PHASE TRANSITIONS FOR RANDOM WALK ASYMPTOTICS ON FREE PRODUCTS OF GROUPS

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Abstract. Suppose we are given finitely generated groups $\Gamma_1, \ldots, \Gamma_m$ equipped with irreducible random walks. Thereby we assume that the expansions of the corresponding Green functions at their radii of convergence contain only logarithmic or algebraic terms as singular terms up to sufficiently large order (except for some degenerate cases). We consider transient random walks on the free product $\Gamma_1 * \ldots * \Gamma_m$ and give a complete classification of the possible asymptotic behaviour of the corresponding n-step return probabilities. They either inherit a law of the form $\varrho^{n\delta} n^{-\lambda_i} \log^{\kappa_i} n$ from one of the free factors Γ_i or obey a $\varrho^{n\delta} n^{-3/2}$-law, where $\varrho < 1$ is the corresponding spectral radius and δ is the period of the random walk. In addition, we determine the full range of the asymptotic behaviour in the case of nearest neighbour random walks on free products of the form $\mathbb{Z}^{d_1} * \ldots * \mathbb{Z}^{d_m}$. Moreover, we characterize the possible phase transitions of the non-exponential types $n^{-\lambda_i} \log^{\kappa_i} n$ in the case $\Gamma_1 * \Gamma_2$.

1. Introduction

In this article we investigate transient random walks on free products $\Gamma_1 * \ldots * \Gamma_m$, where $m \geq 2$ and $\Gamma_1, \ldots, \Gamma_m$ are finitely generated groups. These random walks arise from convex combinations of probability measures on the factors $\Gamma_1, \ldots, \Gamma_m$. Our aim is to compute the asymptotic behaviour of the n-step return probabilities on the free product. In a general setting, one has a typical asymptotic behaviour of the form $\mu^{(n)}(x) \sim C_x\, \varrho^{n\delta} n^{-\lambda}$, where $\mu^{(n)}(x)$ is the probability of being at x at time n, ϱ is the spectral radius and δ the period of the random walk, and C_x some constant depending on x. If e is the group identity and starting point of the random walk, then $\mu^{(n)}(e)$ is called the n-step return probability. Gerl [8] conjectured that the n-step return probabilities of two symmetric measures on a group satisfying such a limit law have the same $n^{-\lambda}$, that is, λ is a group invariant. Cartwright [2] came to the astonishing result that for random walks on $\mathbb{Z}^d * \mathbb{Z}^d$ with $d \geq 5$ there are at least two possible types of asymptotic behaviour, namely $n^{-3/2}$ and $n^{-d/2}$. In relation with his joint work with Chatterji and Pittet [5], L. Saloff-Coste asked whether the range of different asymptotic behaviours can still be wider than in the case considered by Cartwright. In this article we will pick up this question by investigating more general laws of the form $C\, \varrho^{n\delta}\, n^{-\lambda} \log^{\kappa} n$. In this case, we speak of the factor $n^{-\lambda} \log^{\kappa} n$ as the non-exponential type of the return probabilities.

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The starting point for the present investigation was Woess [22, Section 17.B], where the result of Cartwright [2] is explained that simple random walk on $\mathbb{Z}^d * \mathbb{Z}^d$ for $d \geq 5$ satisfies a $n^{-d/2}$-law. In this article we will prove the following more general theorem:

Theorem 1.1. *Let $m \in \mathbb{N}$ with $m \geq 2$ and $d_1, \dots, d_m \in \mathbb{N}$. Consider any irreducible nearest neighbour random walk on the free product $\mathbb{Z}^{d_1} * \dots * \mathbb{Z}^{d_m}$, where e denotes the identity, i.e. the empty word. Then the return probabilities $\mu^{(2n)}(e)$ behave asymptotically either like $C_i \cdot \varrho^{2n} \cdot n^{-d_i/2}$ for $i \in \{1, \dots, m\}$ or like $C_0 \cdot \varrho^{2n} \cdot n^{-3/2}$. In particular, if all exponents d_i are different and $\min\{d_1, \dots, d_m\} \geq 5$ then exactly $m+1$ different asymptotic behaviours may occur by choosing the random walk adequately.*

We will consider more general free products which go beyond free products of lattices. For this purpose, we will present a new approach in order to be able to deal with irreducible random walks on any free product of the form $\Gamma_1 * \ldots * \Gamma_m$. At this point we assume that the Green functions of the random walks on the single factor Γ_i have singular expansions (i.e. in a neighbourhood of their radii of convergence $\mathbf{r}_i$) containing only singular terms of the form $(\mathbf{r}_i - z)^q \log^k(\mathbf{r}_i - z)$ with $q \in (1, \infty)$ and $k \in \mathbb{N}_0$ up to sufficiently large order, whenever the Green functions on the factors are differentiable. The latter property is satisfied for several well-known groups as e.g. $\mathbb{Z}^d$ or $\mathbb{Z}^d \times (\mathbb{Z}/n\mathbb{Z})$ with $d \geq 5$ and $n \geq 2$. If the asymptotic n-step return probabilities of the random walk on Γ_i satisfy a $\mathbf{r}_i^{-n\delta} n^{-\lambda_i} \log^{\kappa_i} n$-law then we will show that only one of the following non-exponential types may occur for the random walk on the free product: $n^{-\lambda_i} \log^{\kappa_i} n$ for some $i \in \{1, \dots, m\}$, or $n^{-3/2}$. That is, we may have up to $m+1$ different types of asymptotic behaviour for (symmetric or non-symmetric) random walks, and Theorem 1.1 shows that one can have indeed exactly $m+1$ different behaviours. Moreover, for the case $\Gamma_1 * \Gamma_2$ equipped with the probability measure $\mu = \alpha_1\mu_1 + (1-\alpha_1)\mu_2$, where μ_1 and μ_2 are probability measures on Γ_1 and Γ_2 and $\alpha_1 \in (0,1)$, we characterize the phase transitions of the non-exponential types in terms of α_1. We split the $(0,1)$-interval, i.e. the interval of possible values for α_1, into three distinct subintervals such that, in each of them, we have exactly one of the non-exponential types $n^{-\lambda_1} \log^{\kappa_1}(n)$, $n^{-\lambda_2} \log^{\kappa_2} n$ or $n^{-3/2}$.

Let us briefly recall some results regarding the asymptotic behaviour of return probabilities. Work in this direction has been done since the 1970's by Gerl, Sawyer, Woess, Cartwright, Soardi and Lalley, see e.g. [9], [18], [21], [3], [13]. Sawyer [18] applies Fourier analysis to isotropic random walks on trees (free groups), which uses in a crucial way methods from complex analysis. For finite range random walks on free groups, it is known from [9] and [13] that the n-step return probabilities behave asymptotically like $C\varrho^n n^{-3/2}$, where $\varrho < 1$. In [8], [20] and [21] free products of finite groups are considered, which have a very tree-like structure and where random walks obey a $n^{-3/2}$-law. In the more general case of free products of arbitrary groups the interior structure of each free factor is more complicated. Woess [21], Cartwright and Soardi [3], Voiculescu [19] and McLaughlin [16] found independently a method to determine the Green function of a free product in terms of functional equations of the Green functions defined on the free factors. We will study these equations carefully, in order to obtain – with the help of the well-known *method of Darboux* – the asymptotic behaviour of the power series' coefficients, which are the sought return probabilities. We refer also to the survey of Woess [23], which outlines the use of generating functions. More recently, random walks on free products have also been studied by Mairesse and Mathéus [15] and Gilch [10], [11], regarding boundary theory, entropy and rate of escape. For more details and references we refer to Woess [22], which serves also as reference text for our work.

The structure of this paper is as follows: in Section 2 we introduce some basic facts and notations. In Section 3 we prove our main result for the case $\Gamma_1 * \Gamma_2$, while in Sections 4 and 5 we are completing the list of degenerate cases, which, in particular, may occur if the Green functions of the random walks on the single factors are non-differentiable at their radii of convergence. In Section 5.3 we are proving inductively the proposed asymptotic behaviour for multi-factor free products of the form $\Gamma_1 * \ldots * \Gamma_m$ with $m \geq 3$. Section 6 discusses some examples, including the case of free products of the form $\mathbb{Z}^{d_1} * \ldots * \mathbb{Z}^{d_m}$ where we give a full classification of the asymptotic behaviour of the return probabilities, which proves Theorem 1.1. For $\Gamma_1 * \Gamma_2$, we give in Section 7 a full characterization of the possible phase transition behaviour of the non-exponential types of the return probabilities in terms of the weight α_1 of the probability measure given on Γ_1. Finally, Section 8 gives some concluding remarks about higher asymptotic order terms.

2. Random Walks on Free Products

Let $m \in \mathbb{N}$ with $m \geq 2$. Suppose we are given finitely generated groups $\Gamma_1, \ldots, \Gamma_m$, and we denote by e_i the identity of Γ_i. We consider the *free product* $\Gamma := \Gamma_1 * \ldots * \Gamma_m$, which consists of all finite words of the form

$$x_1 x_2 \ldots x_n, \tag{2.1}$$

where $x_1, \ldots, x_n \in \bigcup_{i=1}^m \Gamma_i \setminus \{e_i\}$ and two consecutive letters are not from the same free factor Γ_i. In the case $\Gamma_i = \Gamma_j$ we may think that the elements of Γ_i and Γ_j have different colours to distinguish their origin. Observe that each factor Γ_i can be naturally embedded into Γ, and therefore $e_i \in \Gamma_i$ can be identified with the empty word $e \in \Gamma$. The free product is a group with e as identity: the product of two elements is given by concatenation followed by iterated contractions and cancellations of redundant terms in the middle, in order to obtain the requested form (2.1). For example, if $a, b \in \Gamma_1 \setminus \{e_1\}$ and $c \in \Gamma_2 \setminus \{e_2\}$, such that $c^2 \neq e$, then $(aca^{-1})(aca) = ac^2a$. For further details about free products see e.g. Lyndon and Schupp [14].

We recall and introduce some notation: for any function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ with $f(z_0) = 0$ for $z_0 \in D$, $0 < q \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we use the notation $f(z) = \mathbf{o}\big((z_0 - z)^q \log^k(z_0 - z)\big)$, $f(z) = \mathcal{O}_c\big((z_0 - z)^q \log^k(z_0 - z)\big)$ or $f(z) = \mathcal{O}\big((z_0 - z)^q \log^k(z_0 - z)\big)$, if for $z \to z_0$ the function $f(z)$ divided by $(z_0 - z)^q \log^k(z_0 - z)$ tends to zero, has a non-zero finite limit or is bounded nearby z_0 (that is, the quotient has a finite limes superior) respectively. Furthermore, we write $(z_0 - z)^{q_1} \log^{k_1}(z_0 - z) \preceq (z_0 - z)^{q_2} \log^{k_2}(z_0 - z)$ if and only if $(z_0 - z)^{q_2} \log^{k_2}(z_0 - z) = \mathcal{O}\big((z_0 - z)^{q_1} \log^{k_1}(z_0 - z)\big)$. The value z_0 will be obvious from the context.

Suppose we are given probability measures μ_i on Γ_i with $\langle \mathrm{supp}(\mu_i) \rangle = \Gamma_i$ for each $i \in \{1, \ldots, m\}$. These measures μ_i govern random walks on Γ_i, that is, the single step transition probabilities are given by $p_i(x_i, y_i) = \mu_i(x_i^{-1} y_i)$ for all $x_i, y_i \in \Gamma_i$. We lift now μ_i to a probability measure $\bar{\mu}_i$ on Γ by defining $\bar{\mu}_i(x) := \mu_i(x)$ if $x \in \Gamma_i$; otherwise we set $\bar{\mu}_i(x) := 0$. Let $\alpha_1, \dots, \alpha_m > 0$ such that $\sum_{i=1}^m \alpha_i = 1$. Consider now the probability measure $\mu := \sum_{i=1}^m \alpha_i \bar{\mu}_i$ on the free product Γ, which arises as a convex combination of the $\bar{\mu}_i$'s. Then the single step transition probabilities on Γ given by $p(x,y) := \mu(x^{-1}y)$ for $x, y \in \Gamma$ define a random walk on Γ, which is an irreducible Markov chain. We denote by $\mu_1^{(n)}, \dots, \mu_m^{(n)}$ and $\mu^{(n)}$ the n-fold convolution power of $\mu_1, \dots, \mu_m$ and μ, that is, the distribution after n steps with start at the identity. For $z \in \mathbb{C}$, the associated *Green functions* of the random walks on Γ_i and Γ are given by

$$G_i(z) := \sum_{n=0}^{\infty} \mu_i^{(n)}(e_i) z^n \quad \text{and} \quad G(z) := \sum_{n=0}^{\infty} \mu^{(n)}(e) z^n.$$

The corresponding radii of convergence are denoted by $\mathbf{r}_i$ and $\mathbf{r}$ respectively, which are singularities according to Pringsheim's Theorem. Note that $\mathbf{r} > 1$, since Γ is non-amenable unless $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (see e.g. [22, Theorem 10.10, Corollary 12.5]; in the latter case the random walk on Γ is recurrent). In the following we assume that $G_i(z)$ is exactly d_i-times differentiable at $z = \mathbf{r}_i$, where $d_i \in \mathbb{N}_0$. At this point we make the *basic assumption* that – whenever $G_i'(\mathbf{r}_i) < \infty$ – the expansions of the Green functions $G_i(z)$ in a neighbourhood of $z = \mathbf{r}_i$ have the form

$$G_i(z) = \sum_{k=0}^{d_i} g_k^{(i)} (\mathbf{r}_i - z)^k + \sum_{(q,k) \in \mathcal{T}_i} g_{(q,k)}^{(i)} (\mathbf{r}_i - z)^q \log^k(\mathbf{r}_i - z) + \mathcal{O}\big((\mathbf{r}_i - z)^{d_i+2}\big), \tag{2.2}$$

where $\mathcal{T}_i$ is a finite subset of $\{(q,k) \in \mathbb{R} \times \mathbb{N}_0 \mid d_i < q \leq d_i + 2\}$. In other words, the expansions contain only logarithmic and algebraic terms as singular terms up to order $(\mathbf{r}_i - z)^{d_i+2}$. As we will see, higher order terms are not necessary for the computation of the non-exponential type of the n-step return probabilities of the random walk on Γ. In the following we want to motivate this assumption on $G_i(z)$. This property for the expansion is satisfied in several well-known cases: for example, the Green functions of nearest neighbour random walks on lattices $\mathbb{Z}^d$ have such an expansion; see Proposition 6.1. With some effort, such an expansion can be deduced for $\mathbb{Z}^d \times (\mathbb{Z}/n\mathbb{Z})$ via the same methods used for $\mathbb{Z}^d$. In particular, we will prove our main result by induction on the number m of free factors of Γ: we will see that the assumptions stated in (2.2) are stable under free products (except for some degenerate cases), that is, $G(z)$ has again a similar expansion whenever $G'(\mathbf{r}) < \infty$. If the Green function $G_i(z)$ has the form (2.2) then the well-known *method of Darboux* yields that the n-step return probabilities of the random walk on Γ_i (governed by μ_i) behave asymptotically like the coefficients of the Taylor expansion of the leading singular term in (2.2) in a neighbourhood of 0. Assume that $S_i(z) := (\mathbf{r}_i - z)^{q_i} \log^{k_i}(\mathbf{r}_i - z)$ is the *smallest* (or *leading*) singular term in (2.2) w.r.t. $\preceq$, that is, $q > q_i$ or $(q = q_i \wedge k < k_i)$ for all $(q,k) \in \mathcal{T}_i \setminus \{(q_i, k_i)\}$; then the coefficients of its expansion in a neighbourhood of 0 behave asymptotically like the n-step return probabilities on Γ_i (the proof of this fact is completely analogous to the one of Theorem 3.1). More precisely, they behave like $C_i \mathbf{r}_i^{-n\delta_i} n^{-\lambda_i} \log^{\kappa_i}(n)$, where $\delta_i := \gcd\big\{n \in \mathbb{N} \mid \mu_i^{(n)}(e_i) > 0\big\}$ is the period of the random walk on Γ_i and

$$\lambda_i := q_i + 1 \quad \text{and} \quad \kappa_i := \begin{cases} k_i, & \text{if } q_i \notin \mathbb{N}, \\ k_i - 1 & \text{if } q_i \in \mathbb{N}; \end{cases} \tag{2.3}$$

see e.g. Flajolet and Sedgewick [7, Chapter VI.2] for the asymptotic behaviour of the coefficients in the expansion of $(\mathbf{r}_i - z)^{q_i} \log^{k_i}(\mathbf{r}_i - z)$ in a neighbourhood of 0. Analogously, $\delta := \gcd\{n \in \mathbb{N} \mid \mu^{(n)}(e) > 0\} = \gcd\{\delta_1, \ldots, \delta_m\}$ is the period of the random walk on Γ. Note that the method of Darboux needs some differentiability assumptions at $z = \mathbf{r}_i$; therefore, we need the expansions of $G_i(z)$ up to terms of order $(\mathbf{r}_i - z)^{d_i+2}$. For more details about Darboux's method we refer to the comments in the proof of Theorem 3.1. We remark that another – modern – tool to handle singular expansions as in (2.2) is *Singularity Analysis*, which was developed by Flajolet and Odlyzko [6]. However, in our context it turns out that the verification of the specific requirements of singularity analysis is quite cumbersome as one can also see in Lalley [12]. Let us also point out that, in the case $G_i'(\mathbf{r}_i) = \infty$, we do *not* need any assumptions on the singularity type at $z = \mathbf{r}_i$.

In the following we look at free products of the form $\Gamma_1 * \Gamma_2$ different from $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (it is well-known that random walks (in our context) on $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ obey a $n^{-1/2}$-law). Free products with more than two factors are discussed in Section 5.3. We introduce the following *first visit generating functions* for $z \in \mathbb{C}$, $i \in \{1, 2\}$ and all $s_i \in \operatorname{supp}(\mu_i)$, $s \in \operatorname{supp}(\mu) = \operatorname{supp}(\mu_1) \cup \operatorname{supp}(\mu_2)$:

$$\begin{aligned} F_i\big(s_i \big| z\big) &:= \sum_{n \geq 0} \mathbb{P}\big[X_n^{(i)} = e_i, \forall m < n : X_m^{(i)} \neq e_i \,\big|\, X_0^{(i)} = s_i\big]\, z^n, \\ F\big(s \big| z\big) &:= \sum_{n \geq 0} \mathbb{P}\big[X_n = e, \forall m < n : X_m \neq e \,\big|\, X_0 = s\big]\, z^n, \end{aligned}$$

where $\big(X_n^{(i)}\big)_{n \in \mathbb{N}_0}$ is a random walk on Γ_i governed by μ_i. By conditioning on the number of visits of e_i the functions $F_i\big(s_i \big| z\big)$ are directly linked with $G_i(z)$ via

$$G_i(z) = \frac{1}{1 - \sum_{s_i \in \operatorname{supp}(\mu_i)} \mu_i(s_i)\, z\, F_i\big(s_i \big| z\big)}. \tag{2.4}$$

In the following we will summarize some further important basic facts, where we will refer to Woess [22] for further details. Define

$$\begin{aligned} \zeta_1(z) &:= \frac{\alpha_1 z}{1 - \alpha_2 z \sum_{s_2 \in \operatorname{supp}(\mu_2)} \mu_2(s_2) F(s_2|z)} \quad \text{and} \\ \zeta_2(z) &:= \frac{\alpha_2 z}{1 - \alpha_1 z \sum_{s_1 \in \operatorname{supp}(\mu_1)} \mu_1(s_1) F(s_1|z)}. \end{aligned} \tag{2.5}$$

Note that $\zeta_i(1)$ is the probability of starting at e and making a step from e w.r.t. μ_i after finite time. Observe that $F\big(s_i \big| z\big) = F_i\big(s_i \big| \zeta_i(z)\big)$ for $s_i \in \operatorname{supp}(\mu_i)$; see [22, Proposition 9.18c)]. By [22, Equation (9.20)] and (2.4), the functions $F_i\big(s_i \big| \zeta_i(z)\big)$, $G_i(z)$ and $G(z)$ can be linked as follows:

$$G(z) = \frac{\zeta_i(z)}{\alpha_i\, z} G_i\big(\zeta_i(z)\big) = \frac{\zeta_i(z)}{\alpha_i\, z \Big(1 - \sum_{s_i \in \operatorname{supp}(\mu_i)} \mu_i(s_i)\, \zeta_i(z)\, F_i\big(s_i \big| \zeta_i(z)\big)\Big)}. \tag{2.6}$$

Hence, our aim will be to determine an expansion of $\zeta_i(z)$ in a neighbourhood of $z=\mathbf{r}$, in order to get a singular expansion for $G(z)$ in a neighbourhood of $z=\mathbf{r}$. By [22, Proposition 9.10], there are functions Φ_i, $i\in\{1,2\}$, and Φ such that

$$G_i(z)=\Phi_i\bigl(zG_i(z)\bigr) \quad\text{and}\quad G(z)=\Phi\bigl(zG(z)\bigr) \tag{2.7}$$

for all $z\in\mathbb{C}$ in an open neighbourhood of the intervals $[0,\mathbf{r}_i)$ and $[0,\mathbf{r})$ respectively. In particular, the functions Φ_i and Φ are analytic in an open neighbourhood of the intervals $[0,\theta_i)$ and $[0,\theta)$ respectively, where $\theta_i:=\mathbf{r}_iG_i(\mathbf{r}_i)$ and $\theta:=\mathbf{r}G(\mathbf{r})$. Φ_i and Φ are also strictly increasing and strictly convex in $[0,\theta_i)$ and $[0,\theta)$ respectively. Furthermore, we define

$$\Psi_i(t):=\Phi_i(t)-t\Phi_i'(t) \qquad\text{and}\qquad \Psi(t):=\Phi(t)-t\Phi'(t). \tag{2.8}$$

By [22, Theorem 9.19],

$$\Phi(t)=\Phi_1(\alpha_1 t)+\Phi_2(\alpha_2 t)-1 \qquad\text{and}\qquad \Psi(t)=\Psi_1(\alpha_1 t)+\Psi_2(\alpha_2 t)-1. \tag{2.9}$$

We write $\Psi_i(\theta_i):=\lim_{t\to\theta_i-}\Psi_i(t)$. Define

$$\bar{\theta}:=\min\left\{\frac{\theta_1}{\alpha_1},\frac{\theta_2}{\alpha_2}\right\}.$$

We will make a case distinction according to finiteness of $G_i(\mathbf{r}_i)$ and $G_i'(\mathbf{r}_i)$ and also to the sign of $\Psi(\bar{\theta}):=\lim_{t\to\bar{\theta}-}\Psi(t)$. If $\Psi(\bar{\theta})<0$ then the n-step return probabilities of the random walk on Γ behave asymptotically like

$$\mu^{(n\delta)}(e)\sim C\cdot\mathbf{r}^{-n\delta}\cdot n^{-3/2}$$

and the Green function of the random walk on Γ has the form

$$G(z)=A(z)+\sqrt{\mathbf{r}-z}\,B(z), \tag{2.10}$$

where $A(z),B(z)$ are analytic functions in a neighbourhood of $z=\mathbf{r}$ with $B(\mathbf{r})\neq 0$; see [22, Theorem 17.3] or [7, Section VI.7.]. Moreover, if the μ_i's ($i=1,2$) are supported on any finite, symmetric set of generators, where $\mathrm{supp}(\mu_i)$ contains at least one element of order bigger than 2, then α_1 can always be chosen in a suitable way in order to obtain $\Psi(\bar{\theta})<0$; see [22, Corollary 17.10]. In particular, the same asymptotic behaviour (including an expansion of the Green function of the form 2.10) holds if Γ_1 and Γ_2 are finite, see [21]. Therefore, we assume from now on that at least one out of Γ_1 and Γ_2 is infinite, and we may restrict our investigation to the cases $\Psi(\bar{\theta})>0$ and $\Psi(\bar{\theta})=0$.

We remark some important facts for the case $\Psi(\bar{\theta})\geq 0$. If the latter holds, we have $\theta=\bar{\theta}$ and $G(\mathbf{r})<\infty$, see [22, Theorem 9.22]. By [22, Lemma 17.1.a)], we have $\zeta_i(\mathbf{r})\leq\mathbf{r}_i$ for $i\in\{1,2\}$ with equality if and only if $\theta=\theta_i/\alpha_i$.

The proof for the asymptotic behaviours of the return probabilities is split up over the following sections. In Section 3 we calculate the asymptotics in the case when $\Psi(\bar{\theta})>0$, $G_1'(\mathbf{r}_1)<\infty$ and $G_2'(\mathbf{r}_2)<\infty$ hold; see Theorem 3.1. In Section 4 we investigate the case when $\Psi(\bar{\theta})=0$, $G_1'(\mathbf{r}_1)<\infty$ and $G_2'(\mathbf{r}_2)<\infty$ hold; see Theorem 4.1. From the proof of this theorem we will see that even the case $\Psi(\bar{\theta})=0$, $G_1'\bigl(\zeta_1(\mathbf{r}_1)\bigr)<\infty$ and $G_2'\bigl(\zeta_2(\mathbf{r}_2)\bigr)<\infty$ is covered. In Section 5 we treat the remaining cases: Theorem 5.1 covers the case when $G_1(\mathbf{r}_1)<\infty$, $G_1'(\mathbf{r}_1)=\infty$ and $G_2'(\mathbf{r}_2)<\infty$ hold, while Corollary 5.2 answers the question for the asymptotic behaviour when $G_1'(\mathbf{r}_1)=\infty$ and $G_2'(\mathbf{r}_2)=\infty$. Finally, Theorem 5.3 covers the remaining case when $G_1(\mathbf{r}_1)=\infty$ or $G_2(\mathbf{r}_2)=\infty$.

3. The Asymptotic Behaviour in the Case $\Psi(\bar{\theta})>0$

Throughout this section we investigate the case $m=2$ and assume that $\Psi(\bar{\theta})>0$ and $G_1(z)$ and $G_2(z)$ are differentiable at their radii of convergence. That is, the Green functions have an expansion as assumed in (2.2). Recall that the smallest singular term w.r.t. $\preceq$ in the expansion of $G_i(z)$, is denoted by $S_i(z)=(\mathbf{r}_i-z)^{q_i}\log^{k_i}(\mathbf{r}_i-z)$ with $d_i<q_i\leq d_i+1$. Let us remark that Darboux's method yields that the n-step return probabilities of the random walk on Γ_i governed by μ_i behave asymptotically like $C_i\mathbf{r}_i^{-n\delta_i}n^{-\lambda_i}\log^{\kappa_i}n$, where λ_i and κ_i are given by (2.3). We also may assume throughout this section w.l.o.g. that $\theta=\bar{\theta}=\theta_1/\alpha_1$. The aim of this section is to prove the following:

Theorem 3.1. *Assume that $G_1(z)$ and $G_2(z)$ are differentiable at $z=\mathbf{r}_1$, $z=\mathbf{r}_2$ respectively, and have an expansion as in (2.2). If $S_1(z)\preceq S_2(z)$ and $\Psi(\bar{\theta})>0$ then:*

$$\mu^{(n\delta)}(e)\sim\begin{cases}C_1\cdot\mathbf{r}^{-n\delta}\cdot n^{-\lambda_1}\cdot\log^{\kappa_1}(n), & \text{if } \alpha_1\geq\frac{\theta_1}{\theta_1+\theta_2},\\ C_2\cdot\mathbf{r}^{-n\delta}\cdot n^{-\lambda_2}\cdot\log^{\kappa_2}(n), & \text{if } \alpha_1<\frac{\theta_1}{\theta_1+\theta_2}.\end{cases}$$

Recall that $F(s_i|z)=F_i(s_i|\zeta_i(z))$ for all $s_i\in\mathrm{supp}(\mu_i)$. Then we rewrite (2.5) as follows:

$$\alpha_1 z = \zeta_1(z)\Big(1-\alpha_2 z\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2\big(s_2|\zeta_2(z)\big)\Big), \tag{3.1}$$

$$\alpha_2 z = \zeta_2(z)\Big(1-\alpha_1 z\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1\big(s_1|\zeta_1(z)\big)\Big). \tag{3.2}$$

Recall that $\zeta_1(\mathbf{r})=\mathbf{r}_1$ and $\zeta_2(\mathbf{r})\leq\mathbf{r}_2$ with equality if and only if $\theta=\theta_1/\alpha_1=\theta_2/\alpha_2$. We remark also that $\Psi(\bar{\theta})>0$ implies $G'(\mathbf{r})<\infty$: since $\Phi'(\bar{\theta})<\Phi(\bar{\theta})/\bar{\theta}=1/\mathbf{r}$ we get by differentiating (2.7)

$$G'(\mathbf{r})=\lim_{z\to\mathbf{r}}\frac{\Phi'\big(zG(z)\big)\,G(z)}{1-z\,\Phi'\big(zG(z)\big)}=\frac{\Phi'(\bar{\theta})\,G(\mathbf{r})}{1-\mathbf{r}\,\Phi'(\bar{\theta})}<\infty.$$

Furthermore, we define

$$D:=\begin{cases}d_1, & \text{if } \bar{\theta}<\theta_2/\alpha_2,\\ \min\{d_1,d_2\}, & \text{if } \bar{\theta}=\theta_1/\alpha_1=\theta_2/\alpha_2.\end{cases}$$

We denote by $S(z)$ the *main leading singular term*, which is given by

$$S(z)=\begin{cases}S_1(z), & \text{if } \bar{\theta}<\theta_2/\alpha_2,\\ \min\big\{S_1(z),S_2(z)\big\}, & \text{if } \bar{\theta}=\theta_2/\alpha_2.\end{cases}$$

Lemma 3.2. $0<\zeta_1'(\mathbf{r})<\infty$ *and* $0<\zeta_2'(\mathbf{r})<\infty$.

Proof. We prove the lemma only for $\zeta_1'(\mathbf{r})$. We write

$$H_2(z) := \alpha_2 z \sum_{s_2 \in \mathrm{supp}(\mu_2)} \mu_2(s_2) F_2\bigl(s_2 | \zeta_2(z)\bigr).$$

Since $\zeta_1(\mathbf{r}) = \mathbf{r}_1$, we have $H_2(\mathbf{r}) < 1$; compare with the definition of $\zeta_1(z)$. Furthermore, the coefficient of z^n in $H_2(z)$ is just the probability for the random walk on Γ of starting at e, making the first step w.r.t. μ_2 and returning for the first time to e at time n. Thus, this probability is bounded from above by $\mu^{(n)}(e)$, and consequently $H_2'(\mathbf{r}) < G'(\mathbf{r}) < \infty$. Computing the derivative of $\zeta_1(z)$ in a neighbourhood of $z = \mathbf{r}$ gives

$$\zeta_1'(z) = \frac{\alpha_1\bigl(1 - H_2(z)\bigr) + \alpha_1 z H_2'(z)}{\bigl(1 - H_2(z)\bigr)^2} > 0.$$

Finiteness of $\zeta_1'(\mathbf{r})$ follows now directly from the remarks above. $\square$

The functions $F_i(s_i|z)$, where $i \in \{1,2\}$ and $s_i \in \mathrm{supp}(\mu_i)$, are at least d_i-times differentiable at $z = \mathbf{r}_i$, since the same holds for $G_i(z)$ and we can compare the coefficients of z^n in the definitions of $F_i(s_i|z)$ and $G_i(z)$ as follows:

$$\mu_i^{(n)}(e_i) \geq \mu_i(s_i) \cdot \mathbb{P}\bigl[X_n^{(i)} = e_i, \forall m < n : X_m^{(i)} \neq e_i \bigm| X_0^{(i)} = s_i\bigr].$$

Thus, we can rewrite these functions in the form

$$F_i(s_i|z) = \sum_{n=0}^{d_i} f_n(s_i)(\mathbf{r}_i - z)^n + E^{(i)}(s_i|z) \tag{3.3}$$

with coefficients $f_n(s_i) \in \mathbb{R}$ and $E^{(i)}(s_i|z) = \mathbf{o}\bigl((\mathbf{r}_i - z)^{d_i}\bigr)$. If $\zeta_2(\mathbf{r}) < \mathbf{r}_2$ then $F_2(s_2|z)$ is analytic at $z = \zeta_2(\mathbf{r})$ for all $s_2 \in \mathrm{supp}(\mu_2)$ and we can even write

$$F_2(s_2|z) = \sum_{n \geq 0} f_n(s_2)\bigl(\zeta_2(\mathbf{r}) - z\bigr)^n.$$

Now we can prove:

Lemma 3.3. *For $z \in \mathbb{C}$ in a neighbourhood of $\mathbf{r}_i$,*

$$\begin{aligned}
&\sum_{s_i \in \mathrm{supp}(\mu_i)} \mu_i(s_i)\, z\, E^{(i)}(s_i|z) \\
= \;& e^{(i)}_{(q_i,k_i)}(\mathbf{r}_i - z)^{q_i} \log^{k_i}(\mathbf{r}_i - z) + \sum_{(q,k) \in \widehat{\mathcal{T}}_i} e^{(i)}_{(q,k)}(\mathbf{r}_i - z)^q \log^k(\mathbf{r}_i - z) + \mathcal{O}\bigl((\mathbf{r}_i - z)^{d_i+2}\bigr),
\end{aligned}$$

where $e^{(i)}_{(q_i,k_i)} \neq 0$ and $\widehat{\mathcal{T}}_i \subseteq \bigl\{(q,k) \in \mathbb{R} \times \mathbb{N}_0 \mid d_i < q \leq d_i + 2, q > q_i \text{ or } (q = q_i \Rightarrow k < k_i)\bigr\}$ is finite.

Proof. Define

$$U_i(z) := \sum_{s_i \in \mathrm{supp}(\mu_i)} \mu_i(s_i)\, z\, F_i(s_i|z).$$

Observe that the expansions of $U_i(z)$ and $G_i(z)$ have the same leading singular term: indeed, both functions are d_i-times differentiable in a neighbourhood of $z = \mathbf{r}_i$ due to the well-known equation $G_i(z) = 1/\big(1 - U_i(z)\big)$. Therefore, we have expansions

$$G_i(z) = \sum_{k=0}^{d_i} g_k^{(i)} (\mathbf{r}_i - z)^k + R_{G_i}(z) \quad \text{and} \quad U_i(z) = \sum_{k=0}^{d_i} u_k^{(i)} (\mathbf{r}_i - z)^k + R_{U_i}(z),$$

where $R_{G_i}(z) = \mathcal{O}_c\big(S_i(z)\big)$ and $R_{U_i}(z) = \mathbf{o}\big((\mathbf{r}_i - z)^{d_i}\big)$. Substituting these expansions into $G_i(z)(1 - U_i(z)) = 1$, and taking all polynomial terms to one side, we get

$$\big(1 - U_i(\mathbf{r}_i)\big)\, R_{G_i}(z) - G_i(\mathbf{r}_i)\, R_{U_{i}(z)} = p(z) + \mathbf{o}\big((\mathbf{r}_i - z)^{d_i+1}\big),$$

where $p(z)$ is some polynomial. This equation implies that the right hand side is of order $\mathcal{O}\big((\mathbf{r}_i - z)^{d_i+1}\big)$, that is, $R_{U_i(z)} = \mathcal{O}_c\big(S_i(z)\big)$ and we can write

$$U_i(z) = \sum_{k=0}^{d_i} u_k^{(i)} (\mathbf{r}_i - z)^k + u_{(q_i,k_i)}^{(i)} S_i(z) + \widehat{R}_{U_i}(z) \qquad \text{with } \widehat{R}_{U_i}(z) = \mathbf{o}\big(S_i(z)\big).$$

Plugging this expansion once again into $G_i(z)(1 - U_i(z)) = 1$, comparing error terms and iterating the last steps, together with substituting (3.3) in the definition of $U_i(z)$, yields the claim. □

The next goal is to show that $\zeta_1(z)$ and $\zeta_2(z)$ are D-times differentiable at $z = \mathbf{r}$.

Proposition 3.4. *There are real numbers $x_0, x_1, \ldots, x_D$ and $y_0, y_1, \ldots, y_D$ such that*

$$\zeta_1(z) = \sum_{k=0}^{D} x_k\, (\mathbf{r} - z)^k + X_D^{(1)}(z) \quad \textit{and} \quad \zeta_2(z) = \sum_{k=0}^{D} y_k\, (\mathbf{r} - z)^k + X_D^{(2)}(z),$$

where $X_D^{(1)}(z) = \mathbf{o}\big((\mathbf{r} - z)^D\big)$ and $X_D^{(2)}(z) = \mathbf{o}\big((\mathbf{r} - z)^D\big)$.

Proof. We prove the proposition by determining $x_0, x_1, \ldots, x_D$ and $y_0, y_1, \ldots, y_D$ inductively. By Lemma 3.2 and a well-known characterization of differentiability, we can rewrite $\zeta_1(z)$ and $\zeta_2(z)$ in the following way:

$$\begin{aligned} \zeta_1(z) &= \mathbf{r}_1 - \zeta_1'(\mathbf{r})\,(\mathbf{r} - z) + X_1^{(1)}(z), \ \text{where } X_1^{(1)}(z) = \mathbf{o}(\mathbf{r} - z), \\ \zeta_2(z) &= \zeta_2(\mathbf{r}) - \zeta_2'(\mathbf{r})\,(\mathbf{r} - z) + X_1^{(2)}(z), \ \text{where } X_1^{(2)}(z) = \mathbf{o}(\mathbf{r} - z). \end{aligned} \tag{3.4}$$

Thus, we have determined x_0, x_1 and y_0, y_1. Assume now that we can write for some $t < D$

$$\zeta_1(z) = \sum_{k=0}^{t} x_k\, (\mathbf{r} - z)^k + X_t^{(1)}(z) \quad \text{and} \quad \zeta_2(z) = \sum_{k=0}^{t} y_k\, (\mathbf{r} - z)^k + X_t^{(2)}(z), \tag{3.5}$$

where $X_t^{(1)}(z) = \mathbf{o}\big((\mathbf{r} - z)^t\big)$ and $X_t^{(2)}(z) = \mathbf{o}\big((\mathbf{r} - z)^t\big)$. Recall from above that we have expansions of $F_1(s_1|z)$ and $F_2(s_2|z)$ of the form

$$\begin{aligned} F_1(s_1|z) &= \textstyle\sum_{n=0}^{D} a_n(s_1)(\mathbf{r}_1 - z)^n + E^{(1)}(s_1|z) \ \text{ and} \\ F_2(s_2|z) &= \textstyle\sum_{n=0}^{D} b_n(s_2)\big(\zeta_2(\mathbf{r}) - z\big)^n + E^{(2)}(s_2|z), \end{aligned} \tag{3.6}$$

where $E^{(i)}(s_i|z) = \mathbf{o}\big((\zeta_i(\mathbf{r}) - z)^D\big)$. In particular, if $\bar{\theta} < \theta_2/\alpha_2$ then $\zeta_2(\mathbf{r}) < \mathbf{r}_2$ and consequently we can even write $F_2(s_2|z) = \sum_{n\geq 0} b_n(s_2)\big(\zeta_2(\mathbf{r}) - z\big)^n$. Observe that the case $D = d_1 > d_2$ implies $\bar{\theta} < \theta_2/\alpha_2$. We now substitute the expansions (3.5) and (3.6) in equations (3.1) and (3.2), yielding the following system:

$$
\begin{aligned}
\alpha_1 z &= \Big(\textstyle\sum_{k=0}^t x_k\,(\mathbf{r}-z)^k + X_t^{(1)}(z)\Big)\Bigg[1 - \alpha_2\big(\mathbf{r} - (\mathbf{r}-z)\big)\textstyle\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)\cdot \\
&\qquad \cdot\Big[\textstyle\sum_{n=0}^D b_n(s_2)\Big(-\sum_{k=1}^t y_k\,(\mathbf{r}-z)^k - X_t^{(2)}(z)\Big)^n + E^{(2)}\big(s_2\big|\zeta_2(z)\big)\Big]\Bigg], \\
\alpha_2 z &= \Big(\textstyle\sum_{k=0}^t y_k\,(\mathbf{r}-z)^k + X_t^{(2)}(z)\Big)\Bigg[1 - \alpha_1\big(\mathbf{r} - (\mathbf{r}-z)\big)\textstyle\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)\cdot \\
&\qquad \cdot\Big[\textstyle\sum_{n=0}^D a_n(s_1)\Big(-\sum_{k=1}^t x_k\,(\mathbf{r}-z)^k - X_t^{(1)}(z)\Big)^n + E^{(1)}\big(s_1\big|\zeta_1(z)\big)\Big]\Bigg].
\end{aligned}
\tag{3.7}
$$

Observe that $\sum_{s_i\in\mathrm{supp}(\mu_i)}\mu(s_i)zE^{(i)}\big(s_i\big|\zeta_i(z)\big) = \mathbf{o}\big((\zeta_i(\mathbf{r}) - \zeta_i(z))^D\big) = \mathbf{o}\big((\mathbf{r}-z)^D\big)$. We now bring all polynomial and higher order terms to the left hand side and get:

$$
\begin{aligned}
P_t^{(1)}(z) + \mathbf{o}\big((\mathbf{r}-z)^{t+1}\big) &= \Big[1 - \alpha_2\mathbf{r}\textstyle\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2)\Big]X_t^{(1)}(z) \\
&\quad + \Big[\alpha_2\mathbf{r}_1\mathbf{r}\textstyle\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2)\Big]X_t^{(2)}(z), \\
P_t^{(2)}(z) + \mathbf{o}\big((\mathbf{r}-z)^{t+1}\big) &= \Big[\alpha_1\zeta_2(\mathbf{r})\mathbf{r}\textstyle\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1)\Big]X_t^{(1)}(z) \\
&\quad + \Big[1 - \alpha_1\mathbf{r}\textstyle\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1)\Big]X_t^{(2)}(z),
\end{aligned}
\tag{3.8}
$$

where $P_t^{(1)}(z)$ and $P_t^{(2)}(z)$ are polynomials in the variable z. By assumption on $X_t^{(1)}(z)$ and $X_t^{(2)}(z)$, the right hand sides of (3.8) are of order $\mathbf{o}\big((\mathbf{r}-z)^t\big)$. Therefore, the left hand sides have to be of order $\mathcal{O}\big((\mathbf{r}-z)^{t+1}\big)$, and consequently the right hand sides have to be also of order $\mathcal{O}\big((\mathbf{r}-z)^{t+1}\big)$. It remains to show that $X_t^{(1)}(z) = \mathcal{O}\big((\mathbf{r}-z)^{t+1}\big)$ and $X_t^{(2)}(z) = \mathcal{O}\big((\mathbf{r}-z)^{t+1}\big)$. For this purpose, define the matrix $M = (m_{ij})_{1\leq i,j\leq 2}$ by

$$
\begin{aligned}
m_{11} &:= 1 - \alpha_2\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2), \\
m_{12} &:= \alpha_2\mathbf{r}_1\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2), \\
m_{21} &:= \alpha_1\zeta_2(\mathbf{r})\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1), \\
m_{22} &:= 1 - \alpha_1\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1).
\end{aligned}
$$

Then the system (3.8) is equivalent to

$$M\cdot\begin{pmatrix} X_t^{(1)}(z) \\ X_t^{(2)}(z)\end{pmatrix} = \begin{pmatrix} Q_t^{(1)}(z) \\ Q_t^{(2)}(z)\end{pmatrix},$$

where $Q_t^{(1)}(z)=\mathcal{O}\bigl((\mathbf{r}-z)^{t+1}\bigr)$ and $Q_t^{(2)}(z)=\mathcal{O}\bigl((\mathbf{r}-z)^{t+1}\bigr)$. If the matrix M is invertible, then obviously $X_t^{(1)}(z)=\mathcal{O}\bigl((\mathbf{r}-z)^{t+1}\bigr)$ and $X_t^{(2)}(z)=\mathcal{O}\bigl((\mathbf{r}-z)^{t+1}\bigr)$. To this end, we now prove invertibility of M:

Lemma 3.5. $\det(M)\neq 0$.

Proof. We start with differentiating equations (3.1) and (3.2):

$$\begin{aligned}\alpha_1 &= \Bigl(-\alpha_2\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2\bigl(s_2|\zeta_2(z)\bigr)-\alpha_2 z\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2'\bigl(s_2|\zeta_2(z)\bigr)\zeta_2'(z)\Bigr)\zeta_1(z)\\ &\quad +\zeta_1'(z)\Bigl(1-\alpha_2 z\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2\bigl(s_2|\zeta_2(z)\bigr)\Bigr),\end{aligned}$$

$$\begin{aligned}\alpha_2 &= \Bigl(-\alpha_1\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1\bigl(s_1|\zeta_1(z)\bigr)-\alpha_1 z\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1'\bigl(s_1|\zeta_1(z)\bigr)\zeta_1'(z)\Bigr)\zeta_2(z)\\ &\quad +\zeta_2'(z)\Bigl(1-\alpha_1 z\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1\bigl(s_1|\zeta_1(z)\bigr)\Bigr).\end{aligned}$$

Observe that we have $a_0(s_1)=F_1(s_1|\mathbf{r}_1)$, $a_1(s_1)=-F_1'(s_1|\mathbf{r}_1)$, $b_0(s_2)=F_2\bigl(s_2|\zeta_2(\mathbf{r})\bigr)$ and $b_1(s_2)=-F_2'\bigl(s_2|\zeta_2(\mathbf{r})\bigr)$. Substituting these values in the above system and letting $z\to\mathbf{r}$ yields

$$\begin{aligned}\alpha_1 &= \Bigl(-\alpha_2\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2)+\alpha_2\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2)\zeta_2'(\mathbf{r})\Bigr)\mathbf{r}_1+\zeta_1'(\mathbf{r})m_{11},\\ \alpha_2 &= \Bigl(-\alpha_1\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1)+\alpha_1\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1)\zeta_1'(\mathbf{r})\Bigr)\zeta_2(\mathbf{r})+\zeta_2'(\mathbf{r})m_{22}.\end{aligned}$$

Since $\zeta_1(\mathbf{r}),\zeta_2(\mathbf{r})>0$ and $a_1(s_1),b_1(s_2)<0$ the last equations imply $m_{11},m_{22}>0$. We proceed with rewriting the last system:

$$\begin{aligned}\alpha_2\mathbf{r}_1\mathbf{r}\textstyle\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2)\zeta_2'(\mathbf{r}) &= A-\zeta_1'(\mathbf{r})m_{11},\\ \alpha_1\zeta_2(\mathbf{r})\mathbf{r}\textstyle\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1)\zeta_1'(\mathbf{r}) &= B-\zeta_2'(\mathbf{r})m_{22},\end{aligned}\tag{3.9}$$

where

$$A:=\alpha_1+\alpha_2\mathbf{r}_1\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2)\quad\text{and}\quad B:=\alpha_2+\alpha_1\zeta_2(\mathbf{r})\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1).$$

Multiplying both equations in (3.9) yields the equation

$$\zeta_1'(\mathbf{r})\zeta_2'(\mathbf{r})m_{12}m_{21}=AB-\zeta_1'(\mathbf{r})m_{11}B-\zeta_2'(\mathbf{r})m_{22}A+\zeta_1'(\mathbf{r})\zeta_2'(\mathbf{r})m_{11}m_{22}.$$

Assume now that $\det(M)=0$. Then we would get

$$\zeta_1'(\mathbf{r})m_{11}B+\zeta_2'(\mathbf{r})m_{22}A=AB,$$

or equivalently,

$$\zeta_2'(\mathbf{r})=\frac{AB-\zeta_1'(\mathbf{r})m_{11}B}{m_{22}A}. \tag{3.10}$$

Furthermore, (3.9) would imply

$$\zeta_1'(\mathbf{r})=\big(A-C\zeta_2'(\mathbf{r})\big)/m_{11},$$

where $C:=\alpha_2\mathbf{r}_1\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2)<0$. Substituting the last equation in (3.10) would lead to

$$\zeta_2'(\mathbf{r})=\frac{BC}{m_{22}A}\zeta_2'(\mathbf{r}).$$

Observe now that $A,B,m_{22}>0$ and $C<0$. This yields a contradiction in the last equation, since $\zeta_2'(\mathbf{r})>0$. Thus, $\det(M)\neq 0$. $\square$

The last lemma finishes the proof of Proposition 3.4. $\square$

Recall the definition of the main leading singular term $S(z)=S_i(z)=(\mathbf{r}_i-z)^{q_i}\log^{k_i}(\mathbf{r}_i-z)$. The next aim is to show that at least one of the functions $X_D^{(1)}(z)$ and $X_D^{(2)}(z)$ has order $\mathcal{O}_c\big((\mathbf{r}-z)^{q_i}\log^{k_i}(\mathbf{r}-z)\big)$. To this end, we look at the final step of the induction in the proof of Proposition 3.4. For $t=D$, the system (3.7) becomes

$$\begin{aligned}
&\Big[1-\alpha_2\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2)\Big]\cdot X_D^{(1)}(z)+\Big[\alpha_2\mathbf{r}_1\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2)\Big]\cdot X_D^{(2)}(z)\\
&\quad-\alpha_2\mathbf{r}_1\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)\,z\,E^{(2)}\big(s_2|\zeta_2(z)\big)=P_D^{(1)}(z)+\mathbf{o}\big((\mathbf{r}-z)^{D+1}\big),\\
&\Big[\alpha_1\zeta_2(\mathbf{r})\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1)\Big]\cdot X_D^{(1)}(z)+\Big[1-\alpha_1\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1)\Big]\cdot X_D^{(2)}(z)\\
&\quad-\alpha_1\zeta_2(\mathbf{r})\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)\,z\,E^{(1)}\big(s_1|\zeta_1(z)\big)=P_D^{(2)}(z)+\mathbf{o}\big((\mathbf{r}-z)^{D+1}\big),
\end{aligned}$$

where $P_D^{(1)}(z)$ and $P_D^{(2)}(z)$ are polynomials in the variable z. By (3.4), we may conclude $\big(\zeta_i(\mathbf{r})-\zeta_i(z)\big)=\mathcal{O}_c(\mathbf{r}-z)$. Since $\zeta_i'(\mathbf{r}_i)<\infty$ by Lemma 3.2, we have for $1<p\in\mathbb{R}$

$$\big(\zeta_i(\mathbf{r})-\zeta_i(z)\big)^p=\big(\zeta_i'(\mathbf{r}_i)(\mathbf{r}-z)+\mathbf{o}(\mathbf{r}-z)\big)^p=\zeta_i'(\mathbf{r}_i)^p\,(\mathbf{r}-z)^p\,\big(1+\mathbf{o}(1)\big)^p=\mathcal{O}_c\big((\mathbf{r}-z)^p\big)$$

and

$$\begin{aligned}
\log\big(\zeta_i(\mathbf{r})-\zeta_i(z)\big) &= \log\big(\zeta_i'(\mathbf{r}_i)(\mathbf{r}-z)+\mathbf{o}(\mathbf{r}-z)\big)\\
&= \log\big(\zeta_i'(\mathbf{r}_i)\big)+\log(\mathbf{r}-z)+\log\big(1+\mathbf{o}(1)\big)\\
&= \log\big(\zeta_i'(\mathbf{r}_i)\big)+\log(\mathbf{r}-z)+\mathbf{o}(1).
\end{aligned}$$

We remark that $(1+z)^p$ and $\log(1+z)$ are analytic in a neighbourhood of $z=0$. In the following we denote by $i \in \{1,2\}$ the index such that $S(z)=S_i(z)$. Then, the computations above imply with Lemma 3.3 that

$$\sum_{s_i \in \mathrm{supp}(\mu_i)} \mu(s_i)\, z\, E^{(i)}\big(s_i|\zeta_i(z)\big) = \mathcal{O}_c\big((\mathbf{r}-z)^{q_i}\log^{k_i}(\mathbf{r}-z)\big).$$

Since the matrix M from the proof of Proposition 3.4 is invertible, we can conclude analogously that we must have

$$X_D^{(1)}(z) = \mathcal{O}_c\big((\mathbf{r}-z)^{q_i}\log^{k_i}(\mathbf{r}-z)\big) \quad \text{and} \quad X_D^{(2)}(z) = \mathcal{O}_c\big((\mathbf{r}-z)^{q_i}\log^{k_i}(\mathbf{r}-z)\big).$$

Thus, the leading singular term of $\zeta_i(z)$ has the same order as the leading singular term in the expansion of $G_i(z)$ if $S(z)=S_i(z)$. By (2.6), we can conclude that the leading singular term in the expansion of $G(z)$ at $z=\mathbf{r}$ has the same form as the leading singular term in the expansion of $G_i(z)$ at $z=\mathbf{r}_i$, namely $(\mathbf{r}-z)^{q_i}\log^{k_i}(\mathbf{r}-z)$.

Recall that we assumed throughout this section that $G_i(z)$ is exactly d_i-times differentiable at $z=\mathbf{r}_i$. For an application of *Darboux's method* we need in a first step the expansion of $G(z)$ in a neighbourhood of $z=\mathbf{r}$ up to terms of order $(\mathbf{r}-z)^{D+2}$, where $D=d_1$, if $\bar\theta < \theta_2\alpha_2$, and $D=\min\{d_1,d_2\}$, if $\bar\theta=\theta_1\alpha_1=\theta_2\alpha_2$. Thus, by (2.6), we have to extend the expansions of $\zeta_1(z)$ and $\zeta_2(z)$ up to terms of order $(\mathbf{r}-z)^{D+2}$. The next lemma ensures that there are only finitely many terms up to order $(\mathbf{r}-z)^{D+2}$ in these expansions.

Lemma 3.6. *Let $i\in\{1,2\}$, then $\zeta_i(z)$ has an expansion of the form*

$$\sum_{k=0}^{D} x_k(\mathbf{r}-z)^k + \sum_{(q,k)\in\mathcal{T}} x_{(q,k)}(\mathbf{r}-z)^q\log^k(\mathbf{r}-z) + \mathbf{o}\big((\mathbf{r}-z)^{D+2}\big),$$

where $x_k, x_{(q,k)}\in\mathbb{R}$, $\mathcal{T}$ is a finite subset of $\widehat{\mathcal{T}} := \big\{(q,k)\in\mathbb{R}\times\mathbb{N}_0 \mid D<q\leq D+2\big\}$. In particular, $(q_i,k_i)\in\mathcal{T}$ with $x_{(q_i,k_i)}\neq 0$.

Proof. Recall the expansion of $\sum_{s_i\in\mathrm{supp}(\mu_i)}\mu_i(s_i)\,z\,E^{(i)}(s_i|z)$ from Lemma 3.3. Assume that $\zeta_i(z)$ has already an expansion of the form

$$\sum_{k=0}^{D} x_k(\mathbf{r}-z)^k + \sum_{(q,k)\in\mathcal{T}'} x_{(q,k)}(\mathbf{r}-z)^q\log^k(\mathbf{r}-z) + \mathbf{o}(\max\mathcal{T}'), \tag{3.11}$$

where $\mathcal{T}'$ is a finite subset of $\widehat{\mathcal{T}}$ and $\max\mathcal{T}' := \max_{\preceq}\big\{(\mathbf{r}-z)^q\log^k(\mathbf{r}-z) \mid (q,k)\in\mathcal{T}'\big\}$. In particular, $x_{(q_i,k_i)}\in\mathcal{T}'$ with $x_{(q_i,k_i)}\neq 0$. We proceed with expanding the next terms of $\zeta_i(z)$ analogously to the proof of Proposition 3.4. For this purpose, observe that for $p>1$ we can rewrite $\big(\zeta_i(\mathbf{r})-\zeta_i(z)\big)^p$ as

(3.12)

$$(-x_1)^p\,(\mathbf{r}-z)^p\,\left(1+\sum_{k=2}^{D}\frac{x_k}{x_1}(\mathbf{r}-z)^{k-1} + \sum_{(q,k)\in\mathcal{T}'}\frac{x_{(q,k)}}{x_1}(\mathbf{r}-z)^{q-1}\log^k(\mathbf{r}-z) + \mathbf{o}\Big(\frac{\max\mathcal{T}'}{\mathbf{r}-z}\Big)\right)^p$$

and $\log\big(\zeta_i(\mathbf{r})-\zeta_i(z)\big)$ as

$$C+\log(\mathbf{r}-z)+\log\left(1+\sum_{k=2}^{D}\frac{x_k}{x_1}(\mathbf{r}-z)^{k-1}+\sum_{(q,k)\in\mathcal{T}'}\frac{x_{(q,k)}}{x_1}(\mathbf{r}-z)^{q-1}\log^k(\mathbf{r}-z)+\mathbf{o}\Big(\frac{\max\mathcal{T}'}{\mathbf{r}-z}\Big)\right). \tag{3.13}$$

Note that $(1+z)^p$ with $p>1$ and $\log(1+z)$ are analytic in a neighbourhood of $z=0$. We substitute (3.11), (3.12) and (3.13) in Equations (3.1) and (3.2) and compare again the error terms (we will repeat this procedure in each of the following steps). Therefore, if $\max\mathcal{T}'=(\mathbf{r}-z)^{\hat{q}}\log^{\hat{k}}(\mathbf{r}-z)$ then the next possible terms up to order $(\mathbf{r}-z)^{\hat{q}}$ in the expansion may only be

$$(\mathbf{r}-z)^{\hat{q}}\log^{\hat{k}-1}(\mathbf{r}-z),(\mathbf{r}-z)^{\hat{q}}\log^{\hat{k}-2}(\mathbf{r}-z),\ldots,(\mathbf{r}-z)^{\hat{q}}.$$

Analogously to the proof of Proposition 3.4 we determine step by step the corresponding coefficients of these terms. The next term in the expansion of $\zeta_i(z)$ has now the form $(\mathbf{r}-z)^{\check{q}}\log^{\check{k}}(\mathbf{r}-z)$, where $\check{q}>\hat{q}$ is a sum of elements from the finite set

$$\big\{1,q,q-1\mid(q,\cdot)\in\mathcal{T}_1\cup\mathcal{T}_2\big\}$$

with $\mathcal{T}_i$ given as in (2.2). The value of $\check{q}$ is minimal such that $\check{q}>\hat{q}$. Due to (3.12) and (3.13) there is obviously a maximal $\check{k}\in\mathbb{N}_0$ such that $(\mathbf{r}-z)^{\check{q}}\log^{\check{k}}(\mathbf{r}-z)$ may be a non-vanishing next term in the expansion of $\zeta_i(z)$. Thus, we may iterate the last few steps again. Since there are only finitely many possible values for q such that a term of the form $(\mathbf{r}-z)^q\log^k(\mathbf{r}-z)$ may appear in the expansion up to order $(\mathbf{r}-z)^{D+2}$, we have shown that there are only finitely many terms up to order $(\mathbf{r}-z)^{D+2}$ in the expansion of $\zeta_i(z)$. $\square$

With the last lemma we are now able to prove Theorem 3.1:

Proof of Theorem 3.1. We start by expanding $\zeta_1(z)$ and $\zeta_2(z)$ as in Proposition 3.4. If $\alpha_1\geq\theta_1/(\theta_1+\theta_2)$ then $\bar{\theta}=\theta_1/\alpha_1$ and $\zeta_1(\mathbf{r})=\mathbf{r}_1$, and consequently the leading singular term in the expansion of $\zeta_1(z)$ (and $\zeta_2(z)$) is then $(\mathbf{r}-z)^{q_1}\log^{k_1}(\mathbf{r}-z)$. Otherwise, we have $\bar{\theta}=\theta_2/\alpha_2<\theta_1/\alpha_1$, and the leading singular term is then $(\mathbf{r}-z)^{q_2}\log^{k_2}(\mathbf{r}-z)$. For the rest of the proof, we denote by $i\in\{1,2\}$ the index such that $S(z)=S_i(z)$. Therefore, the expansion of the common leading singular term of $\zeta_1(z)$ and $\zeta_2(z)$, namely $S_i(z)$, in a neighbourhood of 0 has coefficients of asymptotic order proportional to $n^{-\lambda_i}\log^{\kappa_i}n$.

We will use the technique which is called *Darboux's method*: recall that the *Riemann-Lebesgue-Lemma* states that if a function $H(z)=\sum_{n\geq 0}h_nz^n$ has radius of convergence $\mathbf{r}_H$ and if H is k-times continuously differentiable on its circle of convergence, then $h_n\mathbf{r}_H^n n^k\to 0$ as $n\to\infty$. Thus, one identifies all singularities on the circle of convergence and subtracts parts of the expansion near them such that the remaining part is sufficiently often differentiable on the circle. The asymptotics of the coefficients arise then from the main leading singular terms. We refer to Olver [17, Chap. 8, §9.2] for more details.

Lemma 3.6 assures that we have a singular expansion of $\zeta_1(z)$ up to terms of order $\lceil\lambda_i\rceil=\lceil q_i\rceil+1=D+2$, which allows us to apply Darboux's method: we get the asymptotic behaviour of $\mu^{(n\delta)}(e)$ by plugging $\zeta_1(z)$ into Equation (2.6). Thus, the leading singular term in the expansion of $G(z)$ in a neighbourhood of $z=\mathbf{r}$ is the same as the one of $\zeta_1(z)$, namely $(\mathbf{r}-z)^{q_i}\log^{k_i}(\mathbf{r}-z)$. We have to show that the expansion of $G(z)$ at every singular point on the disc of convergence has the same form. The singularities are exactly the points $\mathbf{r}\exp(\mathrm{i}2\pi j/\delta)$ with $0\leq j<\delta-1$; see e.g. [22, Theorem 9.4]. Writing $z=\lambda\mathbf{r}\omega_j$, where $\omega_j=\exp(\mathrm{i}2\pi j/\delta)$ and $\lambda\in\mathbb{C}$ with $|\lambda|<1$,

$$G(z)=G(\lambda\mathbf{r}\omega_j)=\sum_{n\geq 0}\mu^{(n\delta)}(e)(\lambda\mathbf{r}\omega_j)^{n\delta}=\sum_{n\geq 0}\mu^{(n\delta)}(e)(\lambda\mathbf{r})^{n\delta}=G(\lambda\mathbf{r})=G(z/\omega_j).$$

Thus, for every $j\in\{0,1,\dots,\delta-1\}$, we have expansions of $G(z)$ in a neighbourhood of $z=\mathbf{r}\omega_j$ given by

$$G(z)=\sum_{k=0}^{D}g_k(\mathbf{r}-z/\omega_j)^k+\sum_{(q,k)\in\mathcal{T}_i}g_{(q,k)}(\mathbf{r}-z/\omega_j)^q\log^k(\mathbf{r}-z/\omega_j)+\mathcal{O}\big((\mathbf{r}\omega_j-z)^{D+2}\big),$$

where $\mathcal{T}_i$ is a finite subset of $\{(q,k)\in\mathbb{R}\times\mathbb{N}\mid D<q\leq D+2, q>q_i\vee(q=q_i\Rightarrow k<k_i)\}$, $g_{(q_i,k_i)}\in\mathcal{T}_i$ and $g_{(q_i,k_i)}\neq 0$. Therefore, the difference

$$G(z)-\sum_{j=0}^{\delta-1}\sum_{(q,k)\in\mathcal{T}_\rangle}g_{(q,k)}(\mathbf{r}-z/\omega_j)^q\log^k(\mathbf{r}-z/\omega_j)$$

is $(D+2)$-times differentiable on the circle of convergence. Observe now that the coefficients of the expansion of $(\mathbf{r}-z/\omega_j)^{q_i}\log^{k_i}(\mathbf{r}-z/\omega_j)$ in a neighbourhood of 0 behave asymptotically like $C_i\,\omega_j^{-n}\,n^{-\lambda_i}\,\log^{\kappa_i}(n)$. We can drop higher order terms in the above difference because the corresponding coefficients have higher asymptotic order. Since $G(z)=\sum_{n\geq 0}\mu^{(n)}z^n$, we can conclude that

$$\mu^{(n)}(e)\sim\sum_{j=0}^{\delta-1}C\,n^{-\lambda_i}\,\log^{\kappa_i}(n)\,\mathbf{r}^{-n}\,\omega_j^{-n}.$$

Observe that $\sum_{j=0}^{\delta-1}\omega_j^{-n}=\delta$ if δ divides n, and this sum is zero otherwise.

We note once again that the asymptotic behaviour of the coefficients in the expansion of the function $(\mathbf{r}-z)^{q_i}\log^{k_i}(\mathbf{r}-z)$ near 0 are well-known; see e.g. Flajolet and Sedgewick [7]. $\square$

Let us remark that the reasoning in the above proof shows analogously the asymptotic behaviour $\mu_i^{(n)}(e_i)\sim C_i\,\mathbf{r}_i^{-n}\,n^{-\lambda_i}\,\log^{\kappa_i}n$. That is, in the presented case of $\Psi(\bar\theta)>0$, $G_1'(\mathbf{r}_1)<\infty$ and $G_2'(\mathbf{r}_2)<\infty$ the asymptotics are directly inherited from the asymptotics of the random walk on Γ_i governed by μ_i.

4. The Case $\Psi(\bar\theta)=0$

We now consider the case $\Gamma=\Gamma_1*\Gamma_2$ and assume that $\Psi(\bar\theta)=0$, $G_1'(\mathbf{r}_1)<\infty$ and $G_2'\big(\zeta_2(\mathbf{r})\big)<\infty$ hold. W.l.o.g. we may also assume $\theta=\bar\theta=\theta_1/\alpha_1$. The aim of this section is to prove the following:

Theorem 4.1. *Assume that* $G_1'(\mathbf{r}_1) < \infty$ *and* $G_2'\big(\zeta_2(\mathbf{r}_2)\big) < \infty$. *If* $\Psi(\bar\theta) = 0$ *then*

$$\mu^{(n\delta)}(e) \sim C \cdot \mathbf{r}^{-n\delta} \cdot n^{-3/2}.$$

In the following we will derive expansions of $\zeta_i(z)$ and $G(z)$ in a neighbourhood of $z = \mathbf{r}$ in order to prove Theorem 4.1. Recall from (2.8) that $\Psi(\bar\theta) = 0$ implies

$$\Phi'(\bar\theta) = \frac{\Phi(\bar\theta)}{\bar\theta} = \frac{\Phi(\theta)}{\theta} = \frac{\Phi\big(\mathbf{r}\,G(\mathbf{r})\big)}{\mathbf{r}\,G(\mathbf{r})} = \frac{G(\mathbf{r})}{\mathbf{r}\,G(\mathbf{r})} = \frac{1}{\mathbf{r}}.$$

Differentiating (2.7) yields

$$G'(z) = \frac{G(z)\Phi'\big(zG(z)\big)}{1 - z\Phi'\big(zG(z)\big)}. \tag{4.1}$$

Therefore, $G'(\mathbf{r}) = \infty$, and consequently we have to proceed differently from the previous section in order to find the expansion of $G(z)$. First, we show positivity of $\Phi''(\bar\theta)$ in the present setting:

Lemma 4.2. *Assume that* $G_1'(\mathbf{r}_1) < \infty$ *and* $G_2'\big(\zeta_2(\mathbf{r}_2)\big)$. *If* $\Psi(\bar\theta) = 0$ *then* $\Phi''(\bar\theta) > 0$.

Proof. Differentiating (2.9) twice yields

$$\Phi''(\bar\theta) = \alpha_1^2 \Phi_1''(\alpha_1\bar\theta) + \alpha_2^2 \Phi_2''(\alpha_2\bar\theta). \tag{4.2}$$

Since $\Phi_1(t)$ and $\Phi_2(t)$ are strictly convex for $t \in [0, \theta_1)$ and $t \in [0, \theta_2)$ respectively, we get $\Phi''(\bar\theta) > 0$ whenever $\theta_1/\alpha_1 \neq \theta_2/\alpha_2$: if $\bar\theta = \theta_1/\alpha_1 < \theta_2/\alpha_2$ then $\alpha_2\bar\theta < \theta_2$, that is, $\Phi_2''(\alpha_2\bar\theta) > 0$.
We consider now the case $\theta_1/\alpha_1 = \theta_2/\alpha_2$, that is $\zeta_2(\mathbf{r}) = \mathbf{r}_2$. Assume now $\Phi''(\bar\theta) = 0$. Then $\Phi_1''(\theta_1) = \lim_{t\to\theta_1} \Phi_1''(t) = 0$ and $\Phi_2''(\theta_2) = \lim_{t\to\theta_2} \Phi_2''(t) = 0$ must hold. For $i \in \{1, 2\}$, differentiating (2.7) yields

$$G_i'(\mathbf{r}_i) = \lim_{z\to\mathbf{r}_i} \frac{G_i(z)\Phi_i'\big(zG_i(z)\big)}{1 - z\Phi_i'\big(zG_i(z)\big)},$$

or equivalently

$$\Phi_i'(\theta_i) = \lim_{z\to\mathbf{r}_i} \frac{G_i'(z)}{zG_i'(z) + G_i(z)} = \frac{G_i'(\mathbf{r}_i)}{\mathbf{r}_i G_i'(\mathbf{r}_i) + G_i(\mathbf{r}_i)} < \infty.$$

In particular, we have $\Phi_i'(\theta_i) < 1/\mathbf{r}_i$ since $G_i'(\mathbf{r}_i) < \infty$ by assumption. If $\Phi_i''(\theta_i) = 0$, differentiating (2.7) twice yields

$$G_i''(\mathbf{r}_i) = \lim_{z\to\mathbf{r}_i} \frac{\Phi_i''\big(zG_i(z)\big)\big(G_i(z) + zG_i'(z)\big)^2 + 2\Phi_i'\big(zG_i(z)\big)G_i(z)}{1 - z\Phi_i'\big(zG_i(z)\big)} = \frac{2\Phi_i'(\theta_i)G_i(\mathbf{r}_i)}{1 - \mathbf{r}_i\Phi_i'(\theta_i)} < \infty.$$

Define the first return generating function as

$$U_i(z) := \sum_{n\geq 1} \mathbb{P}\big[X_n^{(i)} = e_i, \forall m < n : X_m^{(i)} \neq e_i \mid X_0^{(i)} = e_i\big]\, z^n,$$

which satisfies the well-known equation $G_i(z) = 1/\bigl(1-U_i(z)\bigr)$ and is strictly convex. $G_i''(\mathbf{r}_i)<\infty$ implies obviously $U_i''(\mathbf{r}_i)<\infty$. Therefore, we can compute $\Phi_i''(\theta_i)$ as

$$\Phi_i''(\theta_i) = \lim_{z\to\mathbf{r}_i} \frac{G_i(z)^3 U_i''(z)}{\bigl(G_i(z)+zG_i'(z)\bigr)^3} = \frac{G_i(\mathbf{r}_i)^3 U_i''(\mathbf{r}_i)}{\bigl(G_i(\mathbf{r}_i)+\mathbf{r}_i G_i'(\mathbf{r}_i)\bigr)^3} > 0,$$

and consequently $\Phi''(\bar{\theta})>0$ due to (4.2). □

We proceed with expanding $G(z)$ nearby $z=\mathbf{r}$.

Proposition 4.3. *Assume that $\Phi''(\bar{\theta})<\infty$. If $\Psi(\bar{\theta})=0$, $G_1'(\mathbf{r}_1)<\infty$ and $G_2'(\zeta_2(\mathbf{r}))<\infty$ then we can expand $G(z)$ in a neighbourhood of $z=\mathbf{r}$ as follows:*

$$G(z) = g_0 + g_1\sqrt{\mathbf{r}-z} + \mathbf{o}\bigl(\sqrt{\mathbf{r}-z}\bigr),$$

where $g_0, g_1 \in \mathbb{R}$ with $g_1 \neq 0$.

Proof. Consider the auxiliary function $H(z) := \bigl(G(z)-G(\mathbf{r})\bigr)^2$, and its first derivative $H'(z) = 2G'(z)\bigl(G(z)-G(\mathbf{r})\bigr)$. Using Equation (4.1), we get

$$H'(z) = 2\frac{G(z)\Phi'\bigl(zG(z)\bigr)}{1-z\Phi'\bigl(zG(z)\bigr)}\bigl(G(z)-G(\mathbf{r})\bigr).$$

The next aim is to show differentiability of $H(z)$ at $z=\mathbf{r}$. For this purpose, we want to show finiteness of the following limit:

$$\lim_{z\to\mathbf{r}} H'(z) = \lim_{z\to\mathbf{r}} 2G(z)\Phi'\bigl(zG(z)\bigr)\frac{G(z)-G(\mathbf{r})}{1-z\Phi'\bigl(zG(z)\bigr)}.$$

Since $2G(z)\Phi'\bigl(zG(z)\bigr)$ tends to $A := 2G(\mathbf{r})/\mathbf{r} < \infty$, we just look at the following limit:

$$\begin{aligned}
\lim_{z\to\mathbf{r}} \frac{G(z)-G(\mathbf{r})}{1-z\Phi'\bigl(zG(z)\bigr)} &= \lim_{z\to\mathbf{r}} \frac{\Phi\bigl(zG(z)\bigr)-G(\mathbf{r})}{1-z\Phi'\bigl(zG(z)\bigr)}\\
&= \lim_{z\to\mathbf{r}} \frac{\Phi'\bigl(zG(z)\bigr)\bigl(G(z)+zG'(z)\bigr)}{-\Phi'\bigl(zG(z)\bigr)-z\Phi''\bigl(zG(z)\bigr)\bigl(G(z)+zG'(z)\bigr)}.
\end{aligned} \tag{4.3}$$

In the last equation we applied De L'Hôpital's rule. We now write $\mathcal{G}(z) := G(z)+zG'(z)$, which tends to infinity for $z\to\mathbf{r}$. Recall that $\bar{\theta}=\theta=\mathbf{r}G(\mathbf{r})$ if $\Psi(\bar{\theta})=0$. Therefore, Equation (4.3) yields

$$H'(\mathbf{r}) = \lim_{z\to\mathbf{r}} \frac{A\Phi'(\theta)\mathcal{G}(z)}{-\Phi'(\theta)-\mathbf{r}\Phi''(\theta)\mathcal{G}(z)} = \lim_{x\to\infty} \frac{A\Phi'(\theta)x}{-\Phi'(\theta)-\mathbf{r}\Phi''(\theta)x} = \frac{A}{-\mathbf{r}^2\Phi''(\theta)} \in (-\infty,0).$$

Thus,

$$\lim_{z\to\mathbf{r}} \frac{G(\mathbf{r})-G(z)}{\sqrt{\mathbf{r}-z}} = \lim_{z\to\mathbf{r}} \sqrt{\frac{\bigl(G(z)-G(\mathbf{r})\bigr)^2}{\mathbf{r}-z}} = \sqrt{-H'(\mathbf{r})} \in (0,\infty)$$

leads to the proposed expansion, namely

$$G(z) = G(\mathbf{r}) - \sqrt{-H'(\mathbf{r})}\sqrt{\mathbf{r}-z} + \mathbf{o}(\sqrt{\mathbf{r}-z}),$$

where $\sqrt{-H'(\mathbf{r})}\neq 0$. □

The next lemma shows that also $\zeta_1(z)$ and $\zeta_2(z)$ have the same expansion type:

Lemma 4.4. *Assume $\Phi''(\bar{\theta}) < \infty$. If $\Psi(\bar{\theta}) = 0$, $G_1'(\mathbf{r}_1) < \infty$ and $G_2'(\zeta_2(\mathbf{r})) < \infty$ we can expand $\zeta_1(z)$ and $\zeta_2(z)$ in a neighbourhood of $z = \mathbf{r}$ in the following way:*

$$\zeta_1(z) = \mathbf{r}_1 + a_0\sqrt{\mathbf{r} - z} + \mathbf{o}(\sqrt{\mathbf{r} - z}), \qquad \zeta_2(z) = \zeta_2(\mathbf{r}) + b_0\sqrt{\mathbf{r} - z} + \mathbf{o}(\sqrt{\mathbf{r} - z}),$$

where $a_0, b_0 \in \mathbb{R} \setminus \{0\}$.

Proof. Obviously, we can write

$$\zeta_1(z) = \mathbf{r}_1 + X_1(z), \qquad \zeta_2(z) = \zeta_2(\mathbf{r}) + X_2(z), \tag{4.4}$$

where $X_1(\mathbf{r}) = X_2(\mathbf{r}) = 0$. Moreover, for $i \in \{1, 2\}$,

$$G_i\big(\zeta_i(z)\big) = G_i\big(\zeta_i(\mathbf{r})\big) - G_i'\big(\zeta_i(\mathbf{r})\big)\big(-X_i(z)\big) + \mathbf{o}\big(X_i(z)\big). \tag{4.5}$$

Substituting (4.4) and (4.5) in (2.6) yields the claim when comparing all error terms. $\square$

Now we can show that $\Phi''(\bar{\theta}) < \infty$ holds in the present setting¿

Lemma 4.5. *Assume $G_1'(\mathbf{r}_1) < \infty$ and $G_1(\zeta_2(\mathbf{r})) < \infty$. If $\Psi(\bar{\theta}) = 0$ then $\Phi''(\bar{\theta}) < \infty$.*

Proof. Assume now that $\Phi''(\bar{\theta}) = \infty$. We rewrite $\zeta_1(z)$ and $\zeta_2(z)$ as

$$\zeta_1(z) = \mathbf{r}_1 + X_1(z), \quad \text{and} \quad \zeta_2(z) = \zeta_2(\mathbf{r}) + X_2(z), \tag{4.6}$$

with $X_1(\mathbf{r}) = X_2(\mathbf{r}) = 0$. More precisely, if $\Phi''(\bar{\theta}) = \infty$, then the reasoning in Proposition 4.3 yields $H'(\mathbf{r}) = 0$, and consequently $X_1(z), X_2(z) = \mathbf{o}(\sqrt{\mathbf{r} - z})$. Furthermore, $X_1(z), X_2(z) \neq \mathcal{O}\big((\mathbf{r} - z)\big)$, because otherwise $\zeta_1'(\mathbf{r}), \zeta_2'(\mathbf{r}) < \infty$ together with (2.6) would lead to a contradiction with $G'(\mathbf{r}) = \infty$. For $i \in \{1, 2\}$ and $s_i \in \mathrm{supp}(\mu_i)$, we write in the following $F_i(s_i|z) = \sum_{n\geq 1} f_n^{(i)}(s_i)z^n$ with suitable coefficients $f_n^{(i)}(s_i) \in \mathbb{R}$. Our next aim is to find real numbers $C_1^{(i)}$ and $C_2^{(i)}$ such that

$$C_1^{(i)} X_1(z) + C_2^{(i)} X_2(z) + \mathbf{o}(\mathbf{r} - z) = \mathrm{LP}_i, \tag{4.7}$$

where LP_i is a linear polynomial. For this purpose, we rewrite Equations (3.1) and (3.2) with the help of (4.6). In the following j denotes the element of $\{1, 2\}$ which is different from i. We get:

$$\Big(1 - \alpha_j(\mathbf{r} - (\mathbf{r} - z)) \sum_{s_j \in \mathrm{supp}(\mu_j)} \mu_j(s_j) \sum_{n\geq 1} f_n^{(j)}(s_j) \Big(\zeta_j(\mathbf{r}) + X_j(z)\Big)^n\Big)\big(\zeta_i(\mathbf{r}) + X_i(z)\big) = \alpha_i z. \tag{4.8}$$

The coefficients $C_1^{(i)}$ and $C_2^{(i)}$ of $X_1(z)$ and $X_2(z)$ respectively, are

$$\begin{aligned}
C_1^{(1)} &:= 1-\alpha_2\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)\sum_{n\geq 1} f_n^{(2)}(s_2)\,\zeta_2(\mathbf{r})^n\\
&= 1-\alpha_2\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2\big(s_2|\zeta_2(\mathbf{r})\big),\\
C_2^{(1)} &:= -\alpha_2\mathbf{r}_1\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)\sum_{n\geq 1} f_n^{(2)}(s_2)n\,\zeta_2(\mathbf{r})^{n-1}\\
&= -\alpha_2\mathbf{r}_1\mathbf{r}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2'\big(s_2|\zeta_2(\mathbf{r})\big),\\
C_1^{(2)} &:= -\alpha_1\zeta_2(\mathbf{r})\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)\sum_{n\geq 1} f_n^{(1)}(s_1)n\mathbf{r}_1^{n-1}\\
&= -\alpha_1\zeta_2(\mathbf{r})\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1'\big(s_1|\mathbf{r}_1\big),\\
C_2^{(2)} &:= 1-\alpha_1\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)\sum_{n\geq 1} f_n^{(1)}(s_1)\mathbf{r}_1^{n}\\
&= 1-\alpha_1\mathbf{r}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1(s_1|\mathbf{r}_1).
\end{aligned}$$

For $i=1$, the linear polynomial term on the left hand side of (4.8) is

$$\mathbf{r}_1\Big(1-\alpha_2 z\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2\big(s_2|\zeta_2(\mathbf{r})\big)\Big),$$

while on the right hand side it is $\alpha_1 z$. For $i=2$, we have on the left hand side of (4.8)

$$\zeta_2(\mathbf{r})\Big(1-\alpha_1 z\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1(s_1|\mathbf{r}_1)\Big),$$

and on the right hand side $\alpha_2 z$. Therefore, (4.7) holds with

$$\begin{aligned}
\mathrm{LP}_1 &:= \alpha_1 z-\mathbf{r}_1\Big(1-\alpha_2 z\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)F_2\big(s_2|\zeta_2(\mathbf{r})\big)\Big) \quad \text{and}\\
\mathrm{LP}_2 &:= \alpha_2 z-\zeta_2(\mathbf{r})\Big(1-\alpha_1 z\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)F_1(s_1|\mathbf{r}_1)\Big).
\end{aligned}$$

The coefficients $C_1^{(i)}, C_2^{(i)}$ satisfy

$$C_1^{(1)}C_2^{(2)}-C_1^{(2)}C_2^{(1)}=0. \tag{4.9}$$

Indeed, assume that $C_1^{(1)}C_2^{(2)}-C_1^{(2)}C_2^{(1)}\neq 0$. Then the following linear system

$$\begin{aligned}
C_1^{(1)}X_1(z)+C_2^{(1)}X_2(z)+\mathbf{o}(\mathbf{r}-z) &= \mathrm{LP}_1,\\
C_1^{(2)}X_1(z)+C_2^{(2)}X_2(z)+\mathbf{o}(\mathbf{r}-z) &= \mathrm{LP}_2
\end{aligned}$$

has a unique solution for $X_1(z)$ and $X_2(z)$, but this means that both of them are of order $\mathcal{O}(\mathbf{r}-z)$, a contradiction to (4.6).

Evaluating Equation (4.8) with $i=2$ at $z=\mathbf{r}$ gives $C_2^{(2)}>0$. Equation (4.9) yields

$$\mathrm{LP}_1 - \frac{C_2^{(1)}}{C_2^{(2)}}\,\mathrm{LP}_2 = 0. \tag{4.10}$$

Evaluating the last equation at $z=0$ yields

$$-\mathbf{r}_1 + \frac{C_2^{(1)}}{C_2^{(2)}}\cdot \zeta_2(\mathbf{r}) = 0. \tag{4.11}$$

Since $C_2^{(1)}<0$, Equation (4.11) gives us a contradiction, therefore $\Phi''(\bar\theta)=\infty$ cannot hold when $\Psi(\bar\theta)=0$. □

We now proceed analogously to the previous section: we substitute the expansion of the last lemma in Equations (3.1) and (3.2) and determine step by step the next terms in the expansions of $\zeta_1(z)$ and $\zeta_2(z)$. The next lemma shows that we get only a finite number of terms up to order $(\mathbf{r}-z)^2$:

Lemma 4.6. *Let $i\in\{1,2\}$. If $\Psi(\bar\theta)=0$, we can expand $\zeta_i(z)$ in a neighbourhood of $z=\mathbf{r}$ in the following way:*

$$\zeta_i(z) = \zeta_i(r) + c_0\sqrt{\mathbf{r}-z} + \sum_{(q,k)\in\mathcal{T}} c_{(q,k)}(\mathbf{r}-z)^q \log^k(\mathbf{r}-z) + \mathcal{O}\big((\mathbf{r}-z)^2\big),$$

where $\mathcal{T}$ is a finite subset of $\widehat{\mathcal{T}} := \big\{(q,k)\in\mathbb{R}\times\mathbb{N}_0 \mid 1/2<q\leq 2\big\}$ and $c_0, c_{(q,k)}\in\mathbb{R}$ with $c_0\neq 0$.

Proof. We start by plugging $\zeta_i(z)=\zeta_i(\mathbf{r})+c_0\sqrt{\mathbf{r}-z}+X_0^{(i)}(z)$ with $X_0^{(i)}(z)=\mathbf{o}(\sqrt{r-z})$ into Equations (3.1) and (3.2) and determine step by step the next terms inductively analogously to the proof of Lemma 3.6. Assume now that $\zeta_i(z)$ has an expansion of the form

$$\zeta_i(\mathbf{r}) + c_0\sqrt{\mathbf{r}-z} + \sum_{(q,k)\in\mathcal{T}'} c_{(q,k)}(\mathbf{r}-z)^q \log^k(\mathbf{r}-z) + \mathbf{o}(\max\mathcal{T}'),$$

where $\mathcal{T}'=\mathcal{T}''\cup\{(1/2,0\}$ with $\mathcal{T}'\subseteq\widehat{\mathcal{T}}$ finite. For $p>1$, $\big(\zeta_i(\mathbf{r})-\zeta_i(z)\big)^p$ can be rewritten as

$$(-c_0)^p\,(\mathbf{r}-z)^{p/2}\left(1+\sum_{(q,k)\in\mathcal{T}'}\frac{c_{(q,k)}}{c_0}(\mathbf{r}-z)^{q-1/2}\log^k(\mathbf{r}-z)+\mathbf{o}\Big(\frac{\max\mathcal{T}'}{\sqrt{\mathbf{r}-z}}\Big)\right)^p \tag{4.12}$$

and $\log\big(\zeta_i(\mathbf{r})-\zeta_i(z)\big)$ as

$$C+\frac{1}{2}\log(\mathbf{r}-z)+\log\left(1+\sum_{(q,k)\in\mathcal{T}'}\frac{c_{(q,k)}}{c_0}(\mathbf{r}-z)^{q-1/2}\log^k(\mathbf{r}-z)+\mathbf{o}\Big(\frac{\max\mathcal{T}'}{\sqrt{\mathbf{r}-z}}\Big)\right). \tag{4.13}$$

Once again, if $\max \mathcal{T}' = (\mathbf{r}-z)^{\hat{q}} \log^{\hat{k}}(\mathbf{r}-z)$ then the next possible terms up to order $(\mathbf{r}-z)^{\hat{q}}$ in the expansion may only be

$$(\mathbf{r}-z)^{\hat{q}} \log^{\hat{k}-1}(\mathbf{r}-z), (\mathbf{r}-z)^{\hat{q}} \log^{\hat{k}-2}(\mathbf{r}-z), \ldots, (\mathbf{r}-z)^{\hat{q}}.$$

We determine step by step the corresponding coefficients of these terms by plugging the expansions of $\zeta_i(z)$, (4.12) and (4.13) into Equations (3.1) and (3.2) and comparing error terms. The next term has the form $(\mathbf{r}-z)^{\check{q}} \log^{\check{k}}(\mathbf{r}-z)$, where $\check{q} \leq 2$ is now a sum of elements from the finite set $\{1/2, q/2, q/2-1/2 \mid (q,\cdot) \in \mathcal{T}_1 \cup \mathcal{T}_2\}$ such that $\check{q} > \hat{q}$. Due to (4.12) and (4.13) there is obviously a maximal $\check{k} \in \mathbb{N}_0$ such that $(\mathbf{r}-z)^{\check{q}} \log^{\check{k}}(\mathbf{r}-z)$ may be a non-vanishing next term in the expansion of $\zeta_i(z)$. Iterating the last steps yields the claim of the lemma, since there are only finitely many possible values for q such that the term $(\mathbf{r}-z)^{q} \log^{k}(\mathbf{r}-z)$ may appear in the expansion of $\zeta_i(z)$. $\square$

Substituting the obtained expansion of $\zeta_1(z)$ into Equation (2.6) yields the proposed claim of Theorem 4.1.

Remark: The result could also be obtained analogously to Flajolet and Sedgewick [7, Section VI.7.] by singularity analysis, but one still has to prove positivity and finiteness of $\Phi''(\bar{\theta})$.

5. The Remaining Cases

In this section we look at all remaining cases not covered by Section 3 and 4. Moreover we will extend our results to free prodfuncts $\Gamma_1 * \ldots * \Gamma_m$ with $m > 2$.

5.1. Case $G_1(\mathbf{r}_1) < \infty$ and $G_1'(\mathbf{r}_1) = \infty$.

Theorem 5.1. *Consider a free product of the form $\Gamma_1 * \Gamma_2$, where $G_1(\mathbf{r}_1) < \infty$, $G_1'(\mathbf{r}_1) = \infty$ and $G_2'(\mathbf{r}_2) < \infty$. Then:*

$$\mu^{(n\delta)}(e) \sim \begin{cases} C_1 \cdot \mathbf{r}^{-n\delta} \cdot n^{-3/2}, & \text{if } \bar{\theta} = \theta_1/\alpha_1 \quad \text{or} \quad \Psi(\bar{\theta}) \leq 0, \\ C_2 \cdot \mathbf{r}^{-n\delta} \cdot n^{-\lambda_2} \cdot \log^{\kappa_2}(n), & \text{if } \bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1 \text{ and } \Psi(\bar{\theta}) > 0. \end{cases}$$

Proof. For the first part of the proof assume that $\bar{\theta} = \theta_1/\alpha_1$. With

$$U_1(z) := \sum_{g \in \Gamma_1} \mu_1(g)\, z\, F_1(g^{-1}|z)$$

we have the well-known equation $G_1(z) = 1/\big(1 - U_1(z)\big)$. Therefore, $G_1'(\mathbf{r}_1) = \infty$ implies $U_1'(\mathbf{r}_1) = \infty$, and we get due to [22, Equation (9.14)]

$$\Psi_1(\alpha_1\bar{\theta}) = \lim_{z \to \mathbf{r}_1} \Psi_1\big(zG(z)\big) = \lim_{z \to \mathbf{r}_1} \frac{1}{zU_1'(z) + 1 - U_1(z)} = 0. \tag{5.1}$$

Thus,

$$\Psi(\bar{\theta}) = \Psi_1(\alpha_1\bar{\theta}) + \Psi_2(\alpha_2\bar{\theta}) - 1 = \Psi_1(\theta_1) + \Psi_2(\alpha_2\bar{\theta}) - 1 = \Psi_2(\alpha_2\bar{\theta}) - 1.$$

Recall that $\Psi(t)$ is strictly decreasing and $\Psi_2(0) = 1$. Therefore, $\Psi(\bar{\theta}) < 0$, and consequently we obtain the asymptotic behaviour $\mu^{(n\delta)}(e) \sim C_1 \mathbf{r}^{-n\delta} n^{-3/2}$; see [22, Theorem 17.3].

For the case $\bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1$ and $\Psi(\bar{\theta}) = 0$, we refer to Section 4.

In the case $\bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1$ and $\Psi(\bar{\theta}) > 0$ the Green function $G_1(z)$ is analytic at $z = \zeta_1(\mathbf{r}) < \mathbf{r}_1$ and thus we may apply the technique from Section 3 to obtain the proposed asymptotic behaviour. □

At this point, let us remark that the formula for $\Psi(t)$ used in Equation (5.1) always implies $\Psi_i(\theta_i) = 0$ whenever $G_i'(\mathbf{r}_i) = \infty$. Moreover:

Corollary 5.2. *If $G_1'(\mathbf{r}_1) = G_2'(\mathbf{r}_2) = \infty$, then $\mu^{(n\delta)}(e) \sim C \cdot \mathbf{r}^{-n\delta} \cdot n^{-3/2}$.*

Proof. Since $U_1'(\mathbf{r}_1) = U_2'(\mathbf{r}_2) = \infty$, Equation (5.1) implies that at least one of $\Psi_1(\alpha_1\bar{\theta})$ and $\Psi_2(\alpha_2\bar{\theta})$ equals zero, yielding $\Psi(\bar{\theta}) < 0$. □

5.2. **Case $G_1(\mathbf{r}_1) = \infty$.** For finite groups Γ_1 and Γ_2, Woess [21] proved that the n-step return probabilities behave asymptotically like $C\mathbf{r}^{-n\delta} n^{-3/2}$. Moreover, we get the following asymptotic behaviours:

Theorem 5.3. *Consider a free product of the form $\Gamma_1 * \Gamma_2$, where $G_1(\mathbf{r}_1) = \infty$. Then:*

$$\mu^{(n\delta)}(e) \sim \begin{cases} C_1 \cdot \mathbf{r}^{-n\delta} \cdot n^{-3/2}, & \text{if } \Psi(\bar{\theta}) \leq 0, \\ C_2 \cdot \mathbf{r}^{-n\delta} \cdot n^{-\lambda_2} \cdot \log^{\kappa_2}(n), & \text{if } \Psi(\bar{\theta}) > 0. \end{cases}$$

Proof. If $G_2'(\mathbf{r}_2) = \infty$, we have $\Psi(\bar{\theta}) < 0$; see proof of Corollary 5.2.

If $G_2(\mathbf{r}_2) < \infty$ and $G_2'(\mathbf{r}_2) = \infty$ then $\bar{\theta} = \theta_2/\alpha_2$, and $U_2'(\mathbf{r}_2) = \infty$. This implies once again $\Psi(\alpha_2\bar{\theta}) = 0$, and thus $\Psi(\bar{\theta}) < 0$.

If $G_2'(\mathbf{r}_2) < \infty$ then $\bar{\theta} = \theta_2/\alpha_2$ and $\zeta_1(\mathbf{r}) < \mathbf{r}_1$. Therefore, we can follow the argumentation of Section 3 and 4 analogously to prove the proposed claim. □

5.3. **Free Products with more than two Factors.** Let $m \in \mathbb{N}$ with $m \geq 3$. Suppose we are given finitely generated groups $\Gamma_1, \ldots, \Gamma_m$. We consider now a free product of the form $\Gamma := \Gamma_1 * \ldots * \Gamma_m$, on which a random walk is governed by the measure μ defined as $\mu := \sum_{j=1}^m \alpha_j \bar{\mu}_j$; see Section 2. We get the following result:

Theorem 5.4. *Let $m \geq 3$. Consider the free product $\Gamma := \Gamma_1 * \ldots * \Gamma_m$ equipped with a random walk governed by $\mu := \sum_{j=1}^m \alpha_j \bar{\mu}_j$. Assume that the corresponding Green functions $G_i(z)$ on the free factors Γ_i have an expansion as in (2.2) whenever $G_i'(\mathbf{r}) < \infty$. Denote by $\mathbf{r}$ the radius of convergence of the Green function associated with the random walk on Γ. Then the asymptotic behaviour of the corresponding n-step transition probabilities must obey one of the following laws: $C_i\, \mathbf{r}^{-n\delta}\, n^{-\lambda_i} \log^{\kappa_i}(n)$, where λ_i and κ_i are inherited from one of the μ_i, or $C_0\, \mathbf{r}^{-n\delta}\, n^{-3/2}$.*

Proof. In order to prove the theorem, we just remark that – by induction on the number of free factors – the Green function (with radius of convergence $\mathbf{r}^*$) of the random walk on $\Gamma^* := \Gamma_1 * \ldots * \Gamma_{m-1}$ governed by $\mu^* := \sum_{j=1}^{m-1} \frac{\alpha_j}{\alpha_1+\ldots+\alpha_{m-1}} \bar{\mu}_j$ has an expansion either of the form

$$\text{(I)} \qquad G^*(z) = \sum_{k=0}^{D} g_k(\mathbf{r}^* - z)^k + \sum_{(q,k)\in\mathcal{T}} g_{(q,k)}(\mathbf{r}^* - z)^q \log^k(\mathbf{r}^* - z) + \mathcal{O}\big((\mathbf{r}^* - z)^{D+2}\big),$$

where $\mathcal{T}$ is a finite subset of $\{(q,k) \in \mathbb{R} \times \mathbb{N}_0 \mid D < q \leq D+2\}$ and $g_k, g_{(q,k)} \in \mathbb{R}$, or of the form

$$\text{(II)} \qquad G^*(z) = g_0 + g_1\sqrt{\mathbf{r}^* - z} + \sum_{(q,k)\in\mathcal{T}} g_{(q,k)}(\mathbf{r}^* - z)^q \log^k(\mathbf{r}^* - z) + \mathcal{O}\big((\mathbf{r}^* - z)^2\big),$$

where $\mathcal{T}$ is a finite subset of $\{(q,k) \in \mathbb{R} \times \mathbb{N}_0 \mid 1 < q \leq 2\}$ and $g_0, g_1, g_{(q,k))} \in \mathbb{R}$. Thus, we may apply the results from Section 3 to the free product $\Gamma^* * \Gamma_m$ equipped with $\mu = (\alpha_1 + \ldots + \alpha_{m-1})\mu^* + \alpha_m \bar{\mu}_m$ and obtain the proposed result. $\square$

6. Examples

6.1. **Free Products of Lattices.** Let $d_1, \ldots, d_m \in \mathbb{N}$. In this subsection we consider free products of the form $\Gamma := \mathbb{Z}^{d_1} * \ldots * \mathbb{Z}^{d_m}$, equipped with a nearest neighbour random walk. In the following subsection we show that the Green functions of nearest neighbour random walks on $\mathbb{Z}^d$ have an expansion as requested by (2.2). Afterwards we can give a complete classification of the asymptotic behaviour.

6.1.1. *Expansion of the Green Function on $\mathbb{Z}^d$.* Let $d \in \mathbb{N}$. Suppose we are given a probability measure π with $\mathrm{supp}(\pi) = \{\pm e_1, \ldots, \pm e_d\}$, the set of natural generators of $\mathbb{Z}^d$. Then π defines a random walk on $\mathbb{Z}^d$, and we denote by $\pi^{(n)}$ its n-fold convolution power. We write for $1 \leq i \leq d$

$$\beta_i := \pi(e_i) + \pi(-e_i) \quad \text{and} \quad p_i := \frac{\pi(e_i)}{\pi(e_i) + \pi(-e_i)}.$$

Let $\mathbf{0}$ be the zero vector in $\mathbb{Z}^d$. Once again $G_d(z) := \sum_{n\geq 0} \pi^{(n)}(\mathbf{0}) z^n$ denotes the associated Green function, which has radius of convergence $\mathbf{r}_d$. The crucial point for our later discussion is the following:

Proposition 6.1. *The Green function of the random walk on $\mathbb{Z}^d$ has an expansion of the form*

$$G_d(z) = \begin{cases} f(z) + g(z)(\mathbf{r}_d - z)^{(d-2)/2}, & \text{if } d \text{ is odd,} \\ f(z) + g(z)(\mathbf{r}_d - z)^{(d-2)/2} \log(\mathbf{r}_d - z), & \text{if } d \text{ is even,} \end{cases}$$

where the functions $f(z), g(z)$ are analytic in a neighbourhood of $z = \mathbf{r}_d$ and $g(\mathbf{r}_d) \neq 0$.

Remarks: For the case of simple random walks on $\mathbb{Z}^d$ a proof of this proposition can be found in [22, Proposition 17.16]. In our case, we generalize the statement to arbitrary nearest neighbour random walks on $\mathbb{Z}^d$, but we will only give a sketch of the (elementary) proof and refer once again to [22]. From the expansion follows with the help of Darboux's method that $\pi^{(2n)}(\mathbf{0}) \sim C\,\mathbf{r}_d^{-2n}\,n^{-d/2}$; this asymptotic behaviour follows also from Cartwright and Soardi [4].

Proof. First, note that the spectral radius of the random walk on $\mathbb{Z}^d$ is given by

$$\varrho = \sum_{i=1}^{d} \beta_i \sqrt{4p_i(1-p_i)} = \frac{1}{\mathbf{r}_d};$$

compare with [22, Theorem 8.23]. We define random walks on $\mathbb{Z}$ governed by probability measures π_i with $\pi_i(1) := p_i$ and $\pi_i(-1) := 1-p_i$. For $z \in \mathbb{C}$, the *exponential generating function* on $\mathbb{Z}^d$ is given by

$$E(z) := \sum_{n=0}^{\infty} \pi^{(n)}(\mathbf{0}) \frac{z^n}{n!}$$

and on the i-th coordinate axis it is given by

$$E_i(z) := \sum_{n\geq 0} \pi_i^{(n)}(0) \frac{z^n}{n!} = \int_{-1}^{1} e^{\sqrt{4p_i(1-p_i)}tz} \frac{1}{\pi\sqrt{1-t^2}} dt.$$

In the last equation we applied the following relation, which is easy to check:

$$\pi_i^{(n)}(0) = \int_{-1}^{1} \sqrt{4p_i(1-p_i)}^{\,n}\, t^n \frac{1}{\pi\sqrt{1-t^2}} dt.$$

Furthermore, we get $E(z) = \prod_{i=1}^{d} E_i(\beta_i z) = \int_{-\varrho}^{\varrho} e^{tz} \big(\hat{f}_1 * \ldots * \hat{f}_d\big)(t) dt$, where

$$\hat{f}_i(t) := \frac{1}{\beta_i\sqrt{4p_i(1-p_i)}} f_0\Big(\frac{t}{\beta_i\sqrt{4p_i(1-p_i)}}\Big) \text{ and } f_0(t) := \begin{cases} \frac{1}{\pi\sqrt{1-t^2}}, & \text{if } t \in (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

This allows us to rewrite the Green function in the following way:

$$G_d(z) = \int_{-\varrho}^{\varrho} \frac{1}{1-zt} \big(\hat{f}_1 * \ldots * \hat{f}_d\big)(t)\, dt. \tag{6.1}$$

Moreover, there is a function $g_d(t)$, which is analytic in a neighbourhood of $t = \varrho$ and satisfies $g_d(\varrho) \neq 0$ such that

$$\big(\hat{f}_1 * \ldots * \hat{f}_d\big)(t) = (\varrho - t)^{(d-2)/2} g_d(t). \tag{6.2}$$

To prove this, we define $\bar{f}_i(t) := \hat{f}_i\big(\beta_i\sqrt{4p_i(1-p_i)} - t\big)$ and show inductively that we can write

$$(\bar{f}_1 * \ldots * \bar{f}_d)(t) = t^{(d-2)/2} \bar{g}_d(t),$$

where the function $\bar{g}_d(t)$ is analytic in a neighbourhood of $t = 0$ and $\bar{g}_d(0) \neq 0$. Analogously to the proof of [22, Proposition 17.16], we may conclude together with (6.1) and (6.2) that $G_d(z)$ has the proposed expansion. □

6.1.2. *Classification of the Asymptotic Behaviour.* Observe that a nearest neighbour random walk on $\mathbb{Z}^d$ has period 2 since it can come back to the origin only in an even number of steps. Therefore, the period of a nearest neighbour random walk on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ is $\delta = 2$. Now we can give a complete classification of the asymptotic behaviour of n-step return probabilities of nearest neighbour random walks on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$:

Theorem 6.2. *Consider irreducible nearest neighbour random walks on the lattices $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ with $d_1 \leq d_2$. Then the n-step return probabilities of the associated random walk on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ obey one the following laws:*

$$\mu^{(2n)}(e) \sim \begin{cases} C_1 \cdot \mathbf{r}^{-2n} \cdot n^{-d_1/2}, & \textit{if } d_1 \geq 5 \textit{ and } \Psi(\bar{\theta}) > 0 \textit{ and } \bar{\theta} = \theta_1/\alpha_1, \\ C_2 \cdot \mathbf{r}^{-2n} \cdot n^{-d_2/2}, & \textit{if } d_2 \geq 5 \textit{ and } \Psi(\bar{\theta}) > 0 \textit{ and } \bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1, \\ C_3 \cdot \mathbf{r}^{-2n} \cdot n^{-3/2}, & \textit{otherwise.} \end{cases}$$

□

Consider now the multi-factor free product $\mathbb{Z}^{d_1} * \ldots * \mathbb{Z}^{d_m}$. Let μ_i be the simple random walk on $\mathbb{Z}^{d_i}$ for each $i \in \{1, \ldots, m\}$ and choose $\alpha_1, \ldots, \alpha_m > 0$ with $\sum_{j=1}^m \alpha_j = 1$. Let $G_i(z)$ denote the Green function of the simple random walk on $\mathbb{Z}^{d_i}$, which has radius of convergence $\mathbf{r}_i = 1$, and define $\Psi_i(t)$ analogously as in (2.8). Cartwright [1] computed numerically some of the values of $\Psi_i\big(G_i(1)\big)$ and showed that $\Psi_i\big(G_i(1)\big) \to 1$ if $d_i \to \infty$. Thus, for large d_i we have $\Psi_i\big(G_i(1)\big) > 1 - 1/m$. Recall also that $\Psi_i(t)$ is decreasing. Denote by $G(z)$ the Green function of the random walk on $\mathbb{Z}^{d_1} * \ldots * \mathbb{Z}^{d_m}$ and by $\mathbf{r}$ its radius of convergence, and define $\Psi(t)$ analogously as in (2.8). By [22, Equation 9.21],

$$\Psi(\bar{\theta}) = 1 + \sum_{j=1}^{m} \big(\Psi_i(\alpha_i \bar{\theta}) - 1\big),$$

where $\bar{\theta} = \min_{1 \leq i \leq m} \theta_i/\alpha_i$. If all exponents $d_i \geq 5$ are big enough, we get $\Psi(\bar{\theta}) > 0$. Furthermore, if α_i is chosen big enough, we get an asymptotic behaviour of type $C_i\, \mathbf{r}^{-2n}\, n^{-d_i/2}$. Moreover, one can define (symmetric) measures $\mu_1, \ldots, \mu_m$ supported on the natural generators in such a way that we obtain a $C_0\, \mathbf{r}^{-2n}\, n^{-3/2}$-law: one chooses μ_1 and μ_2 such that $\Psi_1(\theta_1), \Psi_2(\theta_2) < 1/2$ and chooses α_1 and α_2 such that $\bar{\theta} = \theta_1/\alpha_1 = \theta_2/\alpha_2$, yielding $\Psi(\bar{\theta}) < 0$; see comments at the end of Section 2. That is, we can have $m+1$ different asymptotic behaviours. This finally proves Theorem 1.1.

For instance, consider $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6 * \mathbb{Z}^7$ equipped with simple random walks μ_1, μ_2 and μ_3 on each free factor. For $i \in \{1, 2, 3\}$, we define $\Psi_i(t)$ analogously to (2.8). Cartwright [1] computed the values $\Psi_1\big(G_1(1)\big) = 0.691$, $\Psi_2\big(G_2(1)\big) = 0.824$ and $\Psi_3\big(G_3(1)\big) = 0.876$. Thus, the random walk on $\mathbb{Z}^5 * \mathbb{Z}^6$ governed by $\mu_{12} := \alpha_1^* \bar{\mu}_1 + \alpha_2^* \bar{\mu}_2$, where $\alpha_1^* = \alpha_1/(\alpha_1+\alpha_2)$ and $\alpha_2^* = \alpha_2/(\alpha_1+\alpha_2)$, satisfies $\Psi(M) \geq 0.515$ with $M := \min\{\theta_1/\alpha_1^*, \theta_2/\alpha_2^*\}$. That is, $M = \mathbf{r}_{1.2} G_{1,2}(\mathbf{r}_{1,2})$, where $G_{1,2}(z)$ is the Green function of the random walk on $\mathbb{Z}^5 * \mathbb{Z}^6$ with radius of convergence $\mathbf{r}_{1,2}$. Since all Ψ_i-functions are strictly decreasing, we obtain for the random walk on $\Gamma = \Gamma_1 * \Gamma_2$ with $\Gamma_1 = \mathbb{Z}^5 * \mathbb{Z}^6$ and $\Gamma_2 = \mathbb{Z}^7$:

$$\Psi(\bar{\theta}) \quad = \quad \Psi_1\big((\alpha_1+\alpha_2)\bar{\theta}\big) + \Psi_2(\alpha_3\bar{\theta}) - 1 \geq 0.515 + 0.876 - 1 > 0.$$

For the simple random walk on Γ, we have then the asymptotic non-exponential type $n^{-7/2}$, if $\alpha_1+\alpha_2 < M/\big(M+G_3(1)\big)$. Otherwise, we have the asymptotic behaviour $n^{-5/2}$ or n^{-3}, if $M=\theta_1/\alpha_1^*$ or $M=\theta_2/\alpha_2^*\neq\theta_1/\alpha_1^*$ respectively.

6.2. $(\mathbb{Z}/m\mathbb{Z})*\mathbb{Z}^d$**.** Consider the groups $\Gamma_1=\mathbb{Z}/m\mathbb{Z}$ and $\Gamma_2=\mathbb{Z}^d$ for any $m\geq 2, d\in\mathbb{N}$. Suppose we are given a probability measure μ_1 on Γ_1 and a probability measure μ_2 on $\mathbb{Z}^d$, which is supported on the natural generators. Then $G_1(1)=\infty$, and thus we get the following classification:

$$\mu^{(n\delta)}(e)\sim\begin{cases} C_1\cdot\mathbf{r}^{-n\delta}\cdot n^{-d/2}, & \text{if } d\geq 5 \text{ and } \Psi(\bar\theta)>0,\\ C_2\cdot\mathbf{r}^{-n\delta}\cdot n^{-3/2}, & \text{otherwise.}\end{cases}$$

6.3. $\Pi_q*\mathbb{Z}^d$**.** Consider the groups $\Gamma_1=\Pi_q:=*_{i=1}^q(\mathbb{Z}/2\mathbb{Z})$ and $\Gamma_2=\mathbb{Z}^d$ for any $q\geq 2$, $d\in\mathbb{N}$. Observe that the Cayley graph of Γ_1 is the homogeneous tree of degree q. Suppose we are given probability measures μ_1 on Γ_1 and μ_2 on $\mathbb{Z}^d$, which are both supported on the natural generators. If $q=2$ then $G_1(1)=\infty$, and thus we get the same classification as in the case $(\mathbb{Z}/m\mathbb{Z})*\mathbb{Z}^d$. If $q\geq 3$, then it is well-known that $G_1(z)$ can be written as

$$G_1(z)=A(z)+\sqrt{\mathbf{r}_1-z}\,B(z),$$

where $A(z),B(z)$ are analytic in a neighbourhood of $z=\mathbf{r}_1$ and $B(\mathbf{r}_1)\neq 0$; see e.g. Woess [23, Equation (4.5)]. Therefore, we get the following classification for the associated random walk on the free product $\Gamma_1*\Gamma_2$:

$$\mu^{(2n)}(e)\sim\begin{cases} C_1\cdot\mathbf{r}^{-2n}\cdot n^{-d/2}, & \text{if } d\geq 5 \text{ and } \bar\theta=\theta_2/\alpha_2<\theta_1/\alpha_1 \text{ and } \Psi(\bar\theta)>0,\\ C_2\cdot\mathbf{r}^{-2n}\cdot n^{-3/2}, & \text{otherwise.}\end{cases}$$

7. Classification of Phase Transitions

Let us return to the case $m=2$, that is, $\Gamma=\Gamma_1*\Gamma_2$. Now we *fix* the measures μ_1 and μ_2, and investigate the variation of $\Psi(\bar\theta)$ as a function of the parameter α_1.

Lemma 7.1. *Assume $\bar\theta<\infty$. Then the function $\Upsilon(\cdot):(0,1)\mapsto\mathbb{R}$ defined by*

$$\Upsilon(\alpha_1):=\Psi_1(\alpha_1\bar\theta)+\Psi_2((1-\alpha_1)\bar\theta)-1$$

is continuous, strictly decreasing in the interval $\big(0,\frac{\theta_1}{\theta_1+\theta_2}\big]$ and strictly increasing in the interval $\big[\frac{\theta_1}{\theta_1+\theta_2},1\big)$. (We set $\frac{c}{c+\infty}:=0$ and $\frac{\infty}{\infty+c}:=1$ for $c\in(0,\infty)$.)

Proof. We leave the proof of continuity of Υ as an easy exercise to the reader, since Ψ_i is analytic in an open neighbourhood of the interval $[0,\theta_i)$.

Note that $\Upsilon(\alpha_1)=\Psi(\bar\theta)$. We divide the proof into two parts, according to finiteness of θ_1 and θ_2.

Case $\theta_1,\theta_2<\infty$. If $0<\alpha_1<\frac{\theta_1}{\theta_1+\theta_2}$ then $\bar\theta=\theta_2/\alpha_2$. Consequently, we have

$$\Upsilon(\alpha_1)=\Psi_1\Big(\frac{\alpha_1}{1-\alpha_1}\theta_2\Big)+\Psi_2(\theta_2)-1.$$

Since the function $\frac{\alpha_1}{1-\alpha_1}$ is strictly increasing, it follows that $\Psi_1(\frac{\alpha_1}{1-\alpha_1}\theta_2)$ is strictly decreasing, implying $\Upsilon(\alpha_1)$ strictly decreasing.
If $\alpha_1 = \frac{\theta_1}{\theta_1+\theta_2}$ we obtain $\bar{\theta} = \theta_1/\alpha_1 = \theta_2/\alpha_2$, that is, $\Upsilon(\alpha_1) = \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1$.
If $\frac{\theta_1}{\theta_1+\theta_2} < \alpha_1 < 1$ we have $\Psi(\bar{\theta}) = \Psi_1(\theta_1) + \Psi_2(\frac{1-\alpha_1}{\alpha_1}\theta_1) - 1$. Since $\frac{1-\alpha_1}{\alpha_1}$ is strictly decreasing, $\Upsilon(\alpha_1)$ is a strictly increasing function in the abovementioned interval.

Case $\theta_1 = \infty$. Then $\bar{\theta} = \frac{\theta_2}{1-\alpha_1}$. The same reasoning as before shows that $\Upsilon(\alpha_1)$ is strictly decreasing in the interval $(0,1)$.

Case $\theta_2 = \infty$. Then $\bar{\theta} = \frac{\theta_1}{\alpha_1}$. Analogously, $\Upsilon(\alpha_1)$ is strictly increasing in the interval $(0,1)$. □

Let us remark that $\bar{\theta} = \infty$ implies $\Psi(\bar{\theta}) < 0$ (see [22, Theorem 9.22]); otherwise we would have a contradiction to ρ-transience.

Now we can give a complete picture of the *phase transition* of the asymptotic behaviour of the return probabilities depending on the parameter α_1, and we present specific examples. In the following we discuss the different possible behaviours of the function $\Upsilon(\alpha_1) = \Psi(\bar{\theta})$. In Figure 1, the dashed line will represent *approximately* the qualitative behaviour of $\Upsilon(\alpha_1)$; we denote its zeros (if they exist) by α_{low} and α_{high} (with $\alpha_{\text{low}} \leq \alpha_{\text{high}}$). Moreover, we write $\alpha_c := \theta_1/(\theta_1+\theta_2)$. We decompose the interval $(0,1)$ into subintervals such that every choice of α_1 in a fixed subinterval leads to the same non-exponential type. With the help of Figure 1 we discuss case by case the different behaviours of $\Upsilon(\alpha_1)$, and for each case we give an example of a nearest neighbour random walk on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$. Recall that $\Psi(0) = \Psi_i(0) = 1$.

Case A: We give an example such that this case holds. We set $\Gamma = \mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ with $d_1, d_2 \geq 5$, and we choose μ_1 and μ_2 such that $\Psi_1(\theta_1) < 1/2$ and $\Psi_2(\theta_2) < 1/2$. Recall that it is possible to find such measures (see end of Section 2 and [22, Lemma 17.9]). We remark that $\Psi_i(\theta_i) > 0$: indeed, $\Psi_i(\theta_i) = 0$ would imply

$$\Phi_i'(\theta_i) = \frac{\Phi_i(\theta_i)}{\theta_i} = \frac{G_i(\mathbf{r}_i)}{\mathbf{r}_i G_i(\mathbf{r}_i)} = \frac{1}{\mathbf{r}_i}.$$

Differentiating (2.7) would yield $G_i'(\mathbf{r}_i) = \infty$, a contradiction to Proposition 6.1, according to which, $G_i'(\mathbf{r}_i)$ must be finite due to $d_i \geq 5$.

- If α_1 is small then $\bar{\theta} = \theta_2/(1-\alpha_1)$ and

$$\Psi(\bar{\theta}) = \underbrace{\Psi_1\Big(\underbrace{\alpha_1\frac{\theta_2}{1-\alpha_1}}_{\xrightarrow{\alpha_1\to 0} 0}\Big)}_{\xrightarrow{\alpha_1\to 0} 1} + \underbrace{\Psi_2(\theta_2)}_{>0} - 1, \tag{7.1}$$

  that is, $\Psi(\bar{\theta}) > 0$ if α_1 is sufficiently small. This yields a $n^{-d_2/2}$-law for small values of α_1.
- If α_1 is close to 1 then $\bar{\theta} = \theta_1/\alpha_1$ and we get analogously an $n^{-d_1/2}$-law.

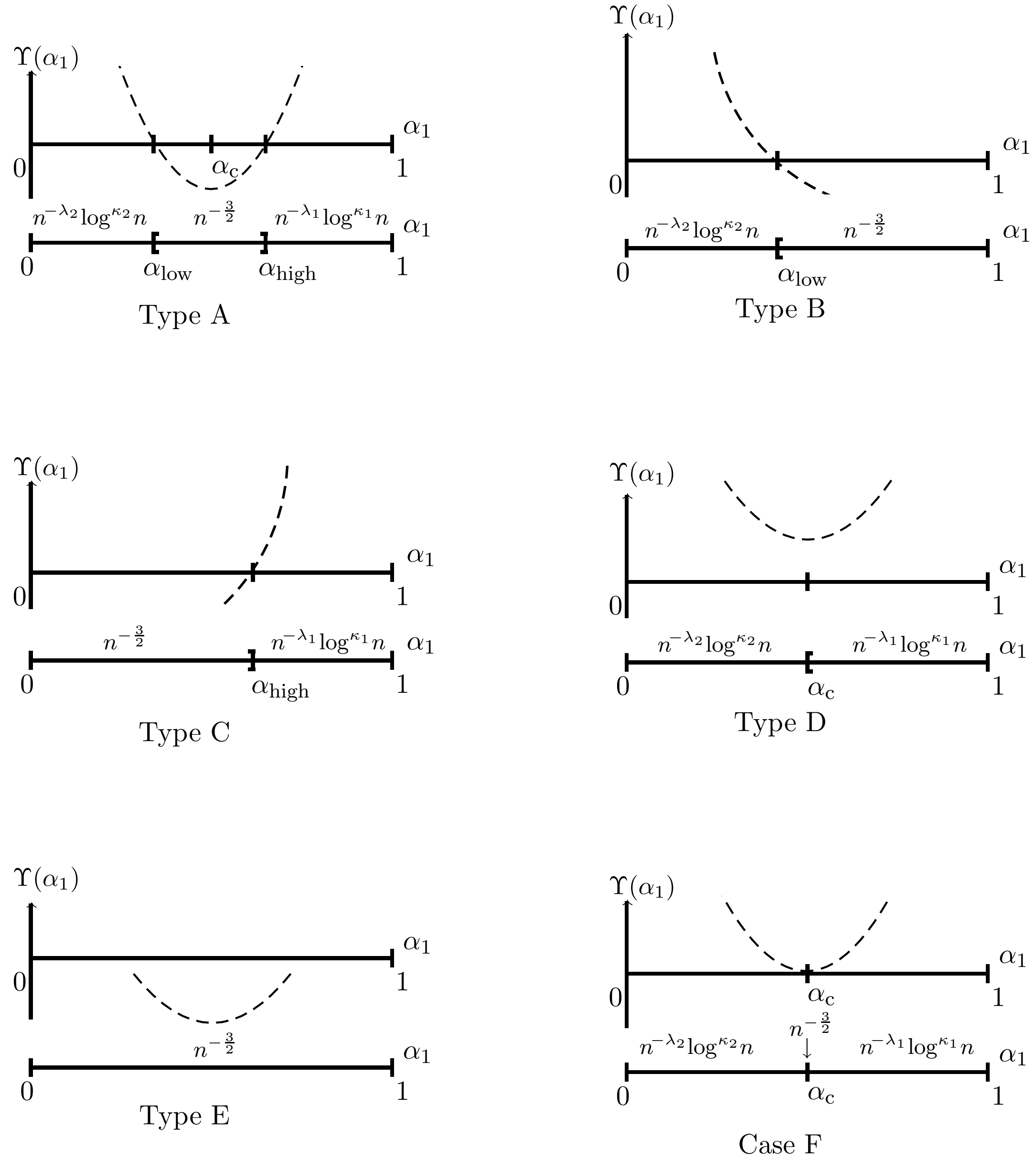

FIGURE 1. The different behaviours of $\alpha_1 \mapsto \Psi(\bar{\theta})$.

- For $\alpha_1 = \alpha_c$, we get $\Psi(\bar{\theta}) = \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1 < 0$, that is, we have a $n^{-3/2}$-law in this case.

Case B: We set $\Gamma = \mathbb{Z}^2 * \mathbb{Z}^7$. By Lemma 7.1, $\Upsilon(\alpha_1)$ is strictly decreasing and $\bar{\theta} = \theta_2/\alpha_2$.

- If α_1 is small then the same reasoning as in (7.1) holds and $\Psi(\bar{\theta}) > 0$, that is, we have a $n^{-d_2/2}$-law for small α_1.

– If α_1 is close to 1 then

$$\Psi(\bar\theta) = \underbrace{\Psi_1\Big(\underbrace{\alpha_1 \frac{\theta_2}{1-\alpha_1}}_{\xrightarrow{\alpha_1\to 1}\infty}\Big)}_{\xrightarrow{\alpha_1\to 1}0} + \underbrace{\Psi_2(\theta_2)}_{<1} - 1 < 0,$$

since $\lim_{t\to\infty}\Psi_1(t) = 0$, which follows analogously to (5.1). That is, we have a $n^{-3/2}$-law for large α_1.

Case C: By setting $\Gamma = \mathbb{Z}^7 * \mathbb{Z}^2$, we have the symmetric situation as in Case B, which gives an example for this case by exchanging the roles of $\mathbb{Z}^2$ and $\mathbb{Z}^7$.

Case D: We set $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6$ and consider simple random walks on the factors $\mathbb{Z}^5$ and $\mathbb{Z}^6$. By Cartwright [1], we have $\Psi_1(\theta_1) = 0.691$ and $\Psi_2(\theta_2) = 0.824$. Since $\Psi_1(z)$ and $\Psi_2(z)$ are strictly decreasing, we have $\Upsilon(\alpha_1) \geq \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1 > 0$ for all $\alpha_1 \in (0,1)$. Thus, we obtain a $n^{-5/2}$-law, if $\alpha_1 \geq \alpha_c$, and a n^{-3}-law, if $\alpha_1 < \alpha_c$.

Case E: We set $\Gamma = \mathbb{Z}^3 * \mathbb{Z}^4$. By Equation (5.1), follows that $\Psi_1(\alpha_1\bar\theta) = 0$ or $\Psi_2(\alpha_2\bar\theta) = 0$, that is, we have $\Upsilon(\alpha_1) < 0$ for all $\alpha_1 \in (0,1)$. This yields a $n^{-3/2}$-law for all $\alpha_1 \in (0,1)$.

We now give an example (see Case F of Figure 1) where the $n^{-3/2}$-interval of case A collapses to a singleton. For this purpose, we have to prove the following:

Lemma 7.2. *Consider $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6$. Then there are probability measures μ_1 and μ_2, supported on the natural generators of $\mathbb{Z}^5$ and $\mathbb{Z}^6$ respectively, such that $\Psi_1(\theta_1) = \Psi_2(\theta_2) = \frac{1}{2}$.*

Proof. Let $i \in \{1,2\}$. We have $d_1 = 5, d_2 = 6$ and choose any $\delta \in (0,1)$. We define

$$\nu_\delta^{(i)}(x) := \begin{cases} (1-\delta)/2, & \text{if } x = (\pm 1, 0, \dots, 0) \in \mathbb{Z}^{d_i} \\ \frac{\delta}{2d_i - 2}, & \text{if } x = (0,\dots,0,\pm 1, 0, \dots, 0) \in \mathbb{Z}^{d_i} \setminus \{(\pm 1, 0, \dots, 0)\} \end{cases}.$$

The Green function associated with the random walk on $\mathbb{Z}^{d_i}$ governed by $\nu_\delta^{(i)}$ is symmetric, that is, its radius of convergence is $\mathbf{r}_i = 1$; see [22, Cor. 8.15]. If $\delta = 1 - 1/d_i$ then $\Psi_1(\theta_1) = 0.691 > 1/2$ and $\Psi_2(\theta_2) = 0.824 > 1/2$; see Cartwright [1]. On the other hand side, if δ is small enough then $\Psi_1(\theta_1) < 1/2$ and $\Psi_2(\theta_2) < 1/2$; see proof of [22, Lemma 17.9]. It remains to show that $\Psi_i(\theta_i)$ varies continuously in dependence of δ, which implies that there is some $\delta_0^{(i)}$ such that $\Psi_i(\theta_i) = 1/2$. We now write $G_i(z) = G_i(\delta|z)$, $U_i(z) = U_i(\delta|z)$ and $\Psi_i(t) = \Psi_i(\delta|t)$. Recall that

$$\Psi_i(\delta|\theta_i) = \frac{1}{U_i'(\delta|1) + 1 - U_i(\delta|1)}.$$

Since $U_i(\delta|1)$ can be rewritten as a power series in the variable δ, the function $\delta \mapsto \Psi_i(\delta|\theta_i)$ is continuous in δ. This finishes the proof. □

We can now present an example, where Case F of Figure 1 holds: we set $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6$ and choose the measures μ_1 and μ_2 such that $\Psi_1(\theta_1) = \Psi_2(\theta_2) = 1/2$. Obviously, we have then $\Upsilon(\alpha_c) = \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1 = 0$. That is, we get a $n^{-5/2}$-law, if $\alpha_1 > \alpha_c$, a n^{-3}-law, if $\alpha_1 < \alpha_c$, and a $n^{-3/2}$-law for $\alpha_1 = \alpha_c$.

As a final remark let us explain that it is not possible that $\Upsilon(\alpha_1)$ is strictly increasing or decreasing with $\Upsilon(\alpha_1) > 0$ for all $\alpha_1 \in (0,1)$. Assume that $\Upsilon(\alpha_1)$ is strictly increasing. Then, by Lemma 7.1, $\theta_2 = \infty$ must hold, that is, $G_2(\mathbf{r}_2) = \infty$. The same reasoning as in Equation (5.1) leads to $\lim_{z\to\mathbf{r}_2} \Psi_2\bigl(zG(z)\bigr) = \lim_{t\to\infty} \Psi_2(t) = 0$. Therefore, we obtain for α_1 small enough

$$\Psi(\bar{\theta}) = \underbrace{\Psi_1(\theta_1)}_{<1} + \underbrace{\Psi_2\Bigl(\underbrace{(1-\alpha_1)\frac{\theta_1}{\alpha_1}}_{\xrightarrow{\alpha_1\to 0}\infty}\Bigr)}_{\xrightarrow{\alpha_1\to 0}0} - 1 < 0.$$

Analogously, if $\Upsilon(\alpha_1)$ is strictly decreasing, then it must have a zero.

8. Higher Asymptotic Orders

The techniques we used for determining the asymptotic behaviour give us not only the leading term $n^{-\lambda}\log^{\kappa} n$, but also the proceeding terms of *higher order*, according to the singular terms in the expansion following the leading one. For instance, consider a nearest neighbour random walk on $\mathbb{Z}^7 * \mathbb{Z}^8$ with $\alpha_1 = \theta_1/(\theta_1+\theta_2)$. Then the associated Green function has the following expansion:

$$\begin{aligned}\sum_{k=0}^{4} g_k(\mathbf{r}-z)^4 &+ \hat{g}_1(\mathbf{r}-z)^{5/2} + \check{g}_1(\mathbf{r}-z)^3\log(\mathbf{r}-z)\\ &+\hat{g}_2(\mathbf{r}-z)^{7/2} + \check{g}_2(\mathbf{r}-z)^4\log(\mathbf{r}-z) + \mathbf{o}\bigl((\mathbf{r}-z)^4\bigr),\end{aligned}$$

where $\hat{g}_1 \neq 0$. That is,

$$\mu^{(2n)}(e) \sim \mathbf{r}^{-2n}\cdot\bigl(C_1\,n^{-7/2} + C_2\,n^{-4} + C_3\,n^{-9/2} + C_4\,n^{-5} + \mathbf{o}(n^{-5})\bigr),$$

where $C_1 \neq 0$.

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