_1 = x_1, \ldots, X_n = x', \eta_n = \eta'\big] > 0 \end{array} \right\}.$$

The limit $\ell_{\mathbb{P}} := \lim_{n\to\infty} d_{\mathbb{P}}\big((\mathbf{0}, e), Z_n\big)/n$ exists almost surely by Kingman's subadditive ergodic theorem and is almost surely constant. It can be shown that if the Cayley graph of G has infinitely many ends, then with respect to the Markovian distance the lamplighter escapes faster to infinity than its projection onto G. If the assumption is dropped one can find counterexamples such that the lamplighter is not faster; e.g. if $G = \mathbb{Z} \times \mathbb{Z}_2$.

5.2. **Multi-State Lamps.** The presented techniques for proving the acceleration of the lamplighter random walks can also be applied to the case that there are more possible lamp states encoded by elements of $\mathbb{Z}/r\mathbb{Z}$ with $r > 2$. In this case one may assign lengths to a set of generators of $\mathbb{Z}/r\mathbb{Z}$. Then the presented results can be proved analogously.

5.3. **Greenian Distance.** Another metric on G is given by the *Greenian distance*

$$d_{\mathrm{Green}}(x, y) := -\ln \mathbb{P}_x[T_y < \infty],$$

where T_y is the hitting time of $y \in G$. Analogously, we can define the Greenian metric for the random walk on $\mathbb{Z}_2 \wr G$. These metrics are not path metrics induced by lengths on the set of generators. Benjamini and Peres [2] proved that the entropy and the rate of escape w.r.t. the Greenian distance of random walks on finitely generated groups with finite support are equal. Blachère, Haïssinsky and Mathieu [4] generalized this result to random walks on countable groups. If the random walk on G is governed by a probability measure μ_0 with $\langle \operatorname{supp} \mu_0 \rangle = G$, then the entropy of the lamplighter random walk on $\mathbb{Z}_2 \wr G$ is strictly bigger than the entropy of the random walk's projection onto G, because the Poisson boundary of the lamplighter random walk projects non-trivially onto the one of the random walk on the base graph; compare with Kaimanovich and Vershik [12, Theorem 3.2]. It follows that with respect to the Greenian distance the lamplighter random walk is faster than its projection onto G.

Acknowledgements

The author is grateful to Wolfgang Woess and Sebastian Müller for numerous discussions, and also to the anonymous referee, who gave several useful remarks regarding content.

References

[1] A. Avez. Entropie des groupes de type fini. *C.R.Acad.Sci.Paris Sér. A-B*, 275:A1363–A1366, 1972.

[2] I. Benjamini and Y. Peres. Tree-indexed random walks on groups and first passage percolation. *Probab. Theory Related Fields*, 98(1):91–112, 1994.

[3] D. Bertacchi. Random walks on Diestel-Leader graphs. *Abh. Math. Sem. Univ. Hamburg*, 71:205–224, 2001.

[4] S. Blachère, P. Haïssinsky, and P. Mathieu. Asymptotic entropy and Green speed for random walks on countable groups. *Annals of Probability*, 36(3):1134–1152, 2008.

[5] K. Chung and D. Ornstein. On the recurrence of sums of random variables. *Bull. Amer. Math. Soc.*, 68(1):30–32, 1962.

[6] Y. Derriennic. Quelques applications du théorème ergodique sous-additif. *Astérisque*, 74:183–201, 1980.

[7] A. Dyubina. Characteristics of random walks on wreath products of groups. *J. of Math. Sciences*, 107(5):4166–4171, 2001.

[8] L. A. Gilch. *Rate of Escape of Random Walks.* PhD thesis, Graz University of Technology, Austria, 2007.

[9] L. A. Gilch. Rate of escape on the lamplighter tree. *J.of Math. Sci. (N.Y.)*, 2008.

[10] Y. Guivarc'h. Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire. *Astérisque*, 74:47–98, 1980.

[11] V. Kaimanovich. Poisson boundary of discrete groups. *Preprint.*

[12] V. Kaimanovich and A. Vershik. Random walks on discrete groups: boundary and entropy. *Ann. of Probab.*, 11:457–490, 1983.

[13] V. Kaimanovich and W. Woess. Boundary and entropy of space homogeneous Markov chains. *Ann. of Probab.*, 30:323–363, 2002.

[14] A. Karlsson and F. Ledrappier. Linear drift and Poisson boundary for random walks. *Pure Appl. Math. Q.*, 3:1027–1036, 2007.

[15] J. Kingman. The ergodic theory of subadditive processes. *J. Royal Stat. Soc., Ser. B*, 30:499–510, 1968.

[16] R. Lyons, R. Pemantle, and Y. Peres. Random walks on the lamplighter group. *Ann. of Probab.*, 24(4):1993–2006, 1996.

[17] D. Revelle. Rate of escape of random walks on wreath products and related groups. *Ann. of Probab.*, 31(4):1917–1934, 2003.

[18] N. T. Varopoulos. Long range estimates for Markov chains. *Bull. Sci. Math.*, 109:225–252, 1985.

Institut für mathematische Strukturtheorie (Math. C), Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria

E-mail address: `gilch@TUGraz.at`

URL: `http://www.math.tugraz.at/~gilch/`