

BRANCHING RANDOM WALKS ON FREE PRODUCTS OF GROUPS

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ABSTRACT. We study certain phase transitions of branching random walks (BRW) on Cayley graphs of free products. The aim of this paper is to compare the size and structural properties of the trace, i.e., the subgraph that consists of all edges and vertices that were visited by some particle, with those of the original Cayley graph. We investigate the phase when the growth parameter λ is small enough such that the process survives but the trace is not the original graph. A first result is that the box-counting dimension of the boundary of the trace exists, is almost surely constant and equals the Hausdorff dimension which we denote by $\Phi(\lambda)$. The main result states that the function $\Phi(\lambda)$ has only one point of discontinuity which is at $\lambda_c = R$ where R is the radius of convergence of the Green function of the underlying random walk. Furthermore, $\Phi(R)$ is bounded by one half the Hausdorff dimension of the boundary of the original Cayley graph and the behaviour of $\Phi(R) - \Phi(\lambda)$ as $\lambda \uparrow R$ is classified.

In the case of free products of infinite groups the end-boundary can be decomposed into words of finite and words of infinite length. We prove the existence of a phase transition such that if $\lambda \leq \tilde{\lambda}_c$ the end boundary of the trace consists only of infinite words and if $\lambda > \tilde{\lambda}_c$ it also contains finite words. In the last case, the Hausdorff dimension of the set of ends (of the trace and the original graph) induced by finite words is strictly smaller than the one of the ends induced by infinite words.

1. INTRODUCTION

A branching random walk (BRW) is a *growing cloud* of particles that move on an underlying graph \mathcal{X} in discrete time. The process starts with one particle in the root e of the graph. Then at each discrete time step a particle produces offspring particles according to some offspring distribution with mean $\lambda > 1$, and then each descendent moves one step according to a random walk on \mathcal{X} . Particles branch and move independently of the other particles and the history of the process. A first natural question is to ask whether the process eventually *fills up* the whole graph, that is, if every finite subset will eventually be occupied or free of particles. If the BRW visits the whole graph it is called recurrent and transient otherwise. As a consequence of Kesten's amenability criterion any BRW is recurrent on the Cayley graph of an amenable group. Furthermore, one observes a phase transition on non-amenable groups; there exists some $\lambda_c > 1$ such that a BRW with $\lambda \leq \lambda_c$ is transient, while it is recurrent otherwise. In the transient case the trace of the BRW, that is, the subgraph that consists of all edges and vertices that were visited by the BRW, is a

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proper random subgraph of the original Cayley graph. Benjamini and Müller [1] studied first general qualitative statements of the trace of BRW on groups. In particular, they proved exponential volume growth of the trace in general. However, their approach is rather qualitative and gives no quantitative results on the growth rate. In this article we study BRW on free products of groups and obtain a precise formula for the growth rate and dimensions of the end boundary of the trace. One motivation to study BRW on this class of structures lies mainly in the fact that they are among the simplest non-amenable groups. This makes them to a reference and starting point for more complicated non-amenable structures such as, for instance, groups with infinitely many ends or hyperbolic groups. Besides this, free products of groups are interesting on their own since they play an important role in some fields of algebraic topology and in Bass–Serre theory.

The starting point of the present investigation of branching random walks was the work of Hueter and Lalley [13], who studied BRW on homogeneous trees. We remark that in their setting and notation weak survival is equivalent to transience in our language. In the transient regime the BRW eventually vacates every finite subset and the particle trails converge to the geometric end boundary Ω of the tree. The *limit set* Λ of the BRW is the random subset of the boundary that consists of all ends, where the BRW accumulates. By this we mean that each neighbourhood of an end in Λ is visited infinitely often by the process. Equivalently, we can define Λ as the geometric end boundary of the trace.

Typical ways of measuring the size of boundaries are by use of the box-counting dimension (also known as Minkowski dimension) or the Hausdorff dimension. In [13] a formula for the Hausdorff dimension of Λ is given for BRW on homogeneous trees. In particular, it is shown there that the limit set has Hausdorff dimension no larger than one half the Hausdorff dimension of the entire boundary Ω . We extend these results to BRW on free products of groups. We prove existence of the box-counting dimension, show that the Hausdorff dimension equals the box-counting dimension and present a formula in terms of generating functions of the underlying random walk, see Theorem 3.5. In the same way we obtain a formula for the Hausdorff dimension of the whole space of ends, see Theorem 3.8. This eventually leads to the result that the Hausdorff dimension of Λ is not larger than one half the Hausdorff dimension of the entire boundary. Another consequence of the formula of the Hausdorff dimension is that the dimension varies continuously in the subcritical regime, see Theorem 3.10. This affirms the conjecture made in [1] for general non-amenable groups that the Hausdorff dimension of the limit set is continuous for $\lambda \neq \lambda_c$ and discontinuous at λ_c . As pointed out in [13] the very same phenomenon hold for other growth processes (e.g. hyperbolic branching Brownian motion, isotropic contact process on homogeneous trees) that exhibit a phase transition between *weak* and *strong survival*.

In [13] the behaviour of the critical BRW on the free group was studied in more detail and two phenomena were observed. First, $\Phi(R) = \text{HD}(\Omega)/2$ if and only if the underlying random walk is the simple random walk. This statement is not true for our more general setting since there are non-simple random walks that attain the maximal Hausdorff dimension $\text{HD}(\Omega)/2$, see Remark 3.12 together with Example 3.14. Second, it was shown in [13] that $\Phi(R) - \Phi(\lambda) \sim C\sqrt{R - \lambda}$ as $\lambda \uparrow R$. For free products of groups this behaviour turns out to be more subtle: $\Phi(R) - \Phi(\lambda)$ may behave like $C(R - \lambda)$ or $C\sqrt{R - \lambda}$ depending on whether the Green function is differentiable at its radius of convergence or not.

The very same phenomena were also studied in the continuous setting. Lalley and Sellke [18] studied the phase transition for branching Brownian motion on the hyperbolic disc and Karpelevich, Pechersky, and Suhov [14] generalized these results to higher dimensional Lobachevsky spaces. Grigor'yan and Kelbert [11] studied recurrence and transience for branching diffusion processes on Riemannian manifolds. In Cammarota and Orsingher [3] first results on a “linear” growing system of particles on the hyperbolic disc are given.

In the case of free products of groups $\Gamma = \Gamma_1 * \dots * \Gamma_r$, where at least one of the factors is infinite, another phase transition occurs. The boundary Ω can be decomposed into up to $r + 1$ direct summands. For $1 \leq i \leq r$, let Ω_i denote the set of ends described by semi-infinite non-backtracking paths, which eventually stay in one copy of Γ_i . The set Ω_∞ consists of all ends described by infinite, non-backtracking paths that change the different copies of the free factors infinitely many times. Now, for all infinite Γ_i , Theorem 3.1 gives a criterion whether $\Lambda \cap \Omega_i \neq \emptyset$ almost surely. In particular, it states that there exists a critical value λ_i such that $\lambda \leq \lambda_i$ is equivalent to $\Lambda \cap \Omega_i = \emptyset$ almost surely. In other words, if we increase the growth parameter λ then more and more different parts of the boundary appear in Λ . However, even if $\Lambda \cap \Omega_i \neq \emptyset$, only the infinite words contribute to the Hausdorff dimension of Λ , see Corollary 3.7.

Finally, for the case of free products of *finite* groups we slightly adapt the metric defined on the boundary and get (following analogously the reasoning in [13]) a simpler formula for the Hausdorff dimension of Λ , see Corollary 3.16. Analogously, we obtain a formula for the Hausdorff dimension of Λ if we have a BRW on free products by amalgamation of *finite* groups, see Corollary 3.18. In both cases the Hausdorff dimension can be expressed through a Perron–Frobenius eigenvalue.

Let us remark that free products have been studied in great variety. Asymptotic behaviour of return probabilities of random walks on free products has been studied in many ways; e.g. Gerl and Woess [7], [24], Sawyer [22], Cartwright and Soardi [5], Lalley [16], and Candellero and Gilch [4]. For free products of finite groups, Mairesse and Mathéus [19] computed an explicit formula for the drift and asymptotic entropy. Gilch [9], [10] computed different formulas for the drift and also for the entropy for random walks on free products of graphs. Our proofs involve in a very crucial way generating functions techniques for free products. These techniques were introduced independently and simultaneously by Cartwright and Soardi [5], Woess [24], Voiculescu [23], and McLaughlin [20]. In particular, we show that the Hausdorff dimension can be computed as the solution of a functional equation in terms of double generating functions.

The structure of the paper is as follows. In Section 2 we give an introduction to random walks on free products, generating functions, and branching random walks. In Section 3 we state our results and illustrate them with sample computations. The proofs are given in Section 4.

2. BRANCHING RANDOM WALKS ON FREE PRODUCTS

2.1. Free Products of Groups and Random Walks. Let $\mathcal{I} = \{1, 2, \dots, r\}$ be a finite index set. Suppose we are given finitely generated groups Γ_i , $i \in \mathcal{I}$, where each Γ_i is

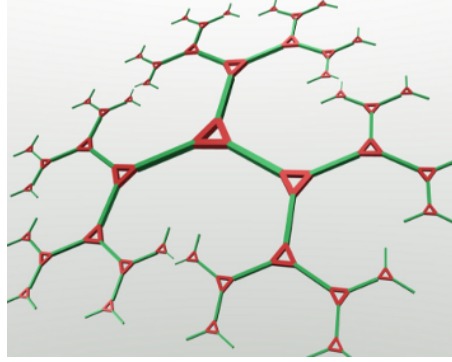


FIGURE 1. Structure of the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$.

generated by a symmetric generating set S_i (that is, $s \in S_i$ implies $s^{-1} \in S_i$) with identity e_i . Let $\Gamma_i^\times := \Gamma_i \setminus \{e_i\}$ for every $i \in \mathcal{I}$ and let $\Gamma_*^\times := \bigcup_{i \in \mathcal{I}} \Gamma_i^\times$. The *free product* $\Gamma := \Gamma_1 * \dots * \Gamma_r$ is defined as the set

$$\left\{ x_1 x_2 \dots x_n \mid n \in \mathbb{N}, x_j \in \Gamma_*^\times, x_j \in \Gamma_k^\times \Rightarrow x_{j+1} \notin \Gamma_k^\times \right\} \cup \{e\}. \quad (2.1)$$

That is, each element of Γ is a *word* $x_1 \dots x_n$ such that each letter (also called *block*) x_i is a non-trivial element of one of the Γ_i 's and two consecutive letters are not from the same free factor Γ_i ; e denotes the empty word. We exclude the trivial cases where Γ_i is the trivial group and the case $r = 2 = |\Gamma_1| = |\Gamma_2|$; see beginning of Subsection 2.2 for further remarks. The group operation on the free product Γ can be described as follows: if $u = u_1 \dots u_m, v = v_1 \dots v_n \in \Gamma$ then uv stands for their concatenation as words with possible contractions and cancellations in the middle in order to get the form of (2.1). For instance, if $u = aba$ and $v = abc$ with $a, c \in \Gamma_1^\times, b \in \Gamma_2^\times$ and $a^2 = e_1, b^2 \neq e_2$, then $uv = (aba)(abc) = a(b^2)c$. In particular, we set $ue_i := u$ for all $i \in \mathcal{I}$ and $eu := u$. Note that $\Gamma_i \subseteq \Gamma$ and e_i as a word in Γ is identified with e . The *block length* of a word $u = u_1 \dots u_m \in \Gamma$ is given by $\|u\| := m$. Additionally, we set $\|e\| := 0$. The *type* $\tau(u)$ of u is defined to be i if $u_m \in \Gamma_i^\times$; we set $\tau(e) := 0$.

To help visualizing the structure of a free product we may interpret the set Γ as the vertex set of its Cayley graph \mathcal{X} (with respect to the generating set $\bigcup_{i \in \mathcal{I}} S_i$), which is constructed as follows: consider Cayley graphs $\mathcal{X}_1, \dots, \mathcal{X}_r$ of $\Gamma_1, \dots, \Gamma_r$ w.r.t. the (finite) symmetric generating sets S_1, \dots, S_r ; take copies of $\mathcal{X}_1, \dots, \mathcal{X}_r$ and glue them together at their identities to one single common vertex, which becomes e ; inductively, at each vertex $v = v_1 \dots v_k$ with $v_k \in \Gamma_i$ attach a copy of every $\mathcal{X}_j, j \neq i$, where v is identified with e_j of the new copy of \mathcal{X}_j ; see Figure 1. The natural graph distance on \mathcal{X} is also used for elements of Γ and we write $l(u)$ for the *graph distance* or *length* of $u \in \Gamma$ to e . A *geodesic* of u is a shortest path from e to u . We remark that the length of an element may differ drastically from its block length.

We construct in a natural way a random walk on Γ from some given random walks on its free factors. Suppose we are given (symmetric, finitely supported) probability measures μ_i on Γ_i with $\langle \text{supp}(\mu_i) \rangle = \Gamma_i$ for each $i \in \mathcal{I}$. For $x, y \in \Gamma_i$, the corresponding single

step transition probabilities of a random walk on Γ_i are given by $p_i(x, y) := \mu_i(x^{-1}y)$ and the n -step transition probabilities are denoted by $p_i^{(n)}(x, y) := \mu_i^{(n)}(x^{-1}y)$, where $\mu_i^{(n)}$ is the n -th convolution power of μ_i . Each of these random walks is irreducible. For sake of simplicity, we also assume $\mu_i(e_i) = 0$ for every $i \in \mathcal{I}$. We lift μ_i to a probability measure $\bar{\mu}_i$ on Γ by defining $\bar{\mu}_i(x) := \mu_i(x)$, if $x \in \Gamma_i$, and $\bar{\mu}_i(x) := 0$ otherwise. Let $\alpha_i > 0$, $i \in \mathcal{I}$, with $\sum_{i \in \mathcal{I}} \alpha_i = 1$. We now obtain a new finitely supported probability measure on Γ given by

$$\mu = \sum_{i \in \mathcal{I}} \alpha_i \bar{\mu}_i.$$

The random walk on Γ starting at e , which is governed by μ , is described by the sequence of random variables $(X_n)_{n \in \mathbb{N}_0}$. For $x, y \in \Gamma$, the associated single and n -step transition probabilities are denoted by $p(x, y) := \mu(x^{-1}y)$ and $p^{(n)}(x, y) := \mu^{(n)}(x^{-1}y)$, where $\mu^{(n)}$ is the n -th convolution power of μ . The Cayley graph under consideration will always be with respect to the set of generators $\text{supp}(\mu) = \bigcup_{i \in \mathcal{I}} \text{supp}(\mu_i)$. We refer to Remark 3.11 for a short discussion for the case of non-nearest neighbour random walks.

2.2. Generating Functions. One key ingredient of the proofs is the study of the following generating functions. The most common among these generating functions are the *Green functions* related to μ_i and μ which are defined by

$$G_i(x_i, y_i | z) := \sum_{n \geq 0} p_i^{(n)}(x_i, y_i) z^n \quad \text{and} \quad G(x, y | z) := \sum_{n \geq 0} p^{(n)}(x, y) z^n,$$

where $z \in \mathbb{C}$, $i \in \mathcal{I}$, $x_i, y_i \in \Gamma_i$ and $x, y \in \Gamma$. We note that the free product Γ is non-amenable and that the radius of convergence R of $G(\cdot, \cdot | z)$ is strictly larger than 1; see e.g. [25, Thm. 10.10, Cor. 12.5]. In particular, this implies *transience* of our random walk on Γ . At this point let us remark that the case $r = 2 = |\Gamma_1| = |\Gamma_2|$ leads to a recurrent random walk (and therefore to a recurrent branching random walk), which is the reason why we excluded this case. Moreover, non-amenableity of Γ yields $G(e, e | R) < \infty$; see e.g. [15, Proposition 2.1].

The *first visit generating functions* related to μ_i and μ are given by

$$\begin{aligned} F_i(x_i, y_i | z) &:= \sum_{n \geq 0} \mathbb{P}[Y_n^{(i)} = y_i, \forall m \leq n-1 : Y_m^{(i)} \neq y_i \mid Y_0^{(i)} = x_i] z^n \quad \text{and} \\ F(x, y | z) &:= \sum_{n \geq 0} \mathbb{P}[X_n = y, \forall m \leq n-1 : X_m \neq y \mid X_0 = x] z^n, \end{aligned}$$

where $(Y_n^{(i)})_{n \in \mathbb{N}_0}$ describes a random walk on Γ_i governed by μ_i . For $M \subseteq \Gamma$, we also define

$$F(x, M | z) := \sum_{n \geq 0} \mathbb{P}[X_n \in M, \forall m \leq n-1 : X_m \notin M \mid X_0 = x] z^n$$

and the *first return generating function*

$$U(x, M | z) := \sum_{n \geq 1} \mathbb{P}[X_n \in M, \forall 1 \leq m \leq n-1 : X_m \notin M \mid X_0 = x] z^n.$$

By a Harnack-type inequality the generating functions $F(\cdot, \cdot|z)$ and $U(\cdot, \cdot|z)$ have also radii of convergence of at least $R > 1$ and $U(x, M|z) = F(x, M|z)$ if $x \notin M$. By transitivity, we have $G_i(x_i, x_i|z) = G_i(e_i, e_i|z)$ and $G(x, x|z) = G(e, e|z)$ for all $x_i \in \Gamma_i$ and $x \in \Gamma$. For $x \in \Gamma \setminus \{e\}$, we have

$$G(e, e|z) > F(e, x|z)G(x, x|z)F(x, e|z); \quad (2.2)$$

indeed, while on the left hand side we take into account all paths from e to e , on the right hand side we only take into account all random walk paths from e to e which pass through x ; therefore, strict inequality follows from irreducibility of the random walk which ensures always existence of random walk paths from e to e not passing through x . Symmetry of the μ_i 's now implies that $F(e, x|z) < 1$ for all $|z| \leq R$ and all $x \in \Gamma \setminus \{e\}$. The *last visit generating functions* related to μ_i and μ are given by

$$\begin{aligned} L_i(x_i, y_i|z) &:= \sum_{n \geq 0} \mathbb{P}[Y_n^{(i)} = y_i, \forall 1 \leq m \leq n : Y_m^{(i)} \neq x_i \mid Y_0^{(i)} = x_i] z^n \quad \text{and} \\ L(x, y|z) &:= \sum_{n \geq 0} \mathbb{P}[X_n = y, \forall 1 \leq m \leq n : X_m \neq x \mid X_0 = x] z^n. \end{aligned}$$

We have the following important equations, which follow by conditioning on the first visits of y_i and y , the last visits of x_i and x respectively:

$$\begin{aligned} G_i(x_i, y_i|z) &= F_i(x_i, y_i|z) \cdot G_i(y_i, y_i|z) = G_i(x_i, x_i|z) \cdot L_i(x_i, y_i|z), \\ G(x, y|z) &= F(x, y|z) \cdot G(y, y|z) = G(x, x|z) \cdot L(x, y|z). \end{aligned} \quad (2.3)$$

Thus, by transitivity we obtain

$$F(x, y|z) = L(x, y|z) \text{ for any } x, y \in \Gamma \text{ and } |z| \leq R. \quad (2.4)$$

Let $x, y, w \in \Gamma$ such that all (random walk) paths from x to w pass through y . Then

$$F(x, w|z) = F(x, y|z) \cdot F(y, w|z) \quad \text{and} \quad L(x, w|z) = L(x, y|z) \cdot L(y, w|z); \quad (2.5)$$

this can be checked by conditioning on the first/last visit of y when walking from x to w . For $i \in \mathcal{I}$ and $z \in \mathbb{C}$, we define the functions

$$\xi_i(z) := U(e, \text{supp}(\mu_i)|z) = U(e, \Gamma_i^\times|z) = F(e, \text{supp}(\mu_i)|z), \quad (2.6)$$

which have also radii of convergence of at least $R > 1$. We remark that $\xi_i(1) < 1$; see e.g. [9, Lemma 2.3]. Moreover, we have $F(x_i, y_i|z) = F_i(x_i, y_i|\xi_i(z))$ and $L(x_i, y_i|z) = L_i(x_i, y_i|\xi_i(z))$ for all $x_i, y_i \in \Gamma_i$; see [25, Prop. 9.18c] and [9, Lemma 2.2]. Thus, by conditioning on the number of visits of e before finally making a step from e to Γ_i^\times we get the following formula:

$$\xi_i(z) = \frac{\alpha_i z}{1 - \sum_{j \in \mathcal{I} \setminus \{i\}} \sum_{s \in \Gamma_j} \alpha_j \mu_j(s) z F_j(s, e_j|\xi_j(z))}. \quad (2.7)$$

Finally, we define the following power series that will lead to a useful expression for the Hausdorff dimension. Let

$$\mathcal{F}(\lambda|z) := \sum_{x \in \Gamma} F(e, x|\lambda) z^{l(x)}, \quad (2.8)$$

and define for $i \in \mathcal{I}$:

$$\mathcal{F}_i^+(\lambda|z) := \sum_{x \in \Gamma_i^\times} F(e, x|\lambda) z^{l(x)} = \sum_{x \in \Gamma_i^\times} F_i(e_i, x|\xi_i(\lambda)) z^{l(x)}, \quad (2.9)$$

$$\mathcal{F}_i(\lambda|z) := \sum_{n \geq 1} \sum_{\substack{x=x_1 \dots x_n \in \Gamma: \\ x_1 \in \Gamma_i^\times}} F(e, x|\lambda) z^{l(x)} = \mathcal{F}_i^+(\lambda|z) \left(1 + \sum_{j \in \mathcal{I} \setminus \{i\}} \mathcal{F}_j(\lambda|z) \right). \quad (2.10)$$

The latter functions satisfy the following relation:

$$\mathcal{F}(\lambda|z) = 1 + \sum_{i \in \mathcal{I}} \mathcal{F}_i(\lambda|z). \quad (2.11)$$

2.3. Branching Random Walks. In this subsection we introduce discrete-time branching random walks on free products and recall some basic results.

There are two different main descriptions or constructions of a branching random walk (BRW). The first defines the process inductively as a *growing cloud* of particles moving in (discrete) time and space. The second, via tree-indexed random walks, uses the fact that the branching distribution does not depend on the space. For that reason one can separate branching and movement into two steps. First, one generates the whole genealogy of the process and then one maps the corresponding genealogical tree into the Cayley graph. In both cases we need the following definition. A Galton–Watson process is characterized through an *offspring distribution* ν . This is a probability measure on $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ with mean (or also called growth parameter) $\lambda = \sum_{k \geq 1} k \nu(k) \in (0, \infty)$. We assume that ν has *finite second moment*, that is, $\sum_{k \geq 1} k^2 \nu(k) < \infty$. Moreover we exclude the cases where $\nu(0) > 0$ and $\nu(1) = 1$; this guarantees that the process survives almost surely and that the BRW is not reduced to a (non-branching) random walk.

The BRW on Γ is defined inductively: at time 0 we have one particle at e (if not mentioned otherwise). Between time n and $n + 1$ the process performs two steps: branching and movement. First, each particle, independently of all others and the previous history of the process, produces descendants according to ν and dies. Second, each of these descendants, independently of all others and the past, moves to a neighbour vertex in Γ according to μ . A particle located at some vertex $x \in \Gamma$ at time n has a unique direct ancestor at time $n - 1$. Consequently, each particle has a unique finite sequence of ancestors, the family history, which traces back to the original starting particle at e . The sequence of the locations of its ancestors (chronologically ordered) gives a path from e to x , which we call the *trail* of the particle.

Sometimes it will be convenient to work with the interpretation of a BRW as a tree-indexed random walk, see [2]. Let \mathcal{T} be a rooted infinite tree. The root is denoted by \mathbf{r} and other vertices by v and let $|v|$ be the (graph) distance from v to the root \mathbf{r} . The random walk on Γ indexed by \mathcal{T} is the collection of Γ -valued random variables $(S_v)_{v \in \mathcal{T}}$ defined as follows. Label the edges of \mathcal{T} with i.i.d. random variables η_v with distribution μ ; the random variable η_v is the label of the edge (v^-, v) . Define $S_v = e \cdot \prod_{i=1}^{|v|} \eta_{v_i}$ where $\langle v_0 = \mathbf{r}, v_1, \dots, v_n = v \rangle$ is the unique geodesic (also called ancestry of v) from \mathbf{r} to v at level n . A tree-indexed random walk becomes a BRW if the underlying tree is a Galton–Watson tree induced by

ν . We refer to \mathcal{T} as the family tree and to \mathcal{X} as the base graph of the BRW. Furthermore, a vertex $v \in \mathcal{T}$ is called a particle of the BRW and \mathcal{T}_n denotes the vertices of \mathcal{T} on level n or equivalently the particles in generation n .

A useful variation of the first description of a BRW is the *coloured branching random walk*, see [13]. This process behaves like a standard BRW where in addition each particle is either blue or red. In order to define this coloured version we choose a subset M of Γ that plays the role of a “paint bucket”. We start the BRW with one blue particle at e . Blue particles located outside of M produce blue offspring. A blue particle that hits the paint bucket is frozen there and will be replaced by a red particle. The new red particle starts an ordinary (red-coloured) branching random walk. As a consequence, every red particle has exactly one “frozen” ancestor in M .

We denote by $Z_\infty(M) \in \mathbb{N} \cup \{\infty\}$ the random number of frozen (blue) particles in M during the whole branching process. If $M = \{x\}$ then we just write $Z_\infty(x)$.

For ease of presentation we will switch freely between the different definitions of a BRW; nevertheless it will always be clear from the context which description we are using.

A BRW on a Cayley graph is called *recurrent* if each vertex is visited infinitely many times and *transient* if any finite subset is eventually free of particles. The recurrence/transience behaviour is well understood. In fact, we have the following classification in recurrence and transience, see [2] for the sub- and supercritical case and [6] for the critical case. We also refer to [11] for the corresponding result in the continuous setting.

Theorem 2.1. *The BRW is transient if and only if $\lambda \leq R$.*

Recall that in the language of [13] transience is equivalent to *weak survival* if $\lambda > 1$. For the rest of this paper we will restrict our investigation to the case of transience or weak survival. Since in this case the process eventually vacates every finite subset of Γ almost surely the investigation of the *convergence* of the BRW to the geometric boundary is meaningful.

2.4. Ends of Graphs, Box-Counting Dimension and Hausdorff Dimension. Let us first recall some basic notations on infinite graphs. Let \mathcal{G} be an infinite, connected, locally finite graph with countable vertex set and root e . For ease of presentation, we will identify \mathcal{G} or a subgraph with its vertex set. A *path* of length n in \mathcal{G} is a finite sequence of vertices $[x_0, x_1, \dots, x_n]$ such that there is an edge from x_{i-1} to x_i for each $i \in \{1, \dots, n\}$. Recall that a *geodesic* of a vertex $x \in \mathcal{G}$ is a shortest path from e to x in \mathcal{G} . A *ray* is a semi-infinite path $[e = x_0, x_1, x_2, \dots]$, which does not backtrack, that is, $x_i \neq x_j$ if $i \neq j$. Two rays η_1 and η_2 are *equivalent* if there is a third ray which shares infinitely many vertices with η_1 and η_2 . An equivalence class of rays is called an *end*. The set of equivalence classes of rays is called the *end boundary* of \mathcal{G} , denoted by $\partial\mathcal{G}$. For further details we refer to [25, Section 21].

In the case of free products we have different types of ends occurring in the Cayley graph \mathcal{X} of Γ : ends arising from ends in one of the \mathcal{X}_i , and “infinite words”. More precisely, denote by $\Omega_i^{(0)}$ the set of ends of \mathcal{X}_i . For $\omega_i \in \Omega_i^{(0)}$, let $\eta = [e_i, y_1, y_2, \dots] \in \omega_i$ and let $x \in \Gamma$, where $[x_0, x_1, \dots, x_n]$ is a geodesic from x_0 to $x = x_n$. Then, the ray $x\eta := [x_0, x_1, \dots, x_n, x_n y_1, x_n y_2, \dots]$ describes an end in Γ . The end described by $x\eta$ is denoted

by $x\omega_i$. We set $\Omega_i := \{x\omega_i \mid x \in \Gamma, \omega_i \in \Omega_i^{(0)}\}$. Moreover, the set of infinite words is given by

$$\Omega_\infty = \{x_1x_2x_3\dots \in (\Gamma_*^\times)^\mathbb{N} \mid x_j \in \Gamma_k^\times \Rightarrow x_{j+1} \notin \Gamma_k^\times\}.$$

It is easy to see that the set Ω of ends of \mathcal{X} can be decomposed in the following way:

$$\Omega = \Omega_\infty \uplus \Omega_1 \uplus \Omega_2 \uplus \dots \uplus \Omega_r.$$

Observe that Ω_i is empty if and only if Γ_i is finite. Thus, if all groups Γ_i are finite then $\Omega = \Omega_\infty$.

In order to measure the size of Ω we define a metric on Ω . We say that an end $\omega_1 \in \Omega$ is contained in a subset of the graph if all representatives have all but finitely many vertices in this subset. Now, if we remove from \mathcal{X} any finite vertex subset $F \subseteq \mathcal{X}$ (including the removal of edges to vertices in F) then there is exactly one connected component in the reduced graph $\mathcal{X} \setminus F$ containing the end ω_1 . We call this component the ω_1 -component and say that ω_1 ends up in this component. Denote by $B_m := \{x \in \Gamma \mid l(x) \leq m\}$ the ball centered at e with radius m ; we also set $B_{-1} := \emptyset$. Let $\omega_2 \in \Omega$ be another end with $\omega_1 \neq \omega_2$. Obviously, there is some maximal $m \in \mathbb{N}_0$ such that ω_1 and ω_2 end up in the same connected component of $\mathcal{X} \setminus B_{m-1}$. We write $c(\omega_1, \omega_2)$ for this maximal integer m . We now define a metric on Ω by

$$d_\Omega(\omega_1, \omega_2) := \alpha^{c(\omega_1, \omega_2)},$$

where $\alpha \in (0, 1)$ is arbitrary, but fixed. Additionally, we set $d_\Omega(\omega_1, \omega_1) := 0$. The ball $B(\omega, \varepsilon)$ centered at $\omega \in \Omega$ with radius $\varepsilon \geq 0$ is given by all ends $\hat{\omega} \in \Omega$ with $d_\Omega(\omega, \hat{\omega}) \leq \varepsilon$. In other words, if $\varepsilon = \alpha^m$ then $\hat{\omega} \in B(\omega, \varepsilon)$ if and only if ω and $\hat{\omega}$ end up in the same component of $\mathcal{X} \setminus B_{m-1}$.

A *cover* of a subset $\Omega' \subseteq \Omega$ is a finite or countable set of balls of the form $B(\omega, \varepsilon_\omega)$ with $\omega \in \Omega'$ and $\varepsilon_\omega > 0$ such that the union of these balls include Ω' . For any $\varepsilon > 0$ let $N_\varepsilon(\Omega')$ be the minimal number of balls of the form $B(\omega, \varepsilon_\omega)$ with $\omega \in \Omega'$ and $0 < \varepsilon_\omega \leq \varepsilon$, which cover Ω' . Apparently, $N_\varepsilon(\Omega')$ is bounded from above by the number of elements in Γ at graph distance $m = \lceil \log(\varepsilon) / \log(\alpha) \rceil$. The *lower* and *upper box-counting dimension* (also called *Minkowski dimension*) of Ω' are defined as

$$\underline{\text{BD}}(\Omega') := \liminf_{\varepsilon \downarrow 0} \frac{\log N_\varepsilon(\Omega')}{-\log \varepsilon} \quad \text{and} \quad \overline{\text{BD}}(\Omega') := \limsup_{\varepsilon \downarrow 0} \frac{\log N_\varepsilon(\Omega')}{-\log \varepsilon}. \quad (2.12)$$

If both limits are equal then the common value is called the *box-counting dimension* $\text{BD}(\Omega')$ of Ω' .

Another well-known measure for the size of Ω' is given by the Hausdorff dimension. For $\delta > 0$, the δ -dimensional *Hausdorff measure* of Ω' is defined by

$$\mathcal{H}_\delta(\Omega') := \liminf_{\varepsilon \downarrow 0} \left\{ \sum_i \varepsilon_i^\delta \mid \{B(\cdot, \varepsilon_i)\}_i \text{ is a cover of } \Omega' \text{ with } \varepsilon_i < \varepsilon \right\}.$$

Then the *Hausdorff dimension* of Ω' is defined as

$$\text{HD}(\Omega') := \inf \{ \delta \geq 0 \mid \mathcal{H}_\delta(\Omega') = 0 \}. \quad (2.13)$$

Since \mathcal{X} has bounded vertex degrees we have $\text{HD}(\Omega') < \infty$. It is well-known that, for all $\Omega' \subseteq \Omega$,

$$\text{HD}(\Omega') \leq \underline{\text{BD}}(\Omega').$$

One of our main goals is to investigate to which kind of ends the branching random walk converges and to compare the dimensions of the whole space of ends with the set of ends which are “hit” by the BRW. More precisely, for any $\omega \in \Omega$, if we remove any finite vertex subset $F \subseteq \mathcal{X}$ then there is exactly one connected component in the reduced graph $\mathcal{X} \setminus F$ containing ω . We say that the branching random walk *accumulates* at the end ω if for every finite vertex subset $F \subseteq \mathcal{X}$ there is at least one particle visiting the connected ω -component in $\mathcal{X} \setminus F$. The set of accumulation points is denoted by Λ . If the BRW is recurrent then $\Omega = \Lambda$; thus, we restrict our investigation to the more interesting case of transience and therefore assume $1 < \lambda \leq R$. Note that $\Lambda \cap \Omega_\infty$ is almost surely non-empty; each infinite ancestry line converges to some element in $g_1 g_2 \cdots \in \Omega_\infty$ with convergence in the sense that the length of the common prefix of the particle’s location and $g_1 g_2 \dots$ tends to infinity, see e.g. [9, Proposition 2.5]. We remark also that the Hausdorff dimensions of Λ and $\Lambda \cap \Omega_\infty$ are almost surely constant, which can be shown analogously as explained in [13, Sec. 1, Remark (C)].

3. RESULTS

In this section we summarize our results about branching random walks on free products and present several explicit examples.

3.1. Main Results. The first result describes how the structure of Λ gets richer when increasing the growth parameter λ and that there are up to $r = |\mathcal{I}|$ possible phase transitions.

Theorem 3.1. *Let $\lambda \in (1, R]$. Then $\mathbb{P}[\Lambda \cap \Omega_i \neq \emptyset] \in \{0, 1\}$, and $\mathbb{P}[\Lambda \cap \Omega_i \neq \emptyset] = 1$ if and only if $\xi_i(\lambda) > 1$. More precisely:*

- (1) *If $\xi_i(\lambda) \leq 1$ then $\emptyset \subsetneq \Lambda \subseteq \Omega_\infty$.*
- (2) *If $\xi_i(\lambda) > 1$ then $\emptyset \subsetneq \Omega_\infty \cap \Lambda \subset \Lambda$ with $\Lambda \cap \Omega_i \neq \emptyset$ and $|\Lambda \cap \Omega_i| = \infty$.*

Remark 3.2. *In the case where one of the free factors is an infinite amenable group its ends do not appear in Λ . In other words, if $R_i = 1$ is the radius of convergence of $G_i(e_i, e_i|z)$ then $\xi_i(\lambda) \leq 1$ for all $\lambda \in (1, R]$; see [25, Lemma 17.1a]. Consequently, no ends in Ω_i contribute to Λ , that is, $\Lambda \cap \Omega_i = \emptyset$ almost surely.*

We illustrate the above described behaviour in the following two examples:

Example 3.3: Consider $\Gamma = \mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ and let μ_1 and μ_2 be two symmetric probability measures on \mathbb{Z}^{d_1} and \mathbb{Z}^{d_2} . Due to Kesten’s amenability criterion we have $R_1 = R_2 = 1$. Consequently, $\Lambda \subseteq \Omega_\infty$ almost surely for all $\lambda \leq R$.

Example 3.4: Consider $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1 and Γ_2 are non-amenable groups, and let μ_i define a symmetric random walk on Γ_i for $i \in \{1, 2\}$. Due to the non-amenability we

have that $R_1, R_2 > 1$ and $G_i(e_i, e_i | R_i) < \infty$. In the case where

$$\alpha_1 = \frac{R_1 G_1(e_1, e_1 | R_1)}{R_1 G_1(e_1, e_1 | R_1) + R_2 G_2(e_2, e_2 | R_2)}$$

we obtain by [25, Lemma 17.1] that $\xi_1(R), \xi_2(R) > 1$. Therefore, there are numbers $\lambda_1, \lambda_2 \in (1, R)$ with $\xi_1(\lambda_1) = \xi_2(\lambda_2) = 1$ which leads to phase transitions at λ_1 and λ_2 .

Now we state our first main result.

Theorem 3.5. *Suppose that ν has finite second moment. Then the box-counting dimension of $\Lambda, \Lambda \cap \Omega_\infty$ respectively, exists and equals the Hausdorff dimension of $\Lambda, \Lambda \cap \Omega_\infty$ respectively. Furthermore:*

$$\text{BD}(\Lambda) = \text{BD}(\Lambda \cap \Omega_\infty) = \text{HD}(\Lambda) = \text{HD}(\Lambda \cap \Omega_\infty) = \frac{\log z^*}{\log \alpha},$$

where z^* is the smallest real positive number with

$$\sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda | z^*)}{1 + \mathcal{F}_i^+(\lambda | z^*)} = 1. \quad (3.1)$$

Remark 3.6. *The proof of Theorem 3.5 directly applies to BRW on free products of finite graphs and a corresponding result holds verbatim; see e.g. [25, Sec. 9.C] for a formal definition of general free products and random walks on them.*

As a first consequence we obtain that only infinite words contribute to the dimension of Λ .

Corollary 3.7. *For $i \in \mathcal{I}$, $\text{HD}(\Lambda \cap \Omega_i) < \text{HD}(\Lambda \cap \Omega_\infty)$.*

For $i \in \mathcal{I}$, $m \in \mathbb{N}$ and $z \in \mathbb{C}$, we define $S_i(m) := |\{x \in \Gamma_i \mid l(x) = m\}|$ and

$$\mathcal{S}_i^+(z) := \sum_{m \geq 1} S_i(m) z^m.$$

Analogously to Theorem 3.5, we can prove existence of the box-counting dimension of the whole boundary Ω and express the dimension as the solution of a functional equation.

Theorem 3.8. *The box-counting dimensions of Ω and Ω_∞ exist and satisfy*

$$\text{BD}(\Omega) = \text{BD}(\Omega_\infty) = \text{HD}(\Omega) = \text{HD}(\Omega_\infty) = \frac{\log z_{\mathcal{S}}^*}{\log \alpha},$$

where $z_{\mathcal{S}}^*$ is the smallest real positive number with

$$\sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^+(z_{\mathcal{S}}^*)}{1 + \mathcal{S}_i^+(z_{\mathcal{S}}^*)} = 1. \quad (3.2)$$

Analogously to Corollary 3.7 we obtain that the Hausdorff dimension of Ω arises only from the ends in Ω_∞ .

Corollary 3.9. *For all $i \in \mathcal{I}$, $\text{HD}(\Omega_i) < \text{HD}(\Omega_\infty)$.*

Beyond these first consequences of Theorems 3.5 and 3.8, the expressions for the Hausdorff dimensions allow us to study first regularity properties. For any fixed free product Γ , let us consider the function

$$\Phi : [1, \infty) \rightarrow \mathbb{R} : \lambda \mapsto \text{HD}(\Lambda),$$

which assigns to every value λ the Hausdorff dimension of Λ of a BRW with growth parameter λ . The limit case $\lambda = 1$ corresponds to the degenerate case of a non-branching random walk; in this case the Hausdorff dimension is just zero.

Theorem 3.10. *The function $\Phi(\lambda)$ has the following properties:*

- (1) $\Phi(\lambda)$ is strictly increasing on $[1, R]$, $\Phi(1) = 0$ and $\Phi(\lambda) = \text{HD}(\Omega)$ for all $\lambda > R$.
- (2) $\Phi(\lambda)$ is continuous in $[1, \infty) \setminus \{R\}$ and continuous from the left at $\lambda = R$ with

$$\Phi(R) \leq \frac{1}{2} \text{HD}(\Omega).$$

- (3) $\Phi(\lambda)$ has the following behaviour as $\lambda \uparrow R$:

$$\Phi(R) - \Phi(\lambda) \sim \begin{cases} C_1 \cdot (R - \lambda), & \text{if } G'(R) < \infty, \\ C_2 \cdot \sqrt{R - \lambda}, & \text{if } G'(R) = \infty \end{cases}$$

for a suitable constant C_1, C_2 respectively.

Remark 3.11. *The last theorem states that $\text{HD}(\Lambda)$ does not exceed $\text{HD}(\Omega)/2$ unless the BRW is recurrent. We always assumed the random walk to be of nearest neighbour type. However, we feel confident that our techniques work well in the case of finite range random walks and that the equality $\text{HD}(\Lambda) \leq \text{HD}(\Omega)/2$ does not depend on the choice of the metric. This type of phenomenon was already conjectured for the contact process on the homogeneous tree in [17]. We also refer to Section 8 in [18] for a discussion how the value $1/2$ can be explained through the “backscattering principle”.*

Remark 3.12. *In [13] it was shown that $\text{HD}(\Lambda) = \text{HD}(\Omega)/2$ only if $\lambda = R$ and if the underlying walk is a simple random walk. In our more general setting this is no longer true, since the maximal Hausdorff dimension can also be attained by a non-simple random walk, see Example 3.14. More generally, we conjecture that one has maximal dimension for the BRW (with λ being the critical growth value) for every choice of $\alpha_1 \in (0, 1)$ if we consider a general free product $\Gamma = \Gamma_1 * \Gamma_2$ with μ_1 and μ_2 governing positive recurrent random walks on the single factors Γ_1 and Γ_2 .*

Remark 3.13. *Recall that we always assume that the random walk on Γ is symmetric. This assumption can be dropped for free products of finite groups/graphs. In this case we always have the crucial property $F(e, x|R) < 1$ for all $x \in \Gamma \setminus \{e\}$ (compare with (2.2)). In fact, if $x = x_1 \dots x_m \in \Gamma \setminus \{e\}$ then*

$$F(e, x_1 \dots x_m|R) = \prod_{j=1}^m F_{\tau(x_j)}(e_{\tau(x_j)}, x_j \mid \xi_{\tau(x_j)}(R)) < 1,$$

as $\xi_i(R) < 1$ due to [25, Lemma 17.1, Theorem 9.22].

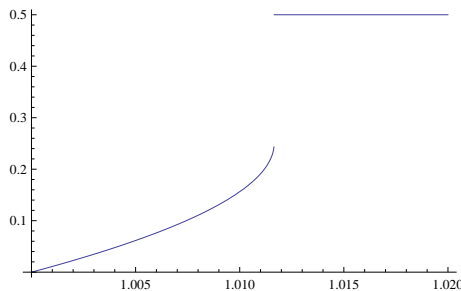


FIGURE 2. Hausdorff dimension $\text{HD}(\Lambda)$ of a BRW on $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ in dependence of λ on the x -axis

Theorem 3.5 allows explicit calculations in all cases where formulas for the involved generating functions are known. In the following examples we set the exponent of the metric on Ω equal to $1/2$, i.e., $d_\Omega(\cdot, \cdot) = 2^{-c(\cdot, \cdot)}$.

Example 3.14: Consider the free product $\Gamma = \Gamma_1 * \Gamma_2 = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/3\mathbb{Z} = \{e_1, a, a^2\}$, with $\text{supp}(\mu_1) = \{a, a^2\}$. The required generating functions $F(e, x|\lambda)$, $x \in \Gamma_*^\times$, may e.g. be obtained by solving the finite systems of equations given in [24, Prop. 3c], and therefore $\text{HD}(\Lambda)$ can be computed via Equation (3.1). Solving Equation (3.2) leads to $\text{HD}(\Omega) = 1/2$. Figure 2 shows – with the help of numerical computation by MATHEMATICA – the graph of the function $\lambda \mapsto \text{HD}(\Lambda)$ for simple random walk on $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Let us remark that in this case the critical parameter R can be explicitly calculated by the formula given in [25, (9.29),(3)].

Another interesting phenomenon occurs in this example. If $\mu_1(a) = \mu_1(a^2) = 1/2$ and if we let α_1 vary in the interval $(0, 1)$ and denote by $R(\alpha_1)$ the radius of convergence of $G(e, e|z)$ in dependence of α_1 then we always get $\Phi(R(\alpha_1)) = \frac{1}{2}\text{HD}(\Omega)$, which can be verified by explicit calculations with the help of MATHEMATICA.

Example 3.15: We consider the free product of two infinite “ladders” $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$. We set $\alpha_1 = \alpha_2 = 1/2$ and $\mu_1((\pm 1, 0)) = \mu_1((0, 1)) = \mu_2((\pm 1, 0)) = \mu_2((0, 1)) = 1/3$. The functions $F_1((0, 0), (z, a)|z)$ with $(z, a) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ can be computed by solving a system of equations as it is shown in [10, Section 7.2]. In order to compute the Hausdorff dimension of Λ one has to solve, analogously to [9, Section 6.2]:

$$\frac{\lambda}{2} \frac{\xi_1(\lambda)}{\lambda - \xi_1(\lambda)} = \frac{\xi_1(\lambda)}{1 - \frac{2\xi_1(\lambda)}{3} (F_1((0, 0), (1, 0)|\xi_1(\lambda)) + F_1((0, 0), (-1, 0)|\xi_1(\lambda)) + F_1((0, 0), (0, 1)|\xi_1(\lambda)))}$$

In order to compute $\text{HD}(\Omega)$ we observe that $S_1(1) = 3$ and $S_1(m) = 4$ for $m \geq 2$. Hence, $\mathcal{S}_1^+(z) = \mathcal{S}_2^+(z) = 3z + 4z^2/(1 - z)$. This yields $z_{\mathcal{S}}^* = \sqrt{5} - 2$. Numerical evaluations then lead to a picture qualitatively similar to Figure 2.

3.2. Free Products of Finite Groups. In this subsection we give a more explicit formula for the box-counting dimension with respect to a slightly changed metric on the boundary

in the case of free products of *finite* groups. In this case we have $\Omega = \Omega_\infty$. Throughout the whole subsection we do not need the assumption that the μ_i 's are symmetric. For any $\omega_1 = x_1x_2\dots, \omega_2 = y_1y_2\dots \in \Omega_\infty$ with $\omega_1 \neq \omega_2$, we define the *confluent* $\omega_1 \wedge \omega_2$ of ω_1 and ω_2 as the word $x_1\dots x_k$ of maximal length with $x_i = y_i$ for all $1 \leq i \leq k$. If $x_1 \neq y_1$, then $\omega_1 \wedge \omega_2 := e$. The metric on the boundary Ω_∞ is defined by

$$d_{\Omega}^{\text{fin}}(\omega_1, \omega_2) := \alpha^{\|\omega_1 \wedge \omega_2\|}$$

for any arbitrary but fixed $\alpha \in (0, 1)$. With respect to this metric on Ω_∞ we can define analogously to (2.12) and (2.13) the upper box-counting dimension $\overline{\text{BD}}^{\text{fin}}(\Omega')$, the box-counting dimension $\text{BD}^{\text{fin}}(\Omega')$ and the Hausdorff dimension $\text{HD}^{\text{fin}}(\Omega')$ for any $\Omega' \subseteq \Omega_\infty$. We set $\mathcal{F}_i^+(\lambda) := \mathcal{F}_i^+(\lambda|1)$ and define the matrix $M = (m(i, j))_{i, j \in \mathcal{I}}$ by

$$m(i, j) := \begin{cases} \mathcal{F}_j^+(\lambda), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Since M is irreducible and has non-negative entries, the Perron–Frobenius eigenvalue exists and is denoted by θ .

Furthermore, define the matrix $D = (d(i, j))_{i, j \in \mathcal{I}}$ by $d(i, j) := |\Gamma_j| - 1$, if $i \neq j$, and $d_{i, i} := 0$, and denote by ϱ its Perron–Frobenius eigenvalue. With this notation we get:

Corollary 3.16.

$$\text{BD}^{\text{fin}}(\Lambda) = \text{HD}^{\text{fin}}(\Lambda) = -\frac{\log \theta}{\log \alpha} \quad \text{and} \quad \text{BD}^{\text{fin}}(\Omega) = \text{HD}^{\text{fin}}(\Omega) = -\frac{\log \varrho}{\log \alpha}.$$

□

Let us remark that, in the case of $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| = |\Gamma_2| < \infty$, we get the following explicit formulas for the dimensions:

$$\begin{aligned} \text{BD}^{\text{fin}}(\Lambda) &= \text{HD}^{\text{fin}}(\Lambda) = -\frac{\log \sqrt{\mathcal{F}_1^+(\lambda)\mathcal{F}_2^+(\lambda)}}{\log \alpha} \quad \text{and} \\ \text{BD}^{\text{fin}}(\Omega) &= \text{HD}^{\text{fin}}(\Omega) = -\frac{\log \sqrt{(|\Gamma_1| - 1)(|\Gamma_2| - 1)}}{\log \alpha}. \end{aligned}$$

Example 3.17: Consider $\Gamma = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/3\mathbb{Z} = \{e_1, a, a^2\}$ and $\mathbb{Z}/2\mathbb{Z} = \{e_2, b\}$. We choose $\mu(a) = p \in (0.1, 0.7)$, $\mu(a^2) = q \in (0, 0.9 - p)$ and $\mu(b) = 1 - p - q$. We set $\alpha := 1/2$ and $\lambda = 1.005$. Let us note that this choice of the parameters p and q lead to $R \geq 1.005$, which can be verified by numerical evaluation. For instance, in [8, Section 3.6.1] the required generating functions are computed. In Figure 3 we can see the behaviour of $\text{HD}^{\text{fin}}(\Lambda)$ with $\lambda = 1.005$ in dependence of the parameters p and q . The Hausdorff dimension of the whole space of ends is 0.5; compare with Example 3.14.

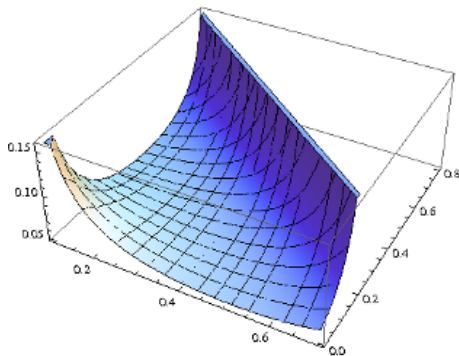


FIGURE 3. Hausdorff dimension $\text{HD}(\Lambda)$ of the branching random walk on $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ with $\lambda = 1.005$ in dependence of p and q .

3.3. Free Products by Amalgamation of Finite Groups. An important generalization of free products are free products by amalgamation (of finite groups). Let $\Gamma_1, \dots, \Gamma_r, H$, be finite groups such that each group Γ_i contains a subgroup H_i that is isomorphic to H . Let $\phi_i : H_i \rightarrow H$ be an isomorphism for each $i \in \{1, \dots, r\}$. Moreover, let S_i be a generating set of Γ_i and R_i its relations. The *free product by amalgamation* with respect to the subgroup H is defined by

$$\begin{aligned} \Gamma_H &:= \Gamma_1 *_H \Gamma_2 *_H \dots *_H \Gamma_r \\ &:= \langle S_1, \dots, S_r \mid R_1, \dots, R_n, \phi_j^{-1}(\phi_i(a)) = a \ \forall a \in H_i \ \forall i, j \in \mathcal{I} \rangle. \end{aligned}$$

For $i \in \mathcal{I}$, the quotient Γ_i/H_i consists of all left co-sets of the form $x_i H_i = \{x_i h \mid h \in H_i\}$, where $x_i \in \Gamma_i$. We fix a set of representatives $\mathcal{R}_i := \{g_{i,1} = e_i, g_{i,2}, \dots, g_{i,n_i}\}$ for the elements of Γ_i/H_i , that is, for each $y_i \in \Gamma_i$ there is a unique $g_{i,k} \in \mathcal{R}_i$ with $y_i \in g_{i,k} H_i$. We write $\hat{\tau}(x) = i$ if $x \in \mathcal{R}_i \setminus \{e_i\}$. The amalgam Γ_H consists of all finite words of the form

$$x_1 x_2 \dots x_n h \tag{3.3}$$

with $n \in \mathbb{N}_0$, $x_i \in \bigcup_{j \in \mathcal{I}} \mathcal{R}_j \setminus \{e_j\}$ and $h \in H$ such that $\hat{\tau}(x_i) \neq \hat{\tau}(x_{i+1})$. Here w.l.o.g. we may identify h with $\phi_1^{-1}(h)$, and e denotes again the empty word. Let Ω be the set of all ends of Γ_H , which consists of all infinite words of the form $w_1 w_2 \dots \in \left(\bigcup_{i \in \mathcal{I}} \mathcal{R}_i \setminus \{e_i\} \right)^\mathbb{N}$ such that $\hat{\tau}(w_i) \neq \hat{\tau}(w_{i+1})$ for all $i \in \mathbb{N}$. For any $\omega_1 = x_1 x_2 \dots, \omega_2 = y_1 y_2 \dots \in \Omega$ with $\omega_1 \neq \omega_2$, we define again the *confluent* $\omega_1 \wedge \omega_2$ of ω_1 and ω_2 as the word $x_1 \dots x_k$ of maximal length with $x_i = y_i$ for all $1 \leq i \leq k$. If $x_1 \neq y_1$, then $\omega_1 \wedge \omega_2 := e$. Again we can define a metric on the boundary Ω :

$$d_\Omega^{(H)}(\omega_1, \omega_2) := \alpha^{\|\omega_1 \wedge \omega_2\|}$$

for any $\alpha \in (0, 1)$. With respect to this metric on Ω we can define analogously to (2.12) and (2.13) the upper box-counting dimension $\overline{\text{BD}}^{(H)}(\Omega')$, the box-counting dimension $\text{BD}^{(H)}(\Omega')$ and Hausdorff dimension $\text{HD}^{(H)}(\Omega')$ for any $\Omega' \subseteq \Omega$.

Suppose we are given symmetric probability measures μ_i on the groups Γ_i and numbers $\alpha_i > 0$ such that $\sum_{i \in \mathcal{I}} \alpha_i = 1$. The random walk on Γ_H is then governed by

$$\mu(x) := \begin{cases} \alpha_i \mu_i(x), & \text{if } x \in \Gamma_i \setminus H_i, \\ \sum_{i \in \mathcal{I}} \alpha_i \mu_i(\phi_i^{-1}(\phi_1(x))), & \text{if } x \in H_1, \\ 0, & \text{otherwise.} \end{cases}$$

For $g_i \in \mathcal{R}_i$, denote by $T_{g_i H}$ the stopping time of the first visit of the set $g_i H_i$. We introduce the following generating functions:

$$F_H(gh|z) := \sum_{n \geq 0} \mathbb{P}[T_{gH} = n, X_n = gh \mid X_0 = e] z^n,$$

where $g \in \bigcup_{i \in \mathcal{I}} \mathcal{R}_i \setminus \{e_i\}$, $h \in H_i$ and $z \in \mathbb{C}$. By symmetry of the μ_i 's, we have $F_H(gh|z) \leq F(e, gh|z) < 1$; compare with (2.2). Conditioning on the first step of the random walk, we get

$$\begin{aligned} F_H(gh|z) &= \mu(gh)z + \sum_{g_0 \in \Gamma_{\tau(g)} \setminus gH_{\tau(g)}} \mu(g_0)z F_H(g_0^{-1}gh|z) + \\ &\quad \sum_{i \in \mathcal{I} \setminus \{\tau(g)\}} \sum_{g_0 \in \Gamma_i} \mu(g_0)z \sum_{h_0 \in H_i} F_H(g_0^{-1}h_0|z) F_H(h_0^{-1}gh|z). \end{aligned} \quad (3.4)$$

Since there are only *finitely* many functions $F_H(\cdot|z)$, one can compute $F_H(\cdot|z)$ by solving the finite system of quadratic equations (3.4). We define also

$$\mathcal{F}_i^{(H)}(z) := \sum_{\substack{g \in \mathcal{R}_i \setminus \{e_i\}, \\ h \in H_i}} F_H(gh|z)$$

and the matrix $N = (n(i, j))_{i, j \in \mathcal{I}}$ with entries

$$n(i, j) := \begin{cases} \mathcal{F}_j^{(H)}(\lambda), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

We denote by θ_H the Perron–Frobenius eigenvalue of N . Furthermore, we denote by ϱ_H the Perron–Frobenius eigenvalue of the matrix $D_H = (d_H(i, j))_{i, j \in \mathcal{I}}$, which is defined by

$$d_H(i, j) := \begin{cases} [\Gamma_j : H_j] - 1, & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Finally, we can state the following formulas for the dimensions:

Corollary 3.18.

$$\text{BD}^{(H)}(\Lambda) = \text{HD}^{(H)}(\Lambda) = -\frac{\log \theta_H}{\log \alpha} \quad \text{and} \quad \text{BD}^{(H)}(\Omega) = \text{HD}^{(H)}(\Omega) = -\frac{\log \varrho_H}{\log \alpha}.$$

Example 3.19: Consider the amalgam $(\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$. Hence, let $\Gamma_1 = \langle a \mid a^6 = e_1 \rangle$, $\Gamma_2 = \langle b \mid b^6 = e_2 \rangle$, and $H = \langle c \mid c^2 = e_H \rangle$, where e_H is the identity in H . The isomorphisms are defined through $\phi_1(a^3) = c = \phi_2(b^3)$. Eventually,

$$(\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z}) = \langle a, b \mid a^6 = b^6 = e, a^3 = b^3 \rangle.$$

We set $\mu_1(a) = \mu_1(a^5) = \mu_2(b) = \mu_2(b^5) = 1/2$, $\alpha_1 = \alpha_2 = 1/2$ and consider the distance with base $\alpha = 1/2$. The system (3.4) becomes then

$$\begin{aligned} F_H(a|z) &= \frac{z}{4} + \frac{z}{4}F_H(a^2|z) + \frac{z}{2}(F_H(a|z)^2 + F_H(a^2|z)^2), \\ F_H(a^2|z) &= \frac{z}{4}F_H(a|z) + \frac{z}{2}(F_H(a|z)F_H(a^2|z) + F_H(a^2|z)F_H(a|z)). \end{aligned}$$

Observe that $F_H(a|z) = F_H(a^5|z)$ and $F_H(a^2|z) = F_H(a^4|z)$. The Hausdorff dimension of the branching random walk is then given by

$$\text{HD}^{(H)}(\Lambda) = \frac{\log(2F_H(a|\lambda) + 2F_H(a^2|\lambda))}{\log 2},$$

while $\text{HD}^{(H)}(\Omega) = 1$. The behaviour of $\text{HD}^{(H)}(\Lambda)$ in function of λ is qualitatively the same as in Figure 2.

4. PROOFS

4.1. Proof of Theorem 3.1. We first introduce some preliminary results on BRW. Using the description of a tree-indexed random walk it is easy to see that the distribution of the location of some particle in generation n has the same distribution as the location of a (non-branching) random walk on Γ after n steps, see [2].

Lemma 4.1. *Let $v \in \mathcal{T}$ with $|v| = n$ for some $n \geq 1$. Then,*

$$\mathbb{P}[S_v = y] = P[X_n = y] = \mu^{(n)}(y).$$

The following lemma will be used several times in our proofs. It gives a formula for the expected number of elements frozen in a set M , in the coloured branching random walk. This observation can be found for example in [21] or [13, Lemma 1]. Nevertheless, we give a short proof since it is one of the essential points where the generating function $F(\cdot, \cdot|z)$ intervenes.

Lemma 4.2. *For any $M \subseteq \Gamma$, we have $\mathbb{E}[Z_\infty(M)] = F(e, M|\lambda)$.*

Proof. For any $v \in \mathcal{T}$, let $\langle v_0 = \mathbf{r}, v_1, \dots, v_{|v|} = v \rangle$ be the unique geodesic from \mathbf{r} to v . Now, we define for any $n \in \mathbb{N}$

$$\text{Fr}_v^{(n)} := \begin{cases} 1, & \text{if } v \in M \text{ and } v_i \notin M \ \forall i \leq n-1, \\ 0, & \text{otherwise.} \end{cases}$$

In words, $\text{Fr}_v^{(n)}$ is the number of particles being frozen in v at time n . Using the well-known fact that $E[|\mathcal{T}_n|] = \lambda^n$ we obtain

$$\begin{aligned} \mathbb{E}\left[\sum_{v \in \mathcal{T}_n} \text{Fr}_v^{(n)}\right] &= \sum_{k \geq 1} \mathbb{E}\left[\sum_{v \in \mathcal{T}_n} \text{Fr}_v^{(n)} \mid |\mathcal{T}_n| = k\right] \mathbb{P}[|\mathcal{T}_n| = k] \\ &= \sum_{k \geq 1} \mathbb{P}[X_n \in M, \forall m \leq n-1 : X_m \notin M] k \mathbb{P}[|\mathcal{T}_n| = k] \\ &= \mathbb{P}[X_n \in M, \forall m \leq n-1 : X_m \notin M] \lambda^n. \end{aligned}$$

Summing over n finishes the proof. \square

The proof of Theorem 3.1 splits up into the proofs of the following Propositions 4.3, 4.4 and 4.5.

Recall from the definition of $\Omega_i^{(0)}$ and Ω_i that $\Omega_i^{(0)} \subseteq \Omega_i \subseteq \Omega$.

Proposition 4.3. *Ends of $\Omega_i^{(0)}$ occur in Λ with positive probability if and only if $\xi_i(\lambda) > 1$, that is, $\mathbb{P}[\Lambda \cap \Omega_i^{(0)} \neq \emptyset] > 0$ if and only if $\xi_i(\lambda) > 1$.*

Proof. It is convenient to work with the *coloured* branching random walk. In fact, the idea of the proof is to define an embedded Galton–Watson process that counts the number of particles that hit Γ_i , where $\xi_i(\lambda)$ will be the growth parameter.

We start the BRW with one particle in $e = e_i$. The first generation of the branching process is formed by those particles that are frozen in Γ_i^\times . Let us check that the number of those particles is almost surely finite. Since μ has finite support every particle visiting Γ_i^\times has to pass through $\text{supp}(\mu_i)$. Hence, $Z_\infty(\Gamma_i^\times) = Z_\infty(\text{supp}(\mu_i))$, which is almost surely finite since the BRW is transient. The second generation of the branching process is constructed as follows. For each particle frozen in some $x \in \Gamma_i^\times$ we start a new BRW where each particle when reaching $\Gamma_i \setminus \{x\}$ is frozen. Now, the second generation of the branching process consists of all these new frozen particles. Further generations are constructed inductively in the same way. Let ψ_n be the number of particles of this process at generation n . Obviously, $(\psi_n)_{n \geq 0}$ turns out to be a Galton–Watson process with mean

$$m_i = \mathbb{E}[Z_\infty(\text{supp}(\mu_i))] = F(e, \text{supp}(\mu_i) | \lambda) = \xi_i(\lambda).$$

Hence, this Galton–Watson process survives with positive probability if and only if $\xi_i(\lambda) > 1$; see e.g. [12, Theorem 6.1]. As a consequence, we have that Γ_i is visited infinitely many times with positive probability if $\xi_i(\lambda) > 1$. That is, $\mathbb{P}[\Lambda \cap \Omega_i^{(0)} \neq \emptyset] > 0$ if $\xi_i(\lambda) > 1$. On the other hand, $\xi_i(\lambda) \leq 1$ implies that Γ_i is almost surely visited only for a finite number of times and hence $\mathbb{P}[\Lambda \cap \Omega_i^{(0)} \neq \emptyset] = 0$. \square

The next step is to show $|\Lambda \cap \Omega_i| = \infty$ if $\xi_i(\lambda) > 1$.

Proposition 4.4. *If $\xi_i(\lambda) > 1$ then there are almost surely infinitely many cosets $x\Gamma_i$, where the branching random walk accumulates. That is, the set*

$$\{x \in \Gamma \mid \tau(x) \neq i, x\Omega_i^{(0)} \cap \Lambda \neq \emptyset\}$$

is almost surely infinite.

Proof. We construct the family tree \mathcal{T} of the BRW with branching distribution ν in the following way. We start with one geodesic line $v_\infty = \langle \mathbf{r}, v_1, v_2, \dots \rangle$ and attach to each of the vertices independent copies of Galton–Watson trees where the distribution of the first generation is $\tilde{\nu}(k) = \nu(k+1)$ for $k \geq 0$ and ν for the other generations. The trajectory along v_∞ has the same distribution as a non-branching random walk, compare with Lemma 4.1. Hence, S_{v_n} converges almost surely to a random infinite word $g_\infty = g_1 g_2 \dots \in \Omega_\infty$ as $n \rightarrow \infty$; here we mean convergence in the sense that the block length of the common prefix of the location of S_{v_n} and g_∞ tends to infinity. Moreover, we define the random indices $n_1 := \min\{m \in \mathbb{N} \mid g_m \in \Gamma_i\}$, and recursively $n_k := \min\{m \in \mathbb{N} \mid m > n_{k-1}, g_m \in \Gamma_i\}$.

Note that these indices are almost surely finite; see e.g. [9, Section 7.I]. Denote by \hat{v}_k the first vertex in v_∞ with $\hat{v}_k = g_1 \dots g_{n_k}$. Let B_k be the set of offspring of $\hat{v}_k = v_s$ different from v_{s+1} and denote by Λ_v the set of accumulation points of the descendants of some $v \in \mathcal{T}$. Moreover, we define A_k as the event that $\Lambda_v \cap S_v \Omega_i^{(0)} \neq \emptyset$ for some $v \in B_k$ with $\tau(v) = i$. Observe that the events A_k are i.i.d. since transitivity yields $\mathbb{P}[\Lambda_v \cap S_v \Omega_i^{(0)} \neq \emptyset] = \mathbb{P}[\Lambda \cap \Omega_i^{(0)} \neq \emptyset]$ for every $v \in \mathcal{T}$. Now, due to Proposition 4.3 and the fact that

$$\begin{aligned} P[B_k \neq \emptyset, \exists v \in B_k : \tau(S_v) = i] &= (1 - \nu(1)) \cdot \mathbb{P}[v \in B_k : \tau(S_v) = i \mid B_k \neq \emptyset] \\ &\geq (1 - \nu(1)) \cdot \alpha_i > 0 \end{aligned}$$

we have $P[A_k] \geq c$ for all k and some $c > 0$. Eventually, the Lemma of Borel–Cantelli yields that an infinite number of A_k 's occurs almost surely. \square

It remains to treat the critical and subcritical case $\xi_i(\lambda) \leq 1$ in order to complete the proof of Theorem 3.1.

Proposition 4.5. *If $\xi_i(\lambda) \leq 1$ then $\mathbb{P}[\Lambda \cap \Omega_i \neq \emptyset] = 0$.*

Proof. Due to Proposition 4.3 we have that $\mathbb{P}[\Lambda \cap x \Omega_i^{(0)} \neq \emptyset] = 0$ for all $x \in \Gamma$: indeed, each $x \in \Gamma$ is almost surely visited finitely often; each particle, which hits x , starts its own branching random walk at x and each of these branching random walk hits $x \Omega_i^{(0)}$ only finitely often with probability one. Since

$$\Lambda \cap \Omega_i = \bigcup_{x \in \Gamma : \tau(x) \neq i} (\Lambda \cap x \Omega_i^{(0)})$$

we conclude

$$\mathbb{P}[\Lambda \cap \Omega_i \neq \emptyset] = \sum_{x \in \Gamma : \tau(x) \neq i} \mathbb{P}[\Lambda \cap x \Omega_i^{(0)} \neq \emptyset] = 0.$$

\square

4.2. Proof of Theorem 3.5 and Corollary 3.7. First, we show that the proposed formula for the dimension is an upper bound for the upper box-counting dimension; see Proposition 4.9 in Subsection 4.2.1. In the second step we show that the proposed formula is also a lower bound for the Hausdorff dimension of Λ ; see Corollary 4.14 in Subsection 4.2.2. Finally, this will imply the proof of Theorem 3.5 and Corollary 3.7.

4.2.1. Upper Bound for the Box-Counting Dimension. In this part we show that $\log z^* / \log \alpha$ is an upper bound for $\overline{\text{BD}}(\Lambda)$. To this end we introduce the following notation: for $n \in \mathbb{N}$, we denote by

$$\mathcal{H}_n := \{x \in \Gamma \mid l(x) = n, x \text{ is visited by the branching random walk}\}$$

the set of visited sites at graph distance n . An important observation is that for each end $\omega \in \Lambda$ and every $m \in \mathbb{N}$, the branching random walk has to visit at least one vertex

$x_\omega \in \mathcal{H}_m$, where x_ω is in the ω -component of $\mathcal{X} \setminus B_{m-1}$. Thus,

$$\Lambda \subseteq \bigcup_{x \in \mathcal{H}_m} \{\omega \in \Omega \mid x \text{ lies in the } \omega\text{-component of } \mathcal{X} \setminus B_{m-1}\}.$$

This implies that Λ can be covered by $|\mathcal{H}_m|$ balls of radius α^m . Our strategy for the upper bound is to study the limit behaviour of $\mathbb{E}|\mathcal{H}_m|^{1/m}$ first and then the resulting limit behaviour of $|\mathcal{H}_m|^{1/m}$ as $m \rightarrow \infty$; see Lemma 4.8. This will eventually lead to the proposed upper bound for $\overline{\text{BD}}(\Lambda)$; see Proposition 4.9.

Observe that $x \in \mathcal{H}_m$ if and only if $Z_\infty(x) \geq 1$. Therefore, by Lemma 4.2,

$$1 \leq \mathbb{E}|\mathcal{H}_m| \leq \sum_{x \in \Gamma:l(x)=m} \mathbb{E}Z_\infty(x) = \sum_{x \in \Gamma:l(x)=m} F(e, x|\lambda) =: H_m.$$

We have that $H_{m+n} \leq H_m H_n$ and hence Fekete's lemma implies that $\lim_{n \rightarrow \infty} H_m^{1/m}$ exists. Recall the definitions of $\mathcal{F}(\lambda|z) = \sum_{m \geq 0} H_m z^m$, $\mathcal{F}_i^+(\lambda|z)$ and $\mathcal{F}_i(\lambda|z)$ in (2.8), (2.9) and (2.10). Due to (2.11) we get the equation

$$\mathcal{F}_i(\lambda|z) = \mathcal{F}_i^+(\lambda|z)(\mathcal{F}(\lambda|z) - \mathcal{F}_i(\lambda|z)),$$

or equivalently

$$\mathcal{F}_i(\lambda|z) = \mathcal{F}(\lambda|z) \frac{\mathcal{F}_i^+(\lambda|z)}{1 + \mathcal{F}_i^+(\lambda|z)}.$$

Hence,

$$\mathcal{F}(\lambda|z) = 1 + \sum_{i \in \mathcal{I}} \mathcal{F}_i(\lambda|z) = 1 + \mathcal{F}(\lambda|z) \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda|z)}{1 + \mathcal{F}_i^+(\lambda|z)},$$

or equivalently

$$\mathcal{F}(\lambda|z) = \frac{1}{1 - \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda|z)}{1 + \mathcal{F}_i^+(\lambda|z)}}. \quad (4.1)$$

This equation holds for every $z \in \mathbb{C}$ with $|z| < R(\mathcal{F})$, where $R(\mathcal{F})$ is the radius of convergence of $\mathcal{F}(\lambda|z)$. Since

$$1 \leq \lim_{m \rightarrow \infty} H_m^{1/m} = 1/R(\mathcal{F}),$$

we have

$$R(\mathcal{F}) \leq 1. \quad (4.2)$$

In order to determine $R(\mathcal{F})$ we have to find – by Pringsheim's Theorem – the smallest singularity point on the positive x -axis of $\mathcal{F}(\lambda|z)$. This smallest singularity point is either one of the radii of convergence $R(\mathcal{F}_i^+)$ of the functions $\mathcal{F}_i^+(\lambda|z)$ or the smallest real positive number z^* with

$$\sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda|z^*)}{1 + \mathcal{F}_i^+(\lambda|z^*)} = 1. \quad (4.3)$$

The next two lemmas imply that in fact $R(\mathcal{F}) = z^*$.

Lemma 4.6. $R(\mathcal{F}) \in (0, 1)$

*Proof.*¹ The fact that $R(\mathcal{F}) > 0$ follows from the fact that the Cayley graph grows not faster than exponentially. To see that $R(\mathcal{F}) < 1$ recall that Equation (2.3) states that the generating functions $F(e, x|z)$ and $G(e, x|z)$ are comparable, i.e., $G(e, x|\lambda) = F(e, x|\lambda)G(e, e|\lambda)$. Hence, for some $C > 0$ we have for all $m \in \mathbb{N}$ that

$$\sum_{x:l(x) \leq m} F(e, x|\lambda) \geq C \sum_{x:l(x) \leq m} G(e, x|\lambda).$$

The sum on the right hand side is the expected number of visits of the BRW in the ball B_m , the set of vertices $x \in \Gamma$ with $l(x) \leq m$. As we assumed the random walk to be of nearest neighbour type all particles up to generation m must be contained in the ball B_m . The expected population size at time m is just λ^m which eventually implies that H_m grows exponentially fast, since $\lim_{m \rightarrow \infty} H_m^{1/m}$ exists and is at least 1. □

Lemma 4.7. *For all $i \in \mathcal{I}$, $R(\mathcal{F}) = z^* < R(\mathcal{F}_i^+)$.*

Proof. Let us first consider the case $\xi_i(\lambda) < 1$, where we obtain

$$\begin{aligned} \mathcal{F}_i^+(\lambda|1) &= \sum_{x \in \Gamma_i^\times} F_i(e_i, x|\xi_i(\lambda)) \\ &= \frac{1}{G_i(e_i, e_i|\xi_i(\lambda))} \sum_{x \in \Gamma_i} G_i(e_i, x|\xi_i(\lambda)) - 1 \\ &= \frac{1}{G_i(e_i, e_i|\xi_i(\lambda))(1-\xi_i(\lambda))} - 1 < \infty. \end{aligned} \quad (4.4)$$

Hence, $\xi_i(\lambda) < 1$ implies $R(\mathcal{F}_i^+) \geq 1 > R(\mathcal{F})$. In the case of $\xi_i(\lambda) \geq 1$ the claim follows from the following inequality:

$$\frac{1}{R(\mathcal{F})} = \limsup_{n \rightarrow \infty} \left(\sum_{\substack{x \in \Gamma: \\ l(x)=n}} F(e, x|\lambda) \right)^{1/n} > \limsup_{n \rightarrow \infty} \left(\sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F(e, x|\lambda) \right)^{1/n} = \frac{1}{R(\mathcal{F}_1^+)}. \quad (4.5)$$

In order to prove (4.5) we define for $n \in \mathbb{N}$

$$a_n := \log \sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F(e, x|\lambda).$$

We have that $a_n \geq 0$ since

$$\sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F(e, x|\lambda) = \sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F_1(e_1, x|\xi_1(\lambda)) \geq \sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F_1(e_1, x|1) \geq \mathbb{P}[T_{S_1(n)} < \infty] = 1,$$

where $S_1(n) := \{x \in \Gamma_1 \mid l(x) = n\}$ and T_M is the stopping time for the random walk on Γ_1 (governed by μ_1) of the first visit of a set $M \subseteq \Gamma_1$. Furthermore, $(a_n)_{n \in \mathbb{N}}$ is a

¹This short proof was suggested by the referee.

subadditive sequence, that is, $a_m + a_n \geq a_{m+n}$ for all $m, n \in \mathbb{N}$. By Fekete's Lemma, the limit $\lim_{n \rightarrow \infty} a_n/n$ exists and is equal to $\inf_{n \in \mathbb{N}} a_n/n$, hence

$$\lim_{n \rightarrow \infty} \left(\sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F(e, x|\lambda) \right)^{1/n} = \frac{1}{R(\mathcal{F}_1^+)} = \inf_{n \in \mathbb{N}} \left(\sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F(e, x|\lambda) \right)^{1/n}.$$

The last equation implies that

$$\left(\sum_{\substack{x \in \Gamma_1: \\ l(x)=n}} F(e, x|\lambda) \right)^{1/n} \geq \frac{1}{R(\mathcal{F}_1^+)} =: q_1 \quad \forall n \in \mathbb{N}.$$

Observe that $\sum_{x \in \Gamma_2: l(x)=1} F(e, x|\lambda) \geq \xi_2(\lambda)$. Then, for all $n \in \mathbb{N}$:

$$\begin{aligned} H_n &= \sum_{\substack{x \in \Gamma: \\ l(x)=n}} F(e, x|\lambda) = \sum_{k=1}^n \sum_{\substack{x=x_1 \dots x_k \in \Gamma: \\ l(x)=n}} \prod_{j=1}^k F(e, x_j|\lambda) \\ &\geq \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\substack{x_1, \dots, x_k \in \Gamma_1: \\ l(x_1) + \dots + l(x_k) + k = n}} \xi_2(\lambda)^k \prod_{j=1}^k F(e, x_j|\lambda) \\ &\geq \sum_{k=1}^{\lfloor n/2 \rfloor} \sum_{\substack{n_1, \dots, n_k \in \mathbb{N}: \\ n_1 + \dots + n_k + k = n}} q_1^{n_1} \xi_2(\lambda) q_1^{n_2} \xi_2(\lambda) q_1^{n_3} \dots \xi_2(\lambda) q_1^{n_k} \xi_2(\lambda) \\ &\geq \sum_{k=1}^{\lfloor n/2 \rfloor} q_1^{n-k} \xi_2(\lambda)^k \binom{n-2k+k-1}{k-1}. \end{aligned}$$

In the last inequality the binomial coefficients arise as follows: we think of counting the number of possibilities of placing $n-k$ (undistinguishable) balls into k urns, where each urn should at least contain one ball. We note that $n-k-2 \geq \lfloor n/2 \rfloor - 1$ for all $k \leq \lfloor n/2 \rfloor - 1$. Therefore, with the help of the Binomial theorem we obtain:

$$\begin{aligned} H_n &\geq q_1^n \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \left(\frac{\xi_2(\lambda)}{q_1} \right)^{k+1} \binom{n-k-2}{k} \\ &\geq q_1^{n-1} \xi_2(\lambda) \sum_{k=0}^{\lfloor n/2 \rfloor - 1} \left(\frac{\xi_2(\lambda)}{q_1} \right)^k \binom{\lfloor n/2 \rfloor - 1}{k} \geq q_1^{n-1} \xi_2(\lambda) \left(1 + \frac{\xi_2(\lambda)}{q_1} \right)^{\lfloor n/2 \rfloor - 1}. \end{aligned}$$

Taking n -th roots on both sides and letting $n \rightarrow \infty$ yields

$$\liminf_{n \rightarrow \infty} \left(\sum_{\substack{x \in \Gamma: \\ l(x)=n}} F(e, x|\lambda) \right)^{1/n} \geq \frac{1}{R(\mathcal{F}_1^+)} \sqrt[1 + \frac{\xi_2(\lambda)}{q_1}]{} > \frac{1}{R(\mathcal{F}_1^+)}. \quad (4.6)$$

□

The next lemma gives an almost sure upper bound for $|\mathcal{H}_m|^{1/m}$ as $m \rightarrow \infty$. Its proof is a straightforward application of Markov's Inequality and the Lemma of Borel–Cantelli.

Lemma 4.8.

$$\limsup_{m \rightarrow \infty} |\mathcal{H}_m|^{1/m} \leq \frac{1}{z^*} \text{ almost surely.}$$

Eventually, we obtain the desired upper box-counting dimension.

Proposition 4.9.

$$\overline{\text{BD}}(\Lambda) \leq \frac{\log z^*}{\log \alpha}$$

Proof. Denote by $N(\alpha^m)$ the number of balls of radius of at most α^m needed to cover Λ . Then, for any $\varepsilon > 0$, $N(\alpha^m) \leq |\mathcal{H}_m| \leq (\frac{1}{z^*} + \varepsilon)^m$ almost surely for sufficiently large m . Therefore,

$$\overline{\text{BD}}(\Lambda) = \limsup_{m \rightarrow \infty} -\frac{\log N(\alpha^m)}{\log \alpha^m} \leq \limsup_{m \rightarrow \infty} -\frac{\log(\frac{1}{z^*} + \varepsilon)^m}{\log \alpha^m} = -\frac{\log(\frac{1}{z^*} + \varepsilon)}{\log \alpha}.$$

Letting $\varepsilon \rightarrow 0$ proves the claim. \square

4.2.2. Lower Bound for Hausdorff Dimension. In this section we will show that $\log z^* / \log \alpha$ is also a lower bound for the Hausdorff dimension of Λ . From this we may then conclude existence of the box-counting dimension since $\text{HD}(\Lambda) \leq \underline{\text{BD}}(\Lambda) \leq \overline{\text{BD}}(\Lambda)$. The main idea of the proof follows [13]. This idea² is to construct a sequence of embedded Galton–Watson trees τ_r in the BRW such that the limit set Λ_{τ_r} of the Galton–Watson trees are subsets of the limit set Λ , see Section 6.3 in [13]. As r goes to infinity we will have that $\text{HD}(\Lambda_{\tau_r}) \rightarrow \text{HD}(\Lambda)$. This approximation property relies mainly on the facts that particles travel essentially along geodesics segments and that limit sets of multi-type Galton–Watson trees are well understood. Both facts hold still true for free products of finite groups and the proof of the lower bound is analogous to the one for free groups in [13], albeit technically more involved. The case of infinite factors need some extra care, since in this case particles do not necessarily travel along geodesics and infinite-type Galton–Watson processes are not so easy to handle. To bypass these difficulties we approximate the infinite factors by increasing sequence of finite subgraphs. These subgraphs $\mathcal{X}_i^{(d)}$ are the subgraphs induced by the balls $B_i(d) := \{y \in \Gamma_i \mid l(y) \leq d\}$, $d \geq 1$. Letting $d \rightarrow \infty$ will give the optimal bound $\log z^* / \log \alpha$.

We add an additional vertex \dagger to $\mathcal{X}_i^{(d)}$, the “tomb”, such that all edges in \mathcal{X}_i exiting $B_i(d)$ now lead to the tomb. The random walk $(Y_n^{(i,d)})_{n \in \mathbb{N}_0}$ on $\mathcal{X}_i^{(d)}$ behaves like the random walk on Γ_i , with the exception that a particle leaving $B_i(d)$ dies. We now build the free product $\mathcal{X}^{(d)}$ from the $\mathcal{X}_i^{(d)}$, whose vertices are given by the set

$$\left\{ x_1 \dots x_n \in \Gamma \mid n \in \mathbb{N}, x_j \in \bigcup_{i \in \mathcal{I}} \mathcal{X}_i^{(d)} \setminus \{e_i, \dagger\}, x_j \in \mathcal{X}_i^{(d)} \Rightarrow x_{j+1} \notin \mathcal{X}_i^{(d)} \right\} \cup \{e, \dagger\},$$

²In this section the parameter r is not identified with $|\mathcal{I}|$ but is used as a parameter of the Galton–Watson trees τ_r as in [13].

where \dagger symbolizes the tomb. We identify $x \in \mathcal{X}^{(d)}$ with the corresponding element in Γ . Analogously to Subsection 2.1, we lift the random walks on the graphs $\mathcal{X}_i^{(d)}$ to a random walk $(X_n^{(d)})_{n \in \mathbb{N}_0}$ on $\mathcal{X}^{(d)}$ and define the associated BRW. We use the same notation (for Green functions, generating functions, etc.) as for the random walk on Γ itself but for reason of distinguishing we add superscripts “ (d) ”, that is, we write, e.g., $G^{(d)}(x, y|z)$ for the corresponding Green function of the random walk on $\mathcal{X}^{(d)}$. All involved generating functions on $\mathcal{X}^{(d)}$ have radii of convergence of at least R .

For any $x, y \in \Gamma$, we define $\overline{x : y}$ to be the set of vertices $w \in \Gamma$ such that there is a geodesic from x to y which passes through w . For $u \in \Gamma$, $d(u, \overline{x : y})$ is defined as the minimal distance w.r.t. the graph metric of u to any element of $\overline{x : y}$. In the case of the coloured branching random walk on $\mathcal{X}^{(d)}$, let $Z_\infty^{(d)}(y|x)$ be the overall number of blue particles arriving and freezing at $y \in \mathcal{X}^{(d)}$ under the assumption that the branching random walk is started with one blue particle at x . For $r \in \mathbb{N}$, we write $Z_{\infty, r}^{(d)}(y|x)$ for the overall number of particles counted in $Z_\infty^{(d)}(y|x)$ whose trail remain within distance r to a geodesic from x to y . In other words, in all sites u with $d(u, \overline{x : y}) > r$ every blue particle is coloured red. In the following we set $x_0 := x_1^{-1}$ for any $x = x_1 \dots x_m \in \mathcal{X}^{(d)}$. The proofs of the two following lemmas are similar to the ones of Lemma 4 and Proposition 7 in [13] and are therefore omitted.³

Lemma 4.10.

$$\lim_{r \rightarrow \infty} \inf_{x=x_1 \dots x_m \in \mathcal{X}^{(d)}} \left(\frac{\prod_{j=1}^m \mathbb{E} Z_{\infty, r}^{(d)}(x_1 \dots x_j | x_1 \dots x_{j-1})}{\mathbb{E} Z_\infty^{(d)}(x|e)} \right)^{1/l(x)} = 1.$$

For $x \in \mathcal{X}^{(d)}$, we define the event $E^{(d)}(x)$ that among the particles counted in $Z_\infty^{(d)}(x|e)$ there is at least one particle whose trail has not entered Γ_1^\times and enters the set

$$\{y \in \mathcal{X}^{(d)} \mid l(y) = l(x)\}$$

first at x . Obviously, $Z_\infty^{(d)}(x|e) \geq 1$ on the event $E^{(d)}(x)$ and hence $\mathbb{P}[E^{(d)}(x)] \leq \mathbb{E} Z_\infty^{(d)}(x|e)$.

Lemma 4.11.

$$\lim_{k \rightarrow \infty} \left(\min_{\substack{x=x_1 \dots x_m \in \mathcal{X}^{(d)}: \\ m \in \mathbb{N}, x_1 \notin \Gamma_1, l(x)=k}} \frac{\mathbb{P}[E^{(d)}(x)]}{\mathbb{E} Z_\infty^{(d)}(x|e)} \right)^{1/k} = 1.$$

³The reader may find all the details in the *arxiv.org* version of this paper.

Analogously to (2.9) and (2.10), we define for $i \in \mathcal{I}$ and $d \in \mathbb{N}$

$$\begin{aligned} \mathcal{L}_i^{(d)+}(\lambda|z) &:= \sum_{x \in \Gamma_i^\times} L^{(d)}(e, x|\lambda) z^{l(x)} = \sum_{x \in \Gamma_i^\times} L_i^{(d)}(e_i, x|\xi_i^{(d)}(\lambda)) z^{l(x)}, \\ \mathcal{L}_i^{(d)}(\lambda|z) &:= \sum_{n \geq 1} \sum_{\substack{x=x_1 \dots x_n \in \mathcal{X}^{(d)}: \\ \tau(x_1)=i}} L^{(d)}(e, x|\lambda) z^{l(x)} \\ &= \mathcal{L}_i^{(d)+}(\lambda|z) \left(1 + \sum_{j \in \mathcal{I} \setminus \{i\}} \mathcal{L}_j^{(d)}(\lambda|z) \right). \end{aligned} \quad (4.7)$$

Writing $\mathcal{L}^{(d)}(\lambda|z) := 1 + \sum_{i \in \mathcal{I}} \mathcal{L}_i^{(d)}(\lambda|z)$ we get analogously to Equation (4.1):

$$\mathcal{L}^{(d)}(\lambda|z) = \frac{1}{1 - \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z)}{1 + \mathcal{L}_i^{(d)+}(\lambda|z)}}.$$

Since every function $\mathcal{L}_i^{(d)+}(\lambda|z)$ is convergent and strictly increasing for all $z \geq 0$ there is some unique $z_{d,\mathcal{L}}^* > 0$ such that $\sum_{i \in \mathcal{I}} \mathcal{L}_i^{(d)+}(\lambda|z_{d,\mathcal{L}}^*) / (1 + \mathcal{L}_i^{(d)+}(\lambda|z_{d,\mathcal{L}}^*)) = 1$. The radius of convergence of $\mathcal{L}^{(d)}(\lambda|z)$ is then given by $z_{d,\mathcal{L}}^*$.

We define for $k \in \mathbb{N}$

$$S_k^* := \{x_1 \dots x_s \in \mathcal{X}^{(d)} \mid s \in \mathbb{N}, l(x) = k, x_1 \notin \Gamma_1, x_s \in \Gamma_1\}.$$

Since we excluded the case $|\mathcal{I}| = 2 = |\Gamma_1| = |\Gamma_2|$ we have that $S_2^* \neq \emptyset$ and $S_3^* \neq \emptyset$. Therefore, $S_k^* \neq \emptyset$ for all $2 \leq k \in \mathbb{N}$.

Lemma 4.12.

$$\limsup_{k \rightarrow \infty} \left(\sum_{x \in S_k^*} \mathbb{P}[E^{(d)}(x)] \right)^{1/k} = \frac{1}{z_{d,\mathcal{L}}^*}.$$

Proof. By Lemma 4.11, we have $\mathbb{P}[E^{(d)}(x)] \geq (1 - \varepsilon)^k \mathbb{E}Z_\infty^{(d)}(x|e)$ uniformly for all x with $l(x) = k$ if k is large enough. Recall also $\mathbb{P}[E^{(d)}(x)] \leq \mathbb{E}Z_\infty^{(d)}(x|e)$. Thus, it is sufficient to prove

$$\limsup_{k \rightarrow \infty} \left(\sum_{x \in S_k^*} \mathbb{E}Z_\infty^{(d)}(x|e) \right)^{1/k} = \frac{1}{z_{d,\mathcal{L}}^*}.$$

Since

$$\sum_{x \in S_k^*} \mathbb{E}Z_\infty^{(d)}(x|e) = \sum_{x \in S_k^*} F^{(d)}(e, x|\lambda) = \sum_{x \in S_k^*} \frac{G^{(d)}(e, e|\lambda)}{G^{(d)}(x, x|\lambda)} L^{(d)}(e, x|\lambda)$$

and $1 \leq G^{(d)}(x, x|\lambda) \leq G(x, x|\lambda) = G(e, e|\lambda) < \infty$ we have

$$\limsup_{k \rightarrow \infty} \left(\sum_{x \in S_k^*} L^{(d)}(e, x|\lambda) \right)^{1/k} = \limsup_{k \rightarrow \infty} \left(\sum_{x \in S_k^*} \mathbb{E}Z_\infty^{(d)}(x|e) \right)^{1/k}. \quad (4.8)$$

To determine the left-hand side of (4.8) we define further generating functions:

$$\begin{aligned}\mathcal{L}_{-1,1}^{(d)}(\lambda|z) &:= \sum_{n \geq 2} \sum_{\substack{x=x_1 \dots x_n \in \mathcal{X}^{(d)}: \\ x_1 \notin \Gamma_1^\times, x_n \in \Gamma_1^\times}} L^{(d)}(e, x|\lambda) z^{l(x)}, \\ \mathcal{L}^{(d)*}(\lambda|z) &:= \sum_{n \geq 1} \sum_{\substack{x=x_1 \dots x_n \in \mathcal{X}^{(d)}: \\ x_1, x_2, \dots, x_n \notin \Gamma_1}} L^{(d)}(e, x|\lambda) z^{l(x)}.\end{aligned}$$

For $k \in \mathbb{N}$, the coefficient of z^k in $\mathcal{L}_{-1,1}^{(d)}(\lambda|z)$ is just $\sum_{x \in S_k^*} L^{(d)}(e, x|\lambda)$. Due to Equation (4.7) we have

$$\mathcal{L}_1^{(d)}(\lambda|z) = \mathcal{L}_1^{(d)+}(\lambda|z) \cdot \left(1 + \sum_{i \in \mathcal{I} \setminus \{1\}} \mathcal{L}_i^{(d)}(\lambda|z)\right),$$

and hence the function $\mathcal{L}_{-1}^{(d)}(\lambda|z) := 1 + \sum_{i \in \mathcal{I} \setminus \{1\}} \mathcal{L}_i^{(d)}(\lambda|z)$ must have the same radius of convergence as $\mathcal{L}^{(d)}(\lambda|z)$, which is $z_{d,\mathcal{L}}^*$. Moreover, we have the following relations:

$$\begin{aligned}\mathcal{L}_{-1}^{(d)}(\lambda|z) &= 1 + \mathcal{L}_{-1,1}^{(d)}(\lambda|z)(1 + \mathcal{L}^{(d)*}(\lambda|z)) + \mathcal{L}^{(d)*}(\lambda|z), \\ \mathcal{L}_{-1,1}^{(d)}(\lambda|z) &\geq \mathcal{L}^{(d)*}(\lambda|z) \cdot \mathcal{L}_1^{(d)+}(\lambda|z).\end{aligned}$$

Since $\mathcal{L}_{-1,1}^{(d)}(\lambda|z), \mathcal{L}^{(d)*}(\lambda|z) \leq \mathcal{L}_{-1}^{(d)}(\lambda|z)$, the function $\mathcal{L}_{-1,1}^{(d)}$ has also radius of convergence of $z_{d,\mathcal{L}}^*$. \square

Now we show that $z_{d,\mathcal{L}}^*$ tends to z^* as $d \rightarrow \infty$. Since $z_{d,\mathcal{L}}^*$ is strictly decreasing as d grows and due to

$$\lim_{d \rightarrow \infty} L^{(d)}(e, x|\lambda) = L(e, x|\lambda) = F(e, x|\lambda) \quad (4.9)$$

we have $z_\infty = \lim_{d \rightarrow \infty} z_{d,\mathcal{L}}^* \geq z^*$. Assume now for a moment that $z^* < z_\infty$. Then $\mathcal{F}_i^+(\lambda|z_\infty) < \infty$: indeed, assume that $\lim_{d \rightarrow \infty} \mathcal{L}_j^{(d)+}(\lambda|z_\infty) = \mathcal{F}_j^+(\lambda|z_\infty) = \infty$ for some $j \in \mathcal{I}$. Then we get the following contradiction:

$$1 = \lim_{d \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z_{d,\mathcal{L}}^*)}{1 + \mathcal{L}_i^{(d)+}(\lambda|z_{d,\mathcal{L}}^*)} \geq \lim_{d \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z_\infty)}{1 + \mathcal{L}_i^{(d)+}(\lambda|z_\infty)} > 1, \quad (4.10)$$

since $\mathcal{L}_j^{(d)+}(\lambda|z_\infty)/(1 + \mathcal{L}_j^{(d)+}(\lambda|z_\infty))$ is arbitrarily close to 1 if d is large enough. Hence, $\mathcal{F}_i^+(\lambda|z_\infty) < \infty$. Now $z_\infty > z^*$ yields the following contradiction:

$$1 = \lim_{d \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z_{d,\mathcal{L}}^*)}{1 + \mathcal{L}_i^{(d)+}(\lambda|z_{d,\mathcal{L}}^*)} \geq \limsup_{d \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z_\infty)}{1 + \mathcal{L}_i^{(d)+}(\lambda|z_\infty)} = \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda|z_\infty)}{1 + \mathcal{F}_i^+(\lambda|z_\infty)} > 1.$$

which produces a contradiction. Thus,

$$\lim_{d \rightarrow \infty} z_{d,\mathcal{L}}^* = z^*. \quad (4.11)$$

Let $2 \leq k \in \mathbb{N}$ arbitrary, but fixed. Similar to [13] we define an embedded Galton–Watson process of the BRW on the free product $\mathcal{X}^{(d)}$. For $n \in \mathbb{N}_0$, we define generations $\text{gen}(n)$ S_{nk}^* and distinguished particles ζ_x associated to vertices $x \in \text{gen}(n)$ inductively as follows:

- (1) $\text{gen}(0) := \{e\}$ consists of one particle ζ_e located at e .
- (2) $y \in S_{(n+1)k}^*$ belongs to $\text{gen}(n+1)$ if and only if there exists a distinguished particle ζ_x in $\text{gen}(n)$ such that some of its offspring particles counted in $Z_\infty^{(d)}(y|x)$ has a trail which
 - (a) remains in the set

$$\Gamma(x) := \{y \in \Gamma \mid y \text{ has the form } xw_1 \dots w_s \text{ with } w_1 \notin \Gamma_1, s \geq 1\} \cup \{x\},$$
 - (b) hits the set $\{w \in \mathcal{X}^{(d)} \mid l(w) = (n+1)k\}$ first at y .
- (3) The first particle hitting $y \in S_{(n+1)k}^*$ becomes the distinguished particle ζ_y .

Let ϕ_n denote the number of particles in generation n . Since we have the same offspring distribution at every $x \in S_{nk}^*$, $(\phi_n)_{n \geq 0}$ defines a Galton–Watson process with mean $M_{d,k}$.

Corollary 4.13.

$$\limsup_{k \rightarrow \infty} M_{d,k}^{1/k} = \frac{1}{z_{d,\mathcal{L}}^*}.$$

Proof. The claim follows directly with Lemma 4.12 since $M_{d,k} = \sum_{x \in S_k^*} \mathbb{P}[E^{(d)}(x)]$. \square

Applying Hawkes’ Theorem as in Corollary 7 in [13] together with Equation (4.11) yields the following Corollary.

Corollary 4.14. $\text{HD}(\Lambda \cap \Omega_\infty) \geq \frac{\log z^*}{\log \alpha}$.

Proof of Theorem 3.5. The following chains of inequalities summarize the previous results and finish the proof of the theorem:

$$\begin{aligned} \frac{\log z^*}{\log \alpha} &\leq \text{HD}(\Lambda) \leq \underline{\text{BD}}(\Lambda) \leq \overline{\text{BD}}(\Lambda) \leq \frac{\log z^*}{\log \alpha}, \\ \frac{\log z^*}{\log \alpha} &\leq \text{HD}(\Lambda \cap \Omega_\infty) \leq \underline{\text{BD}}(\Lambda \cap \Omega_\infty) \leq \overline{\text{BD}}(\Lambda \cap \Omega_\infty) \leq \overline{\text{BD}}(\Lambda) \leq \frac{\log z^*}{\log \alpha}. \end{aligned}$$

\square

Proof of Corollary 3.7. It is well-known that the Hausdorff dimension of a countable union $\bigcup_i B_i$ of sets $B_i \subseteq \Omega$ equals the supremum of the Hausdorff dimensions of the single sets B_i . Thus,

$$\text{HD}(\Lambda \cap \Omega_i) = \sup_{x \in \Gamma: \tau(x) \neq i} \text{HD}(\Lambda \cap x\Omega_i^{(0)}) \leq \sup_{x \in \Gamma: \tau(x) \neq i} \overline{\text{BD}}(\Lambda \cap x\Omega_i^{(0)}).$$

For arbitrary, but fixed $x \in \Gamma$ with $\tau(x) \neq i$, denote by $\mathcal{H}_m^{(x)}$ the vertices $y \in x\Gamma_i$ with $l(y) = l(x) + m$, which are visited by the branching random walk. Therefore,

$$\mathbb{E}|\mathcal{H}_m^{(x)}| \leq \sum_{y \in \Gamma_i: l(y)=m} F(e, xy|\lambda) = F(e, x|\lambda) \sum_{y \in \Gamma_i: l(y)=m} F(e, y|\lambda).$$

Define

$$\mathcal{F}_{x,i}^+(\lambda|z) := F(e, x|\lambda) \sum_{m \geq 1} \sum_{y \in \Gamma_i: l(y)=m} F(e, y|\lambda) z^m.$$

The radius of convergence of $\mathcal{F}_{x,i}^+(\lambda|z)$ is obviously $R(\mathcal{F}_i^+)$. Therefore, Lemma 4.7 yields $\limsup_{m \rightarrow \infty} (\mathbb{E}|\mathcal{H}_m^{(x)}|)^{1/m} \leq 1/R(\mathcal{F}_i^+) < 1/z^*$. The rest follows analogously to the proofs of Lemma 4.8 and Proposition 4.9. \square

4.3. Proof of Theorem 3.8 and Corollary 3.9. In order to prove Theorem 3.8 we can follow the argumentation of the proof of Theorem 3.5. For this purpose, we define for $m \in \mathbb{N}$ and $i \in \mathcal{I}$

$$S_i(m) := |\{x \in \Gamma_i \mid l(x) = m\}|, \quad S(m) := |\{x \in \Gamma \mid l(x) = m\}|,$$

$$S^{(i)}(m) := |\{x = x_1 \dots x_n \in S(m) \mid n \in \mathbb{N}, x_1 \in \Gamma_i\}|.$$

To cover Ω by balls of radius α^m we need at least $S(m-1)$ balls: indeed, for all $x, y \in \Gamma$, $x \neq y$, with $l(x) = l(y) = m-1$ we can choose $v_x \in \Gamma_*^\times \setminus \Gamma_{\tau(x)}$ and $v_y \in \Gamma_*^\times \setminus \Gamma_{\tau(y)}$; then all balls of the form $B(\omega_1, \alpha^m)$ and $B(\omega_2, \alpha^m)$, where xv_x lies in the ω_1 -component of $\mathcal{X} \setminus B_{m-1}$ and yv_y in the ω_2 -component, do not intersect. Apparently, we need at most $S(m)$ balls of radius α^m to cover Ω . Obviously, the same holds for covering Ω_∞ . We are now interested in the behaviour of $S(m)^{1/m}$ as $m \rightarrow \infty$. We define

$$\begin{aligned} \mathcal{S}_i^+(z) &:= \sum_{m \geq 1} S_i(m) z^m, & \mathcal{S}_i(z) &:= \sum_{m \geq 1} S^{(i)}(m) z^m, \\ \mathcal{S}(z) &:= \sum_{m \geq 0} S(m) z^m = 1 + \sum_{i \in \mathcal{I}} \mathcal{S}^{(i)}(z). \end{aligned}$$

Analogously to the computations in Section 4.2.1 – we just replace the functions $\mathcal{F}_i^+(\lambda|z)$, $\mathcal{F}_i(\lambda|z)$ and $\mathcal{F}(\lambda|z)$ by the functions $\mathcal{S}_i^+(z)$, $\mathcal{S}_i(z)$ and $\mathcal{S}(z)$ – we get

$$\mathcal{S}(z) = \frac{1}{1 - \sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^+(z)}{1 + \mathcal{S}_i^+(z)}}. \quad (4.12)$$

Lemma 4.15.

$$\lim_{m \rightarrow \infty} S(m)^{1/m} = \frac{1}{z_{\mathcal{S}}^*} < 1,$$

where $z_{\mathcal{S}}^*$ is the smallest positive real number with

$$\sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^+(z_{\mathcal{S}}^*)}{1 + \mathcal{S}_i^+(z_{\mathcal{S}}^*)} = 1.$$

Proof. Obviously, $R(\mathcal{S}) \leq R(\mathcal{F}) < 1$ since $F(e, x|\lambda) < 1$ for all $x \in \Gamma \setminus \{e\}$. The equation $R(\mathcal{S}) = z_{\mathcal{S}}^*$ follows now analogously to the proof of Lemma 4.7. This yields

$$\limsup_{m \rightarrow \infty} S(m)^{1/m} = \frac{1}{z_{\mathcal{S}}^*} = \frac{1}{R(\mathcal{S})} > 1.$$

Thus, it is sufficient to prove convergence of $S(m)^{1/m}$ as $m \rightarrow \infty$. By transitivity of Γ , we have $S(m)S(n) \geq S(m+n)$ for all $m, n \in \mathbb{N}$. Therefore, $\log S(m) + \log S(n) \geq \log S(m+n)$, that is, $(\log S(m))_{m \in \mathbb{N}}$ forms a subadditive sequence. By Fekete's Lemma, $\frac{1}{m} \log S(m) = \log S(m)^{1/m}$ converges to some constant s , that is, $S(m)^{1/m}$ converges to e^s , which must equal $1/z_{\mathcal{S}}^*$. \square

Remark 4.16. *One can show analogously to Lemma 4.7 that $z_{\mathcal{S}}^* < R(\mathcal{S}_i^+)$, where $R(\mathcal{S}_i^+)$ is the radius of convergence of $\mathcal{S}_i^+(z)$. In particular, $z_{\mathcal{S}}^*$ is the radius of convergence of $\mathcal{S}(z)$.*

We can conclude by giving a formula for $\text{BD}(\Omega)$ and observing that the box-counting dimension of Ω results from the dimension of Ω_∞ .

Proposition 4.17.

$$\text{BD}(\Omega) = \text{BD}(\Omega_\infty) = \frac{\log z_{\mathcal{S}}^*}{\log \alpha}.$$

Proof. Recall the remarks at the beginning of this section concerning the minimal and maximal number of balls needed to cover Ω_∞ . This yields

$$\begin{aligned} \underline{\text{BD}}(\Omega) &\geq \underline{\text{BD}}(\Omega_\infty) \geq \liminf_{m \rightarrow \infty} -\frac{\log S(m-1)}{\log \alpha^m} \\ &= \liminf_{m \rightarrow \infty} -\frac{\log S(m-1)^{1/(m-1)} m-1}{\log \alpha} = \frac{\log z_{\mathcal{S}}^*}{\log \alpha}. \end{aligned}$$

Analogously,

$$\overline{\text{BD}}(\Omega_\infty) \leq \overline{\text{BD}}(\Omega) \leq \limsup_{m \rightarrow \infty} -\frac{\log S(m)}{\log \alpha^m} = \limsup_{m \rightarrow \infty} -\frac{\log S(m)^{1/m}}{\log \alpha} = \frac{\log z_{\mathcal{S}}^*}{\log \alpha}.$$

Both inequality chains together yield the formula for the box-counting dimension. \square

Finally, we can prove the formula for the Hausdorff dimensions of Ω and Ω_∞ .

Proof of Theorem 3.8. It is sufficient to show that $\text{HD}(\Omega_\infty) \geq \frac{\log z_{\mathcal{S}}^*}{\log \alpha}$. Define for $d, k \in \mathbb{N}$ and $i \in \mathcal{I}$

$$\begin{aligned} S_i^{(d)+}(k) &= |\{x \in \mathcal{X}_i^{(d)} \setminus \{\dagger\} \mid l(x) = k\}|, & S^{(d)}(k) &= |\{x \in \mathcal{X}^{(d)} \setminus \{\dagger\} \mid l(x) = k\}|, \\ S_i^{(d)}(k) &= |\{x_1 \dots x_s \in \mathcal{X}^{(d)} \setminus \{\dagger\} \mid s \in \mathbb{N}, l(x) = k, x_1 \in \Gamma_i\}|, \\ S_{-1}^{(d)}(k) &= |\{x_1 \dots x_s \in \mathcal{X}^{(d)} \setminus \{\dagger\} \mid s \in \mathbb{N}, l(x) = k, x_1 \notin \Gamma_1\}|, \\ S_{-1,-1}^{(d)}(k) &= |\{x_1 \dots x_s \in \mathcal{X}^{(d)} \setminus \{\dagger\} \mid s \in \mathbb{N}, l(x) = k, x_1, x_s \notin \Gamma_1\}|, \\ S_{-1,1}^{(d)}(k) &= |\{x_1 \dots x_s \in \mathcal{X}^{(d)} \setminus \{\dagger\} \mid s \in \mathbb{N}, l(x) = k, x_1 \notin \Gamma_1, x_s \in \Gamma_1\}|. \end{aligned}$$

The associated generating functions are given by

$$\begin{aligned} S_i^{(d)+}(z) &= \sum_{k \geq 1} S_i^{(d)+}(k) z^k, & S^{(d)}(z) &= \sum_{k \geq 0} S^{(d)}(k) z^k, \\ S_i^{(d)}(z) &= \sum_{k \geq 1} S_i^{(d)}(k) z^k, & S_{-1}^{(d)}(z) &= \sum_{k \geq 1} S_{-1}^{(d)}(k) z^k, \\ S_{-1,-1}^{(d)}(z) &= \sum_{k \geq 1} S_{-1,-1}^{(d)}(k) z^k, & S_{-1,1}^{(d)}(z) &= \sum_{k \geq 1} S_{-1,1}^{(d)}(k) z^k. \end{aligned}$$

Once again we can write

$$\mathcal{S}^{(d)}(z) = \frac{1}{1 - \sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^{(d)+}(z)}{1 + \mathcal{S}_i^{(d)+}(z)}}$$

and obtain

$$\mathcal{S}_2^{(d)+}(z) \mathcal{S}_1^{(d)+}(z) \mathcal{S}_{-1,-1}^{(d)}(z) \mathcal{S}_1^{(d)+}(z) \leq \mathcal{S}_{-1,1}^{(d)}(z) = \mathcal{S}_{-1}^{(d)}(z) - \mathcal{S}_{-1,-1}^{(d)}(z).$$

Thus, $\mathcal{S}_{-1,1}^{(d)}(z)$ and $\mathcal{S}_{-1}^{(d)}(z)$ have the same radius of convergence. Moreover,

$$\mathcal{S}_2^{(d)+}(z) \mathcal{S}_1^{(d)}(z) \leq \mathcal{S}_{-1}^{(d)}(z) = \mathcal{S}^{(d)}(z) - \mathcal{S}_1^{(d)}(z) - 1.$$

That is, $\mathcal{S}_{-1,1}^{(d)}(z)$ and $\mathcal{S}^{(d)}(z)$ have the same radius of convergence, which is given by $z_{d,\mathcal{S}}^*$, the smallest positive solution satisfying

$$1 = \sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^{(d)+}(z)}{1 + \mathcal{S}_i^{(d)+}(z)}.$$

Since $z_{d,\mathcal{S}}^*$ is strictly decreasing as $d \rightarrow \infty$ we have that $\lim_{d \rightarrow \infty} z_{d,\mathcal{S}}^* = z_{\mathcal{S}}^*$. This can be seen by contradiction. Indeed, if $\lim_{d \rightarrow \infty} z_{d,\mathcal{S}}^* = z_{\infty,\mathcal{S}}^* > z_{\mathcal{S}}^*$ then $\mathcal{S}_i^+(z_{\infty,\mathcal{S}}^*) < \infty$ for all $i \in \mathcal{I}$ (this is proven analogously as explained in Equation (4.10)) and therefore

$$1 = \lim_{d \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^{(d)+}(z_{d,\mathcal{S}}^*)}{1 + \mathcal{S}_i^{(d)+}(z_{d,\mathcal{S}}^*)} \geq \lim_{d \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^{(d)+}(z_{\infty,\mathcal{S}}^*)}{1 + \mathcal{S}_i^{(d)+}(z_{\infty,\mathcal{S}}^*)} = \sum_{i \in \mathcal{I}} \frac{\mathcal{S}_i^+(z_{\infty,\mathcal{S}}^*)}{1 + \mathcal{S}_i^+(z_{\infty,\mathcal{S}}^*)} > 1,$$

a contradiction. Thus,

$$\left(\mathcal{S}_{-1,1}^{(d)}(k) \right)^{1/k} \xrightarrow{k \rightarrow \infty} \frac{1}{z_{d,\mathcal{S}}^*} \xrightarrow{d \rightarrow \infty} \frac{1}{z_{\mathcal{S}}^*}.$$

We can embed a “deterministic” Galton–Watson tree into the free product analogously to Subsection 4.2.2, where each generation has exactly $\mathcal{S}_{-1,1}^{(d)}(k)$ descendants. By Hawkes’s Theorem, the Hausdorff dimension of the boundary of the embedded tree is bounded from below by $\log z_{d,\mathcal{S}}^* / \log \alpha$, and therefore $\text{HD}(\Omega_\infty) \geq \log z_{\mathcal{S}}^* / \log \alpha$. \square

Proof of Corollary 3.9. Analogously to the proof of Corollary 3.7 and by Remark 4.16, we can use the property $\text{HD}(\cup_i B_i) = \sup_i \text{HD}(B_i)$ for all countable unions of sets $B_i \subseteq \Omega$ in order to show that

$$\text{HD}(\Omega_i) = \sup_{x \in \Gamma: \tau(x) \neq i} \text{HD}(x\Omega_i^{(0)}) \leq \overline{\text{BD}}(\Omega_i^{(0)}) < \text{BD}(\Omega_\infty) = \text{HD}(\Omega_\infty).$$

\square

4.4. Proof of Theorem 3.10.

Proof of Theorem 3.10 (1). In the following we write $z^* = z^*(\lambda)$ in order to distinguish the solutions of (3.1) for different values of λ . Note that $z^*(\lambda_1) > z^*(\lambda_2)$ if $\lambda_1 < \lambda_2$. This implies the strictly increasing behaviour of Φ in the interval $(1, R]$. Recall that the BRW does almost surely *not* survive in the limit case $\lambda = 1$, yielding $\Phi(1) = 0$. Moreover, if $\lambda > R$ then the BRW is recurrent and thus $\text{HD}(\Lambda) = \text{HD}(\Omega)$. \square

The proof of Theorem 3.10 (2) splits up into the following two lemmas:

Lemma 4.18. Φ is continuous in $[1, \infty) \setminus \{R\}$ and continuous from the left at $\lambda = R$.

Proof. In order to prove continuity of Φ , it is sufficient to prove continuity of the mapping $\lambda \mapsto z^* = z^*(\lambda)$. First, we prove continuity from the left at $\lambda_0 \in (1, \infty)$. For this purpose, let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of strictly increasing real numbers with $\lambda_n < \lambda_0$ and $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$. We use a proof by contradiction. Assume $z_0 := \lim_{n \rightarrow \infty} z^*(\lambda_n) > z^*(\lambda_0)$ (by simple domination arguments, $z^*(\lambda_n)$ can not be less than $z^*(\lambda_0)$). We have that $z^*(\lambda_n)$ is strictly decreasing and

$$\mathcal{F}_i^+(\lambda_n | z^*(\lambda_0)) + \xi_i(1)(z_0 - z^*(\lambda_0)) \leq \mathcal{F}_i^+(\lambda_n | z_0) < \infty.$$

Here we used the fact that the coefficient of z in $\mathcal{F}_i^+(\lambda | z)$ is at least $\xi_i(1)$. We set $c := \xi_i(1)(z_0 - z^*(\lambda_0))$. Since $f(x)/(1 + f(x))$ is strictly increasing in $[1, \infty)$ if $f(x)$ is a strictly increasing function on $[1, \infty)$ we get the following contradiction:

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_n | z^*(\lambda_n))}{1 + \mathcal{F}_i^+(\lambda_n | z^*(\lambda_n))} \\ &\geq \limsup_{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_n | z_0)}{1 + \mathcal{F}_i^+(\lambda_n | z_0)} \\ &\geq \limsup_{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_n | z^*(\lambda_0)) + c}{1 + \mathcal{F}_i^+(\lambda_n | z^*(\lambda_0)) + c} \\ &= \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_0 | z^*(\lambda_0)) + c}{1 + \mathcal{F}_i^+(\lambda_0 | z^*(\lambda_0)) + c} \\ &> \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_0 | z^*(\lambda_0))}{1 + \mathcal{F}_i^+(\lambda_0 | z^*(\lambda_0))} = 1. \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} z^*(\lambda_n) = z^*(\lambda_0)$.

Since $\text{HD}(\Lambda) = \text{HD}(\Omega)$ for all $\lambda > R$, it remains to prove continuity from the right for $\lambda_0 \in (1, R)$. We make a case distinction whether $\xi_i(\lambda_0) < 1$ or not. If $\xi_i(\lambda_0) < 1$ then $\mathcal{F}_i^+(\lambda_0 + \delta | 1) < \infty$ for all $\delta > 0$ with $\xi_i(\lambda_0 + \delta) < 1$ according to (4.4). Moreover, $z^*(\lambda_0) < 1$. Therefore, continuity from the right follows directly from the Implicit Function Theorem, since $z^* = z^*(\lambda)$ is given by the equation

$$1 = \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda | z^*(\lambda))}{1 + \mathcal{F}_i^+(\lambda | z^*(\lambda))}.$$

We note that the derivative $\partial \mathcal{F}_i^+(\lambda | z) / \partial z$ evaluated at $z = z^*(\lambda)$ is positive and finite, since $z^*(\lambda)$ is strictly smaller than the radius of convergence of $\mathcal{F}_i^+(\lambda | z)$; see Lemma 4.7.

Now we turn to the case $\xi_i(\lambda_0) \geq 1$. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a sequence of strictly decreasing real numbers with $\lambda_0 < \lambda_n < R$ and $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$. Assume $z_0 := \lim_{n \rightarrow \infty} z^*(\lambda_n) < z^*(\lambda_0)$ (by simple domination arguments, $z^*(\lambda_n)$ can not be larger than $z^*(\lambda_0)$). Observe that $z^*(\lambda_n)$ is strictly increasing. By (4.6), there is $C := \sqrt{1 + \xi_2(1)/(2|\text{supp}(\mu_1)|)} > 1$ such

that $Cz^*(\lambda_n) \leq R(\mathcal{F}_i^+)$ for all $n \in \mathbb{N}$. Choose $\tilde{C} \in (1, C)$ such that $\tilde{C}z_0 < z^*(\lambda_0)$ and choose $N \in \mathbb{N}$ large enough such that $\tilde{C}z^*(\lambda_n) \geq z_0$ for all $n \geq N$. Therefore,

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_n | z^*(\lambda_n))}{1 + \mathcal{F}_i^+(\lambda_n | z^*(\lambda_n))} \\ &\leq \lim_{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_n | \tilde{C}z_0)}{1 + \mathcal{F}_i^+(\lambda_n | \tilde{C}z_0)} = \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_0 | \tilde{C}z_0)}{1 + \mathcal{F}_i^+(\lambda_0 | \tilde{C}z_0)} < 1, \end{aligned}$$

a contradiction. Consequently, $\lim_{n \rightarrow \infty} z^*(\lambda_n) = z^*(\lambda_0)$.

It remains to prove continuity from the right at $\lambda_0 = 1$. In this case $\xi_i(1) < 1$. Once again $\mathcal{F}_i^+(\lambda_0 + \delta | 1) < \infty$ for all $\delta > 0$ with $\xi_i(\lambda_0 + \delta) < 1$ according to (4.4). Let $(\lambda_n)_{n \in \mathbb{N}}$ be a strictly decreasing sequence of real numbers with limit 1. We write $z_0 = \lim_{n \rightarrow \infty} z^*(\lambda_n) \leq 1$. Then, for n large enough,

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_n | z^*(\lambda_n))}{1 + \mathcal{F}_i^+(\lambda_n | z^*(\lambda_n))} \\ &\leq \lim_{n \rightarrow \infty} \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(\lambda_n | z_0)}{1 + \mathcal{F}_i^+(\lambda_n | z_0)} = \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(1 | z_0)}{1 + \mathcal{F}_i^+(1 | z_0)}. \end{aligned}$$

In order to finish the proof we verify that $z^*(1) = 1$, from which $z_0 = z^*(1) = 1$ follows. Indeed, by Equation (4.4) we get

$$\sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i^+(1 | 1)}{1 + \mathcal{F}_i^+(1 | 1)} = \sum_{i \in \mathcal{I}} \left(1 - G_i(e_i, e_i | \xi_1(1)) (1 - \xi_i(1)) \right).$$

From [9, Lemma 5.1] follows that $1 - G_i(e_i, e_i | \xi_1(1)) (1 - \xi_i(1))$ is just the probability that a single random walk on Γ tends to an infinite word of the form $x_1 x_2 \cdots \in \Omega_\infty$ with $x_1 \in \Gamma_i^\times$, that is, the above sum equals 1. \square

The next result completes the proof of Theorem 3.10 (2):

Lemma 4.19. *For all $\lambda \in [1, R]$, $\text{HD}(\Lambda) \leq \frac{1}{2} \text{HD}(\Omega)$.*

Proof. Define the function

$$\mathcal{F}^{(2)}(\lambda | z) := \sum_{x \in \Gamma} F(e, x | \lambda)^2 z^{l(x)},$$

whose radius of convergence is denoted by z_2^* . The Cauchy-Schwarz Inequality gives then

$$\begin{aligned} \frac{1}{z^*} &= \limsup_{m \rightarrow \infty} \left(\sum_{x \in \Gamma: l(x)=m} F(e, x|\lambda) \right)^{1/m} \\ &\leq \limsup_{m \rightarrow \infty} \sqrt{\left(\sum_{x \in \Gamma: l(x)=m} F(e, x|\lambda)^2 \right)^{1/m}} \cdot \limsup_{m \rightarrow \infty} \sqrt{\left(\sum_{x \in \Gamma: l(x)=m} 1^2 \right)^{1/m}} \\ &= \sqrt{\frac{1}{z_2^*}} \cdot \sqrt{\frac{1}{z_S^*}}. \end{aligned}$$

To prove the claim of the lemma it suffices (by the formulas given in Theorems 3.5 and 3.8) to show that $z_2^* \geq 1$. First,

$$\begin{aligned} \mathcal{F}^{(2)}(\lambda|1) &= \sum_{x \in \Gamma} F(e, x|\lambda)^2 = \frac{1}{G(e, e|\lambda)^2} \sum_{x \in \Gamma} G(e, x|\lambda)^2 \\ &= \frac{1}{G(e, e|\lambda)^2} \sum_{x \in \Gamma} \left(\sum_{n \geq 0} p^{(n)}(e, x) \lambda^n \right)^2. \end{aligned}$$

For given $x \in \Gamma$, the coefficient of λ^n in the inner squared sum can – by symmetry – be rewritten as

$$\frac{1}{G(e, e|\lambda)^2} \sum_{m=0}^n p^{(m)}(e, x) p^{(n-m)}(x, e). \quad (4.13)$$

Thus, every path $[x_0 = e, x_1, \dots, x_n = e]$ of length n (consisting of $n + 1$ vertices) from e to e is counted $n + 1$ times, since every x_i can play the role of x in Equation (4.13). That is,

$$\mathcal{F}^{(2)}(\lambda|z) = \frac{1}{G(e, e|\lambda)^2} \sum_{n \geq 0} p^{(n)}(e, e) \cdot (n + 1) \cdot \lambda^n = \frac{\lambda G'(e, e|\lambda)}{G(e, e|\lambda)^2} + \frac{1}{G(e, e|\lambda)}.$$

From this follows $z_2^* \geq 1$ whenever $\lambda < R$ or $G'(e, e|R) < \infty$, and thus $\text{HD}(\Lambda) \leq \frac{1}{2} \text{HD}(\Omega)$ for $\lambda < R$. By Lemma 4.18, the proposed inequality holds – due to continuity from the left – also in the case $\lambda = R$. \square

In order to prove Theorem 3.10 (3) we start with the following lemma:

Lemma 4.20. *For all $i \in \mathcal{I}$, $G'_i(e_i, e_i|\xi_i(R)) < \infty$.*

Proof. From [25, Prop. 9.18] follows $\xi_i(R) \leq R_i$, where R_i is the radius of convergence of $G_i(e_i, e_i|z)$. If $\xi_i(R) < R_i$ then the claim of the lemma is obvious. Assume now that $\xi_i(R) = R_i$. Then, by [25, Lemma 17.1.(a)], $R G(e, e|R) = R_i G_i(e_i, e_i|R_i)/\alpha_i$. Therefore, $G_i(e_i, e_i|R_i) < \infty$ since $G(e, e|R) < \infty$ by non-amenability of Γ . If $G'_i(e_i, e_i|R_i) = \infty$ would hold, we would get a contradiction to $\xi_i(R) = R_i$ by [25, Equ. (9.14), Thm. 9.22, Lemma 17.1.(a)]. \square

Let us remark that $F'_i(e_i, x | \xi_i(R)) = F'_i(x, e_i | \xi_i(R)) < \infty$ for all $x \in \Gamma_*^\times$; this can be easily verified with the help of the inequality

$$\mu_i^{(n+|x|)}(e_i) \geq \mu_i^{(|x|)}(x) \cdot \mathbb{P}[Y_n^{(i)} = e_i, \forall m < n : Y_m^{(i)} \neq e_i \mid Y_0^{(i)} = x] \quad \text{for all } n \in \mathbb{N},$$

where $(Y_n^{(i)})_{n \in \mathbb{N}}$ is a random walk on Γ_i governed by μ_i . We proceed now with expanding the Green function $G(z) := G(e, e|z)$ in a neighbourhood of $z = R$. By [25, Prop. 17.4] and [4, Sec. 3 & 4], we have

$$G(z) = \begin{cases} G(R) + g_1 \cdot \sqrt{R-z} + \mathbf{o}(\sqrt{R-z}), & \text{if } G'(R) = \infty, \\ G(R) - G'(R) \cdot (R-z) + \mathbf{o}(R-z), & \text{if } G'(R) < \infty. \end{cases}$$

We write in the following $c := 1/2$, if $G'(R) = \infty$, and $c := 1$ otherwise. The next aim is to show that the functions $F(e, x|z)$, $x \in \Gamma \setminus \{e\}$, have the same expansions.

Lemma 4.21. *For all $x \in \Gamma \setminus \{e\}$, there are constants $f_x \neq 0$ such that*

$$F(e, x|z) = F(e, x|R) + f_x \cdot (R-z)^c + \mathbf{o}((R-z)^c).$$

Proof. We consider the case $c = 1$ first. By [4, Lemma 3.2], we have $0 < \xi'_i(R) < \infty$, that is, we can write

$$\xi_i(z) = \xi_i(R) - \xi'_i(R) \cdot (R-z) + \mathbf{o}(R-z).$$

In the following we write $F_i(e_i, x|z) = \sum_{n \geq 1} f_n(x) z^n$ for $x \in \Gamma_i^\times$. Therefore,

$$F(e, x|z) = F_i(e_i, x | \xi_i(z)) = \sum_{n \geq 1} f_n(x) (\xi_i(R) - \xi'_i(R) \cdot (R-z) + \mathbf{o}(R-z))^n. \quad (4.14)$$

The coefficient of $(R-z)$ is given by

$$-\xi'_i(R) \cdot \sum_{n \geq 1} n \cdot f_n(x) \cdot \xi_i(R)^{n-1} = -\xi'_i(R) F'_i(e_i, x | \xi_i(R)) \in (-\infty, 0).$$

Recall that, for $x = x_1 \dots x_n \in \Gamma \setminus \{e\}$,

$$F(e, x_1 \dots x_n | z) = \prod_{j=1}^n F_{\tau(x_j)}(e_{\tau(x_j)}, x_j | \xi_{\tau(x_j)}(z)).$$

Now, plugging the expansion (4.14) into the above formula gives us the coefficient of $(R-z)$:

$$f_x = \sum_{j=1}^n -\xi'_{\tau(x_j)}(R) F'_{\tau(x_j)}(e_{\tau(x_j)}, x_j | \xi_{\tau(x_j)}(R)) \prod_{\substack{k=1, \\ k \neq j}}^n F_{\tau(x_k)}(e_{\tau(x_k)}, x_k | \xi_{\tau(x_k)}(R)) \in (-\infty, 0).$$

This yields the claim in the case $c = 1$.

We now turn to the case $c = 1/2$. By [25, Equ. (9.20)], we have

$$\alpha_i z G(z) = \xi_i(z) G_i(\xi_i(z)). \quad (4.15)$$

Write $\xi_i(z) = \xi_i(R) + X_i(z)$ with $X_i(R) = 0$. Our aim is to show that $X_i(z)$ is of order $\sqrt{R-z}$, from which we can derive the proposed expansion of $F(e, x|z)$. We rewrite (4.15) as

$$\begin{aligned} & \alpha_i(R - (R - z)) \cdot (G(R) + g_1\sqrt{R-z} + \mathbf{o}(\sqrt{R-z})) \\ &= (\xi_i(R) + X_i(z)) \cdot \sum_{n \geq 0} \mu_i^{(n)}(e_i) (\xi_i(R) + X_i(z))^n. \end{aligned}$$

The constant term on the left hand side of the equation is $\alpha_i R G(R)$, which equals the constant term on the right hand side $\xi_i(R) G_i(\xi_i(R))$ by (4.15). The coefficient of $\sqrt{R-z}$ on the left hand side is $\alpha_i R g_1 \neq 0$. The coefficient of $X_1(z)$ on the right hand side is given by

$$\xi_i(R) G_i'(e_i, e_i | \xi_i(R)) + G_i(e_i, e_i | \xi_i(R)) \in (0, \infty).$$

Thus, $X_1(z) \sim \sqrt{R-z}$ as $z \uparrow R$, and therefore

$$F_i(e_i, x | \xi_i(z)) = \sum_{n \geq 1} f_n(x) (\xi_i(R) + \hat{\xi}_i \cdot \sqrt{R-z} + \mathbf{o}(R-z))^n$$

for some $\hat{\xi}_i < 0$. The rest follows analogously to the case $c = 1$ by replacing $(R-z)$ with $\sqrt{R-z}$. \square

Consider now the following difference for $i \in \mathcal{I}$:

$$\begin{aligned} & \mathcal{F}_i(R|z^*(R)) - \mathcal{F}_i(\lambda|z^*(\lambda)) \\ &= \sum_{m \geq 1} z^*(R)^m \sum_{\substack{x \in \Gamma_i: \\ l(x)=m}} F(e, x|R) \\ & \quad - \sum_{m \geq 1} \left(z^*(R) - (z^*(R) - z^*(\lambda)) \right)^m \sum_{\substack{x \in \Gamma_i: \\ |x|=m}} \left[F(e, x|R) + f_x(R-\lambda)^c + \mathbf{o}((R-\lambda)^c) \right] \\ &= \sum_{m \geq 1} z^*(R)^m \sum_{\substack{x \in \Gamma_i: \\ l(x)=m}} (-f_x(R-\lambda)^c - \mathbf{o}((R-\lambda)^c)) \\ & \quad + (z^*(R) - z^*(\lambda)) \frac{\partial}{\partial z} \mathcal{F}_i^+(\lambda|z^*(R)) + \mathbf{o}(z^*(R) - z^*(\lambda)). \end{aligned}$$

Moreover,

$$\begin{aligned} 0 &= \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i(R|z^*(R))}{1 + \mathcal{F}_i(R|z^*(R))} - \sum_{i \in \mathcal{I}} \frac{\mathcal{F}_i(\lambda|z^*(\lambda))}{1 + \mathcal{F}_i(\lambda|z^*(\lambda))} \\ &= \sum_{i \in \mathcal{I}} \sum_{n \geq 0} \left(-\mathcal{F}_i(\lambda|z^*(\lambda)) \right)^{n+1} - \left(-\mathcal{F}_i(R|z^*(R)) \right)^{n+1}. \end{aligned} \quad (4.16)$$

Write

$$\left(-\mathcal{F}_i(\lambda|z^*(\lambda)) \right)^{n+1} - \left(-\mathcal{F}_i(R|z^*(R)) \right)^{n+1} = \left(\mathcal{F}_i(R|z^*(R)) - \mathcal{F}_i(\lambda|z^*(\lambda)) \right) \cdot g_n(\lambda), \quad (4.17)$$

where $g_n(R) \neq 0$ for every $n \in \mathbb{N}$. Plugging the decomposition of $\mathcal{F}_i(R|z^*(R)) - \mathcal{F}_i(\lambda|z^*(\lambda))$ into (4.16) and comparing all error terms yields in view of (4.17) the following behaviour:

$$z^*(R) - z^*(\lambda) \sim \begin{cases} \hat{C}_1 \cdot (R - \lambda), & \text{if } G'(R) < \infty, \\ \hat{C}_2 \cdot \sqrt{R - \lambda}, & \text{if } G'(R) = \infty \end{cases}$$

for a suitable constant \hat{C}_1, \hat{C}_2 respectively. The statement (3) of Theorem 3.10 follows now from

$$\log z^*(\lambda) - \log z^*(R) = \log\left(1 - \frac{1}{z^*(R)}(z^*(R) - z^*(\lambda))\right)$$

and by the Taylor expansion of $\log(1 - x)$ at $x = 0$.

4.5. Proof of Corollary 3.16. In a first step we show the following lemma:

Lemma 4.22.

$$\overline{\text{BD}}^{\text{fin}}(\Lambda) \leq -\frac{\log \theta}{\log \alpha} \quad \text{and} \quad \text{BD}^{\text{fin}}(\Omega) = -\frac{\log \varrho}{\log \alpha}.$$

Proof. First, we define the matrices $M_0 = (m_0(i, j))_{i, j \in \mathcal{I}}$ and $D_0 = (d_0(i, j))_{i, j \in \mathcal{I}}$ by

$$m_0(i, j) := \begin{cases} \mathcal{F}_i^+(\lambda), & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad d_0(i, j) := \begin{cases} |\Gamma_i| - 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{N}$, denote by $\mathcal{H}_m^{\text{fin}}$ the random number of visited words of the form $w_1 \dots w_m \in \Gamma$. Then

$$\begin{aligned} \mathbb{E}|\mathcal{H}_m^{\text{fin}}| &\leq \sum_{x \in \mathcal{H}_m^{\text{fin}}} \mathbb{E}Z_\infty(x) \leq \sum_{x \in \Gamma: \|x\|=m} F(e, x|\lambda) = \mathbf{1}^T M_0 M^{m-1} \mathbf{1} \quad \text{and} \\ \hat{S}(m) &= |\{x \in \Gamma \mid \|x\| = m\}| = \mathbf{1}^T D_0 D^{m-1} \mathbf{1}. \end{aligned}$$

Let $u \in \mathbb{R}^r$ be an eigenvector w.r.t. the eigenvalue θ such that $u \geq \mathbf{1}$. Then:

$$\mathbb{E}|\mathcal{H}_m^{\text{fin}}| \leq \begin{pmatrix} \mathcal{F}_1(\lambda) \\ \vdots \\ \mathcal{F}_r(\lambda) \end{pmatrix}^T M_1^{m-1} u \leq \begin{pmatrix} \mathcal{F}_1(\lambda) \\ \vdots \\ \mathcal{F}_r(\lambda) \end{pmatrix}^T \theta^{m-1} u.$$

Thus, $\limsup_{m \rightarrow \infty} (\mathbb{E}|\mathcal{H}_m^{\text{fin}}|)^{1/m} \leq \theta$. Similarly, one can show that $\lim_{m \rightarrow \infty} \hat{S}(m)^{1/m} = \varrho$ by taking eigenvectors $v_1 \geq \mathbf{1}$ and $v_2 \leq \mathbf{1}$. Analogously to the proofs of Lemma 4.8 and Propositions 4.9, 4.17 we obtain the claim. \square

Proof of Corollary 3.16. First, we remark that we dropped the assumption on symmetry of the μ_i 's in the case of free products of finite groups. This assumption is needed in the general case to ensure $F(e, x|\lambda) < 1$. This inequality holds also in the present setting: by [25, Equation (9.20)],

$$\alpha_i z G(e, e|z) = G_i(e_i, e_i|\xi_i(z)) \xi_i(z).$$

Since $G(e, e|R) < \infty$ and $G_i(e_i, e_i|1) = \infty$, we must have $\xi_i(R) < 1$, and consequently,

$$F(e, x_1 \dots x_k | \lambda) = \prod_{j=1}^k F_{\tau(x_j)}(e_{\tau(x_j)}, x_j | \xi_{\tau(x_j)}(\lambda)) < \prod_{j=1}^k F_{\tau(x_j)}(e_{\tau(x_j)}, x_j | 1) = 1.$$

In order to show that $-\log \theta / \log \alpha$ is a lower bound for $\text{HD}^{\text{fin}}(\Lambda)$, we can follow the reasoning in [13, Section 6] or also as in Section 4.2.2. Analogously to the proof of Theorem 3.8 we obtain that $\text{HD}^{\text{fin}}(\Omega) = \text{BD}^{\text{fin}}(\Omega)$. \square

4.6. Proof of Corollary 3.18. First, we prove the following lemma:

Lemma 4.23.

$$\overline{\text{BD}^{(H)}}(\Lambda) \leq -\frac{\log \theta_H}{\log \alpha} \quad \text{and} \quad \text{BD}^{(H)}(\Omega) = -\frac{\log \varrho_H}{\log \alpha}.$$

Proof. First, we define the matrices $N_0 = (n_0(i, j))_{i, j \in \mathcal{I}}$ and $D_{0, H} = (d_{0, H}(i, j))_{i, j \in \mathcal{I}}$ by

$$n_0(i, j) := \begin{cases} \mathcal{F}_i^{(H)}(\lambda), & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad d_{0, H}(i, j) := \begin{cases} [\Gamma_i : H_i] - 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{N}$, we denote by $\mathcal{H}_m^{(H)}$ the set of words of the form $g_1 \dots g_m h \in \Gamma$ in the sense of (3.3). Since every path from e to $g_1 \dots g_m h \in \Gamma$ has to pass through points $g_1 \dots g_j h_j \in \Gamma$, where $h_j \in H$ with $h_m = h$, we have

$$\sum_{g_1 \dots g_m h \in \Gamma} F_H(g_1 \dots g_m h | z) = \sum_{g_1 \dots g_m h \in \Gamma} \sum_{h_1, \dots, h_{m-1} \in H} \prod_{i=1}^m F_H(g_i h_i | z) = \mathbf{1}^T N_0 N^{m-1} \mathbf{1}.$$

Choose now an eigenvector $v = (v_1, \dots, v_r)^T \geq \mathbf{1}$ w.r.t. the eigenvalue θ_H of N . Then

$$\mathbb{E}|\mathcal{H}_m^{(H)}| \leq \mathbf{1}^T N_0 N^{m-1} \mathbf{1} \leq \mathbf{1}^T N_0 N^{m-1} v = \theta_H^{m-1} \cdot \left(\sum_{i \in \mathcal{I}} v_i \mathcal{F}_i^{(H)}(\lambda) \right),$$

and therefore, $\limsup_{m \rightarrow \infty} \mathbb{E}|\mathcal{H}_m^{(H)}|^{1/m} \leq \theta_H$. Furthermore, we remark that $\hat{S}_H(m) = |\{x_1 \dots x_m \mid x_i \in \bigcup_{j \in \mathcal{I}} \mathcal{R}_j \setminus \{e_j\}, x_i \in \mathcal{R}_j \Rightarrow x_{i+1} \notin \mathcal{R}_j\}|$ can be written as

$$\hat{S}_H(m) = \mathbf{1}^T D_{0, H} D_H^{m-1} \mathbf{1}.$$

Taking eigenvectors $v_1 \geq \mathbf{1}$ and $v_2 \leq \mathbf{1}$ w.r.t. ϱ_H leads to $\lim_{m \rightarrow \infty} |\hat{S}_H(m)|^{1/m} = \varrho_H$. The same reasoning as used in the proofs of Lemma 4.8 and Propositions 4.9, 4.17 yields the proposed claim. \square

Proof of Corollary 3.18. It is sufficient to show that $-\log \theta_H / \log \alpha$ is also a lower bound for $\text{HD}^{(H)}(\Lambda)$. First, we remark that for $m \in \mathbb{N}$

$$\begin{aligned} \sum_{g_1 \dots g_m h \in \Gamma: g_1 \notin \mathcal{R}_1} \mathbb{E}Z_\infty(g_1 \dots g_m h) &= \sum_{g_1 \dots g_m h \in \Gamma: g_1 \notin \mathcal{R}_1} F(e, g_1 \dots g_m h | \lambda) \\ &= \sum_{g_1 \dots g_m h \in \Gamma: g_1 \notin \mathcal{R}_1} \sum_{h_0 \in H} F_H(g_1 \dots g_m h_0 | \lambda) F(e, h_0^{-1} h | \lambda). \end{aligned}$$

Since $|H| < \infty$, there are constants $d, D > 0$ such that $d \leq F(e, h|\lambda) \leq D$ for all $h \in H$. We write $\mathbf{1}_0 := (0, 1, \dots, 1)^T \in \mathbb{R}^r$ and get:

$$\left(\sum_{g_1 \dots g_m h \in \Gamma: g_1 \notin \mathcal{R}_1} \mathbb{E} Z_\infty(g_1 \dots g_m h) \right)^{1/m} \leq \left(D \cdot \mathbf{1}_0^T N_0 N^{m-1} \mathbf{1} \right)^{1/m} \xrightarrow{m \rightarrow \infty} \theta_H \quad \text{and}$$

$$\left(\sum_{g_1 \dots g_m h \in \Gamma: g_1 \notin \mathcal{R}_1} \mathbb{E} Z_\infty(g_1 \dots g_m h) \right)^{1/m} \geq \left(d \cdot \mathbf{1}_0^T N_0 N^{m-1} \mathbf{1} \right)^{1/m} \xrightarrow{m \rightarrow \infty} \theta_H.$$

This can be easily verified by substituting $\mathbf{1}$ by an eigenvector $v_1 \geq \mathbf{1}$ of θ_H , by an eigenvector $v_2 \leq \mathbf{1}$ of θ_H respectively. With the help of this convergence behaviour and the last lemma, we can prove once again analogously to the reasoning in [13, Section 6] or Section 4.2.2 that the upper bounds in Lemma 4.23 equal the Hausdorff and the Box-Counting dimensions. Analogously to the proof of Theorem 3.8 we obtain that $\text{HD}^{(H)}(\Omega) = \text{BD}^{(H)}(\Omega)$. \square

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