

COUNTING SELF-AVOIDING WALKS ON FREE PRODUCTS OF GRAPHS

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ABSTRACT. The *connective constant* $\mu(G)$ of a graph G is the asymptotic growth rate of the number σ_n of self-avoiding walks of length n in G from a given vertex. We prove a formula for the connective constant for free products of quasi-transitive graphs and show that $\sigma_n \sim A_G \mu(G)^n$ for some constant A_G that depends on G . In the case of finite products $\mu(G)$ can be calculated explicitly and is shown to be an algebraic number.

1. INTRODUCTION

Let $G = (V, E, o)$ be a rooted graph with vertex set V , edge set E , and some distinguished vertex o . The graphs considered in this note are locally finite, connected and simple, i.e. no multiple edges between any pair of vertices. An (undirected) edge $e \in E$ with endpoints $u, v \in V$ is noted as $e = \langle u, v \rangle$. Two vertices u, v are called *adjacent* if there exists an edge $e \in E$ such that $e = \langle u, v \rangle$; in this case we write $u \sim v$. The vertex $o \in V$ is called the root (or origin) of the graph.

A *path* of length $n \in \mathbb{N}$ on a graph G is a sequence of vertices $[v_0, v_1, \dots, v_n]$ such that $v_{i-1} \sim v_i$ for $i \in \{1, \dots, n\}$. An *n -step self-avoiding walk* (SAW) on a rooted graph G is a path of length n starting in o where no vertex appears more than once. The *connective constant* describes the asymptotic growth of the number σ_n of self-avoiding walks of length n on a given graph G ; it is defined as

$$(1) \quad \mu(G) = \lim_{n \rightarrow \infty} \sigma_n^{1/n},$$

provided the limit exists. In fact, if the graph is (quasi-)transitive the existence of the limit is guaranteed by subadditivity of σ_n , see Hammersley [13]. In the case of a finite graph $\mu(G)$ equals trivially zero. In the following we will just write $\mu = \mu(G)$.

Self-avoiding walks were originally introduced on Euclidean lattices as a model for long polymer chains. But in recent years, the study on hyperbolic lattices, see Madras and Wu [16], Swierczak and Guttmann [19], on one-dimensional lattices, see Alm and Janson [1], and on general graphs and Cayley graphs, see Grimmett and Li [8, 9, 10, 11, 12], has received increasing attention from physicists and mathematicians alike. In particular, the connective constant of a d -regular vertex-transitive simple graph is shown to be bounded below by $\sqrt{d-1}$, see [11]. However, exact values of the connective constant are only known for a small class of non-trivial graphs, namely ladder graphs, see [1], and the hexagonal lattice where the connective constant equals $\sqrt{2 + \sqrt{2}}$, see Duminil-Copin and Smirnov [5].

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In the present paper we use generating functions, in particular the function $\mathcal{M}(z) := \sum_{n \geq 0} \sigma_n \cdot z^n$, to give formulas for the connective constant for the class of free products of graphs. Generating functions already played an important role in the theory of self-avoiding walks, e.g. see [1] and Bauerschmidt, Duminil-Copin, Goodman and Slade [2]. The approach of this note is that the involved generating functions satisfy a functional equation of the same type as some generating functions for the ordinary random walk. As a consequence the connective constant can be expressed as the smallest positive root of an equation, see Theorem 3.3. In the case of products of finite graphs this equation can be solved explicitly and it is shown that μ is an algebraic number, see Corollary 3.4.

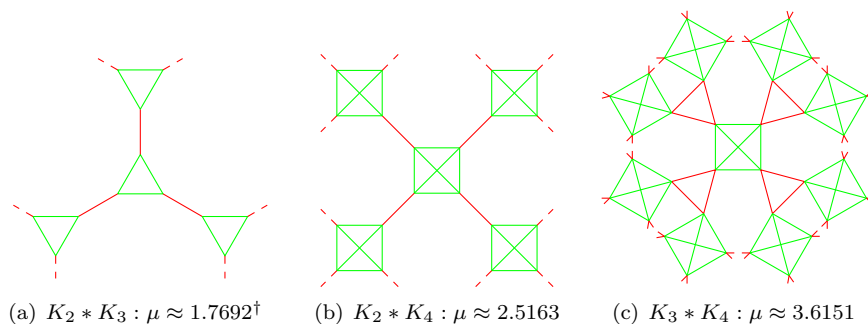


FIGURE 1. Examples for free products of complete graphs K_n .

It is widely believed that on \mathbb{Z}^d , $d \neq 4$, there is a critical exponent γ depending on d such that $\sigma_n \sim A\mu^n n^{1-\gamma}$, where A is a constant depending on d . However, this behaviour is rigorously proven only on lattices in dimensions $d \geq 5$, see Hara and Slade [14], on ladder graphs, see [1], and (in a weaker form) on regular tessellations of the hyperbolic lattice, see [16]. In this note we prove that $\sigma_n \sim A\mu(G)^n n^{1-\gamma}$ with $\gamma = 1$ for free products of graphs, see Theorem 4.1. This result supports the conjecture evoked in [16] that SAW on nonamenable graphs exhibits a mean-field behavior, i.e., that $\gamma = 1$.

The study of stochastic processes on free products has a long and fruitful history. In most of the works generating function techniques played an important role, e.g. Woess [22]. Similar techniques we use for rewriting generating functions in terms of functions on the factors of the free product were introduced independently and simultaneously in Cartwright and Soardi [4], McLaughlin [17], Voiculescu [20] and Woess [21]. Free products owe some of their importance Stallings's Splitting Theorem which states that a finitely generated group has more than one (geometric) end if and only if it admits a nontrivial decomposition as an amalgamated free product or an HNN-extension over a finite subgroup. Furthermore, they constitute an important class of nonamenable graphs where calculations are still possible while they remain completely open on one-ended nonamenable graphs. For example, the spectral radius of random walks, see [22], and the critical percolation probability p_c , see Špakulová [15, 18], can be calculated on free products but still constitute a big challenge on one-ended nonamenable graphs.

[†] The exact value of μ is $6\left(-2 + \sqrt[3]{46 - 6\sqrt{57}} + \sqrt[3]{46 + 6\sqrt{57}}\right)^{-1}$

2. FREE PRODUCTS OF GRAPHS

Let $r \in \mathbb{N}$ with $r \geq 2$ and set $\mathcal{I} := \{1, \dots, r\}$. Let $G_1 = (V_1, E_1, o_1), \dots, G_r = (V_r, E_r, o_r)$ be a finite family of undirected, connected, quasi-transitive, rooted graphs with vertex sets V_i , edge sets E_i and roots o_i for $1 \leq i \leq r$. We recall that a graph is called *quasi-transitive* if its automorphism group acts quasi-transitive, i.e. with finitely many orbits. In particular, every finite graph is quasi-transitive. Furthermore, we shall assume that we have $|V_i| \geq 2$ for every $i \in \mathcal{I}$ and that the vertex sets are distinct.

Let $V_i^\times := V_i \setminus \{o_i\}$ for every $i \in \mathcal{I}$ and set $\tau(x) := i$ if $x \in V_i^\times$. Define

$$V := V_1 * \dots * V_r = \{x_1 x_2 \dots x_n \mid n \in \mathbb{N}, x_i \in \bigcup_{j \in \mathcal{I}} V_j^\times, \tau(x_i) \neq \tau(x_{i+1})\} \cup \{o\},$$

which is the set of ‘words’ over the alphabet $\bigcup_{i \in \mathcal{I}} V_i^\times$ such that no two consecutive letters come from the same V_i^\times . The empty word in V is denoted by o . We extend the function τ on V by setting $\tau(x_1 \dots x_n) := \tau(x_n)$ for $x_1 x_2 \dots x_n \in V$. On the set V we have a partial word composition law: if $x = x_1 \dots x_m, y = y_1 \dots y_n \in V$ with $\tau(x_m) \neq \tau(y_1)$ then xy stands for the concatenation of x and y , which is again an element of V . In particular, if $\tau(x) \neq i \in \mathcal{I}$ then we set $x o_i := o_i x := x$ and $x o := o x := x$. We regard each V_i as a subset of V , identifying each o_i with o .

We now equip the set V with a graph structure by defining the set of edges E as follows: if $i \in \mathcal{I}$ and $x, y \in V_i$ with $x \sim y$ then $w x \sim w y$ for all $w \in V$ with $\tau(w) \neq i$. The *free product* of the graphs G_1, \dots, G_r is then given by the graph

$$G := G_1 * G_2 * \dots * G_r := (V, E, o).$$

In order to visualize this graph take a copy of each graph G_1, \dots, G_r and glue them together at their roots o_1, \dots, o_r , that is, we identify the roots of the single graphs as one common vertex in V , which becomes o . Inductively, at each element $x \in V$ with $\tau(x) = i \in \mathcal{I}$ attach copies of the graphs $G_j, j \neq i$, where x is identified with the roots $o_j, j \neq i$, of each single graph G_j .

For $i \in \mathcal{I}$ and $n \in \mathbb{N}$, let $\sigma_n^{(i)}$ be the number of self-avoiding walks of length n on G_i starting at o_i . Since all graphs G_i are quasi-transitive there are, due to [13], numbers μ_i such that

$$\mu_i = \lim_{n \rightarrow \infty} (\sigma_n^{(i)})^{1/n}.$$

3. FUNCTIONAL EQUATION FOR GENERATING FUNCTIONS

We turn to self-avoiding walks on the free product G . Let σ_n be the number of self-avoiding walks of length $n \in \mathbb{N}$ on the free product G starting at o . We set $\sigma_0 := 1$.

For $i \in \mathcal{I}$ and $z \in \mathbb{C}$, define the generating functions

$$\begin{aligned} \mathcal{M}(z) &:= \sum_{n \geq 0} \sigma_n \cdot z^n, \\ \mathcal{M}_i(z) &:= \sum_{n \geq 1} \sigma_n^{(i)} \cdot z^n. \end{aligned}$$

Furthermore, let $\bar{\sigma}_n^{(i)}$ be the number of self-avoiding walks of length n on the free product G starting at o , which do visit V_i^\times . Define

$$\mathcal{M}_i^*(z) := \sum_{n \geq 1} \bar{\sigma}_n^{(i)} \cdot z^n.$$

Due to the recursive structure of free products we can rewrite

$$\begin{aligned} \mathcal{M}_i^*(z) &= \sum_{m \geq 0} \mathcal{M}_i(z) \sum_{\substack{i_1, \dots, i_m \in \mathcal{I}: \\ i_1 \neq i, i_j \neq i_{j+1}}} \prod_{j=1}^m \mathcal{M}_{i_j}(z) \\ (2) \quad &= \mathcal{M}_i(z) \left(1 + \sum_{j \in \mathcal{I} \setminus \{i\}} \mathcal{M}_j^*(z) \right). \end{aligned}$$

Furthermore, we also have

$$(3) \quad \mathcal{M}(z) = 1 + \sum_{i \in \mathcal{I}} \mathcal{M}_i^*(z).$$

Plugging (3) into (2) yields

$$\mathcal{M}_i^*(z) = \mathcal{M}_i(z) (\mathcal{M}(z) - \mathcal{M}_i^*(z)),$$

which in turn yields

$$\mathcal{M}_i^*(z) = \mathcal{M}(z) \frac{\mathcal{M}_i(z)}{1 + \mathcal{M}_i(z)}.$$

Hence,

$$\mathcal{M}(z) = 1 + \sum_{i \in \mathcal{I}} \mathcal{M}_i^*(z) = 1 + \mathcal{M}(z) \sum_{i \in \mathcal{I}} \frac{\mathcal{M}_i(z)}{1 + \mathcal{M}_i(z)},$$

or equivalently

$$(4) \quad \mathcal{M}(z) = \frac{1}{1 - \sum_{i \in \mathcal{I}} \frac{\mathcal{M}_i(z)}{1 + \mathcal{M}_i(z)}}.$$

The above equations do hold for every $z \in \mathbb{C}$ with $|z| < R(\mathcal{M})$ and $1 + \mathcal{M}_i(z) \neq 0$, where $R(\mathcal{M})$ is the radius of convergence of $\mathcal{M}(z)$.

Remark 3.1. *Analogous functional equations as in (4) appear in different context for random walks on free groups, where one has some generating functions on the single factors and deduces the corresponding generating function on the free product. In Gilch [7, Section 4], a functional equation of the same type was established in order to compute the rate of escape of random walks on free products. In that case the functions $\mathcal{M}_i(z)$ and $\mathcal{M}(z)$ are replaced by probability generating functions. Furthermore, in Candellero, Gilch and Müller [3, Equation (4.1)] a functional equation of the same type was also established in order to compute the Hausdorff dimension of branching random walks on free products; in this case the involved generating functions describe the evolution of particles.*

We have

$$1 \leq \limsup_{m \rightarrow \infty} \sigma_n^{1/n} = 1/R(\mathcal{M}),$$

and therefore that $R(\mathcal{M}) \leq 1$. In order to determine $R(\mathcal{M})$ we have to find – by Pringsheim’s Theorem – the smallest singularity point on the positive x -axis

of $\mathcal{M}(z)$. This smallest singularity point is either one of the radii of convergence $R(\mathcal{M}_i)$ of the functions $\mathcal{M}_i(z)$ or the smallest real positive number z_* with

$$\sum_{i \in \mathcal{I}} \frac{\mathcal{M}_i(z_*)}{1 + \mathcal{M}_i(z_*)} = 1.$$

That is, $R(\mathcal{M}) = \min\{R(\mathcal{M}_1), \dots, R(\mathcal{M}_r), z_*\}$. The next lemma answers which of the candidates is the radius of convergence of $\mathcal{M}(z)$:

Lemma 3.2. $R(\mathcal{F}) = z_* < \min\{R(\mathcal{M}_1), \dots, R(\mathcal{M}_r)\}$.

Proof. If all generating functions $\mathcal{M}_i(z)$ have infinite radius of convergence (that is, $R(\mathcal{M}_i) = \infty$), then $R(\mathcal{F}) = z_*$ since $\mathcal{M}(0) = 0$ and $\lim_{z \rightarrow \infty} \mathcal{M}_i(z) = \infty$. Therefore, w.l.o.g. we may assume from now on that $R(\mathcal{M}_1) < \infty$. Recall that $\mu_1 = \lim_{n \rightarrow \infty} (\sigma_n^{(1)})^{1/n} = 1/R(\mathcal{M}_1)$. Due to quasi-transitivity of the graph G_1 we obtain with [13, Theorem 1] that

$$\log \mu_1 = \inf_{n \in \mathbb{N}} \frac{\log \sigma_n^{(1)}}{n},$$

that is, we have $\sigma_n^{(1)} \geq \mu_1^n$.

We follow a similar reasoning as in [3]. We now consider σ_n and count only those SAW which make at least one step into the copy of G_1 , before they make exactly one step into a copy of G_2 and then at least one step into a copy of G_1 , and so on. This leads to the following estimates for σ_n :

$$\begin{aligned} \sigma_n &\geq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{n_1, \dots, n_k \geq 1: \\ n_1 + \dots + n_k = n-k}} \sigma_{n_1}^{(1)} \cdot \dots \cdot \sigma_{n_k}^{(1)} \\ &\geq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \mu_1^{n-k} \binom{n-2k+k-1}{k-1} \\ &\geq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mu_1^{n-k-1} \binom{n-k-2}{k}. \end{aligned}$$

The binomial coefficient in the second inequality arises as follows: we think of placing $n-k$ balls into k urns, where there should be at least one ball in each urn. If $k \leq \lfloor \frac{n}{2} \rfloor - 1$ then $n-k-2 \geq \lfloor \frac{n}{2} \rfloor - 1$. This yields together with the Binomial Theorem:

$$\begin{aligned} \sigma_n &\geq \mu_1^{n-1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \mu_1^{-k} \binom{\lfloor \frac{n}{2} \rfloor - 1}{k} \\ &= \mu_1^{n-1} (1 + \mu_1^{-1})^{\lfloor \frac{n}{2} \rfloor - 1}. \end{aligned}$$

This gives

$$\liminf_{n \rightarrow \infty} \sigma_n^{1/n} \geq \liminf_{n \rightarrow \infty} \mu_1^{\frac{n-1}{n}} \sqrt{1 + \mu_1^{-1}} = \mu_1 \cdot \sqrt{1 + \mu_1^{-1}} > \mu_1.$$

This finishes the proof, and the only remaining possibility for $R(\mathcal{M})$ is z_* . \square

Eventually, we proved the following theorem.

Theorem 3.3. *The connective constant $\mu = \mu(G)$ of a free product G of quasi-transitive graphs is given as $\mu = 1/z_*$, where z_* is the smallest real positive number with*

$$\sum_{i \in \mathcal{I}} \frac{\mathcal{M}_i(z_*)}{1 + \mathcal{M}_i(z_*)} = 1.$$

Finally, we consider the special case of free products of *finite* graphs. If all single graphs G_i are finite (that is, V_i is finite), then the generating functions $\mathcal{M}_i(z)$ are polynomials with positive coefficients of degree less or equal than $|V_i| - 1$. We then have that z_* is the smallest positive zero of the polynomial

$$(5) \quad D(z) = \prod_{k \in \mathcal{I}} (1 + \mathcal{M}_k(z)) - \sum_{i \in \mathcal{I}} \mathcal{M}_i(z) \cdot \left(\prod_{j \in \mathcal{I} \setminus \{i\}} (1 + \mathcal{M}_j(z)) \right).$$

In the case $r = 2$ we obtain that z_* is the smallest positive zero of the polynomial $1 - \mathcal{M}_1(z)\mathcal{M}_2(z)$. This proves the following:

Corollary 3.4. *If all factors G_i are finite graphs, then μ is algebraic.*

4. THE CRITICAL EXPONENT γ

In this section we want to show that the critical exponent $\gamma = 1$. We say that two functions $f(n), g(n)$ satisfy $f(n) \sim g(n)$, if $f(n)/g(n) \rightarrow 1$ as $n \rightarrow \infty$.

Theorem 4.1. *Let σ_n be the number of self-avoiding walks of length n on a free product G of quasi-transitive graphs and let $\mu(G)$ be the connective constant of G . Then, there exists some constant A_G such that*

$$\sigma_n \sim A_G \mu(G)^n.$$

In order to prove the theorem we make some further observations. We begin by recalling that Equation (4) is valid for all $z \in \mathbb{C}$ with $|z| < z_*$ and $1 + \mathcal{M}_i(z) \neq 0$ for all $i \in \mathcal{I}$. By multiplication of (4) with $\prod_{i \in \mathcal{I}} (1 + \mathcal{M}_i(z))$ we can rewrite $\mathcal{M}(z)$ as

$$\mathcal{M}(z) = \frac{N(z)}{D(z)},$$

where $N(z) := \prod_{i \in \mathcal{I}} (1 + \mathcal{M}_i(z))$ and

$$D(z) := \prod_{k \in \mathcal{I}} (1 + \mathcal{M}_k(z)) - \sum_{i \in \mathcal{I}} \mathcal{M}_i(z) \cdot \prod_{j \in \mathcal{I} \setminus \{i\}} (1 + \mathcal{M}_j(z)).$$

Lemma 4.2. *There exists some function $g(z)$ that is analytic in a neighbourhood of $z = z_*$ such that $g(z_*) \neq 0$ and*

$$\frac{N(z)}{D(z)} = \frac{g(z)}{z - z_*}.$$

Proof. Since $z_* < \min\{R(\mathcal{M}_1), \dots, R(\mathcal{M}_r)\}$ we can expand each $\mathcal{M}_i(z)$ around z_* as a Taylor series of the form

$$\mathcal{M}_i(z) = \sum_{n \geq 0} a_n^{(i)} (z - z_*)^n.$$

In particular, $N(z)$ and $D(z)$ are analytic in a neighbourhood of $z = z_*$. Plugging this expansion into $D(z)$ we must have due to $D(z_*) = 0$ that $D(z)$ can be rewritten as

$$(6) \quad D(z) = \sum_{n \geq 1} d_n (z - z_*)^n.$$

We now claim that z_* is a single zero of (6), that is, we have $d_1 \neq 0$: the derivative of (6) is

$$\begin{aligned} & \sum_{i \in \mathcal{I}} \mathcal{M}'_i(z) \prod_{k \in \mathcal{I} \setminus \{i\}} (1 + \mathcal{M}_k(z)) - \sum_{i \in \mathcal{I}} \mathcal{M}'_i(z) \cdot \prod_{j \in \mathcal{I} \setminus \{i\}} (1 + \mathcal{M}_j(z)) \\ & - \sum_{i \in \mathcal{I}} \mathcal{M}_i(z) \sum_{k \in \mathcal{I} \setminus \{i\}} \mathcal{M}'_k(z) \prod_{l \in \mathcal{I} \setminus \{i, k\}} (1 + \mathcal{M}_l(z)) \\ = & - \sum_{i \in \mathcal{I}} \mathcal{M}_i(z) \sum_{k \in \mathcal{I} \setminus \{i\}} \mathcal{M}'_k(z) \prod_{l \in \mathcal{I} \setminus \{i, k\}} (1 + \mathcal{M}_l(z)). \end{aligned}$$

Since $\mathcal{M}_i(z_*), \mathcal{M}'_i(z_*) > 0$ for all $i \in \mathcal{I}$ the derivative is strictly negative for real $z > 0$. Consequently, $D'(z_*) \neq 0$ and therefore z_* is a single zero of $D(z)$. Since z_* is the smallest positive zero of $D(z)$ we can write $D(z) = (z - z_*) \cdot d(z)$, where $d(z)$ is analytic in a neighbourhood of $z = z_*$ with $d(z_*) \neq 0$. Setting $g(z) := N(z) \cdot d(z)^{-1}$ yields the claim since $g(z)$ is a function with $g(z_*) \neq 0$, which is analytic in a neighbourhood of z_* . \square

We now claim that z_* is the unique dominant singularity of $N(z)/D(z)$.

Lemma 4.3. *The only singularity of $N(z)/D(z)$ on its circle of convergence (of radius z_*) is z_* .*

Proof. Expanding $D(z)$ yields that we can rewrite $D(z) = 1 - S(z)$, where $S(z)$ has the form

$$S(z) = \sum_{k=2}^r \sum_{1=i_1 < i_2 < \dots < i_k=r} a_{i_1, \dots, i_k} \mathcal{M}_{i_1}(z) \dots \mathcal{M}_{i_k}(z) = \sum_{n \geq 1} s_n z^n,$$

where $a_{i_1, \dots, i_k} \geq 0$ and $s_n > 0$ for all $n \in \mathbb{N}$. In particular, $S(z_*) = 1$. It is sufficient to show that there is no $z_0 \in \mathbb{C}$ with $|z_0| = z_*$ and $S(z_0) = 1$. For this purpose, let be $\theta \in [0, 2\pi)$ and consider

$$1 - S(e^{i\theta} z_*) = 1 - \sum_{n \geq 1} s_n z_*^n e^{i\theta n}.$$

Observe that the elements $e^{i\theta n}$ are points on the (complex) unit circle and that $S(e^{i\theta} z_*)$ is a convex combination of these points since $S(z_*) = 1$. Since 1 is an extremal point of the convex unit disc and since the greatest common divisor of the s_n 's equals 1, we must have that $\theta = 0$, which yields the claim. \square

Now we can proceed with the proof of Theorem 4.1 which uses the modern tool of Singularity Analysis developed by Flajolet and Sedgewick [6, Chapter VI].

Proof of Theorem 4.1. Recall that the formula for $\mathcal{M}(z)$ given by Equation (4) is only valid for all $z \in \mathbb{C}$ with $|z| < z_*$, while the quotient $N(z)/D(z)$ is analytic in some domain of the form

$$\Delta = \{z \in \mathbb{C} \mid |z| < R', z \neq z_*, |\arg(z - z_*)| > \varphi\}$$

for some $R' > z_*$ and some $\varphi \in (0, \pi/2)$ due to Lemma 4.3; compare also with [6, Definition VI.1]. Thus, $N(z)/D(z)$ is an analytic continuation of $\mathcal{M}(z)$ to the domain Δ . This is because, if there is some $z_0 \in \mathbb{C} \setminus \{z_*\}$ with $|z_0| = z_*$ and $1 + \mathcal{M}_i(z_0) = 0$ for some $i \in \mathcal{I}$ then $\mathcal{M}(z) \rightarrow 0$ as $z \rightarrow z_0$, and by Riemann's theorem on removable singularities we obtain that $N(z)/D(z)$ is an analytic representation of $\mathcal{M}(z)$ also in a neighbourhood of $z = z_0$, yielding that z_0 is not singular. In view of Lemma 4.3 and the above discussion z_* is the only singularity of $\mathcal{M}(z)$ on the circle with radius z_* . With this observation and with Lemma 4.2 we can rewrite $N(z)/D(z)$ as

$$\mathcal{M}(z) = \frac{a_0}{z - z_*} + \sum_{n \geq 1} a_n (z - z_*)^{n-1},$$

where $g(z) = \sum_{n \geq 0} a_n (z - z_*)^n$. By [6, Chapter VI.2], the singular term $(z - z_*)^{-1}$ leads to the proposed form for the coefficients σ_n . \square

5. EXAMPLES

In this section we collect some examples with explicit calculations for free products of *finite* and *infinite* graphs. Let K_n be the complete graph of n vertices.

Example 5.1. Consider the free product of $K_2 * K_3$, see Figure 1(a). Then

$$\mathcal{M}_1(z) = z, \quad \mathcal{M}_2(z) = 2z + 2z^2.$$

The smallest positive zero of $1 - \mathcal{M}_1(z)\mathcal{M}_2(z) = 1 - 2z^2 - 2z^3$ is given by

$$z_* = \frac{1}{6} \left(-2 + \sqrt[3]{46 - 6\sqrt{57}} + \sqrt[3]{46 + 6\sqrt{57}} \right) \approx 0.565198.$$

Eventually, $\mu = 1/z_* \approx 1.76929$.

Example 5.2. Consider $K_2 * K_3 * K_4$. Then

$$\mathcal{M}_1(z) = z, \quad \mathcal{M}_2(z) = 2z + 2z^2, \quad \mathcal{M}_3(z) = 3z + 6z^2 + 6z^3.$$

The smallest positive zero of the polynomial

$$\prod_{k=1}^3 (1 + \mathcal{M}_k(z)) - \sum_{i=1}^3 \mathcal{M}_i(z) \cdot \left(\prod_{j \neq i} (1 + \mathcal{M}_j(z)) \right).$$

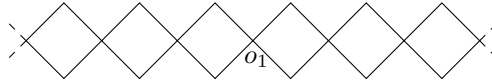
is given by $z_* \approx 0.210631$, which is the smallest positive zero of the polynomial $24z^6 + 60z^5 + 66z^4 + 38z^3 + 11z^2 - 1$. That is, $\mu = 1/z_* \approx 4.74763$.

Example 5.3. Let C_n be the cycle graph with n vertices. Consider free products of the form $C_2 * C_n$ for $n \geq 3$. Then $\mathcal{M}_1(z) = z$ and $\mathcal{M}_2(z) = \sum_{k=1}^{n-1} 2z^k$. The following table shows the approximations of μ for different n :

n	4	5	6	7	8	9	10	∞
μ	1.89932	1.95350	1.97781	1.98920	1.99468	1.99737	1.99869	2

Here, we set C_∞ to be the one-dimensional Euclidean lattice. The free product $C_2 * C_\infty$ is then the regular tree of degree 3 and hence $\mu = 2$.

Example 5.4. Consider the following infinite graph G_1 :



We have $\sigma_1^{(1)} = 4$. Let us consider SAW whose first step is to the right. Every SAW of even length consists only of steps which pass through an edge from the left to the right. At every crossing we have two possibilities to extend some given SAW towards the right hand side. For SAW of odd length each SAW of length $n - 1$ can be extended to the right by two ways, and by one edge back to the left. By symmetry this yields $\sigma_1^{(2n)} = 2 \cdot 2^n$ and $\sigma_1^{(2n+1)} = 2 \cdot 3 \cdot 2^n$ for $n \geq 1$. Hence,

$$\begin{aligned} \mathcal{M}_1(z) &= \sum_{n \geq 1} \sigma_1^{(n)} z^n = 4z + \sum_{n \geq 1} 2^{n+1} z^{2n} + \sum_{n \geq 1} 3 \cdot 2^{n+1} \cdot z^{2n+1} \\ &= \frac{2 + 6z}{1 - 2z^2} - 2z - 2. \end{aligned}$$

That is, $R(\mathcal{M}_1) = \frac{1}{\sqrt{2}}$. Consider again the complete graph of degree 4 denoted by K_4 with

$$\mathcal{M}_2(z) = 3z + 6z^2 + 6z^3.$$

The connective constant of the free product $G_1 * K_4$ is then given as the solution z_* of $D(z) = 0$ which can be calculated numerically:

$$z_* = 0.203143.$$

The connective constant is then given by

$$\mu = \frac{1}{z_*} = 4.92264.$$

REFERENCES

- [1] Alm, S.E., Janson, S.: Random self-avoiding walks on one-dimensional lattices. *Comm. Statist. Stochastic Models* **6**(2), 169–212 (1990). DOI 10.1080/15326349908807144. URL <http://dx.doi.org/10.1080/15326349908807144>
- [2] Bauerschmidt, R., Duminil-Copin, H., Goodman, J., Slade, G.: Lectures on self-avoiding walks. In: Probability and statistical physics in two and more dimensions, *Clay Math. Proc.*, vol. 15, pp. 395–467. Amer. Math. Soc., Providence, RI (2012)
- [3] Candellero, E., Gilch, L.A., Müller, S.: Branching random walks on free products of groups. *Proc. Lond. Math. Soc.* (3) **104**(6), 1085–1120 (2012). DOI 10.1112/plms/pdr060. URL <http://dx.doi.org/10.1112/plms/pdr060>
- [4] Cartwright, D., Soardi, P.: Random walks on free products, quotients and amalgams. *Nagoya Math. J.* **102**, 163–180 (1986)
- [5] Duminil-Copin, H., Smirnov, S.: The connective constant of the honeycomb lattice equals $\sqrt{2 + \sqrt{2}}$. *Ann. of Math.* (2) **175**(3), 1653–1665 (2012). DOI 10.4007/annals.2012.175.3.14. URL <http://dx.doi.org/10.4007/annals.2012.175.3.14>
- [6] Flajolet, P., Sedgewick, R.: Analytic combinatorics. Cambridge University Press, Cambridge (2009). DOI 10.1017/CBO9780511801655. URL <http://dx.doi.org/10.1017/CBO9780511801655>
- [7] Gilch, L.A.: Rate of escape of random walks on free products. *J. Aust. Math. Soc.* **83**(1), 31–54 (2007). DOI 10.1017/S1446788700036375. URL <http://dx.doi.org/10.1017/S1446788700036375>
- [8] Grimmett, G.R., Li, Z.: Counting self-avoiding walks. Preprint, <http://arxiv.org/abs/1304.7216> (2013)
- [9] Grimmett, G.R., Li, Z.: Self-avoiding walks and the Fisher transformation. *Electron. J. Combin.* **20**(3), Paper 47, 14 (2013)
- [10] Grimmett, G.R., Li, Z.: Strict inequalities for connective constants of transitive graphs. *SIAM J. Discrete Math.* **28**(3), 1306–1333 (2014). DOI 10.1137/130906052. URL <http://dx.doi.org/10.1137/130906052>
- [11] Grimmett, G.R., Li, Z.: Bounds on connective constants of regular graphs. *Combinatorica* **35**(3), 279–294 (2015). DOI 10.1007/s00493-014-3044-0. URL <http://dx.doi.org/10.1007/s00493-014-3044-0>

- [12] Grimmett, G.R., Li, Z.: Locality of connective constants, II. Cayley graphs. Preprint, <http://arXiv:1501.00476> (2015)
- [13] Hammersley, J.M.: Percolation processes II. The connective constant. *Proc. Cambridge Philos. Soc.* **53**, 642–645 (1957)
- [14] Hara, T., Slade, G.: Self-avoiding walk in five or more dimensions. I. The critical behaviour. *Comm. Math. Phys.* **147**(1), 101–136 (1992). URL <http://projecteuclid.org/euclid.cmp/1104250528>
- [15] Kozáková, I.: Critical percolation of free product of groups. *Internat. J. Algebra Comput.* **18**(4), 683–704 (2008). DOI 10.1142/S0218196708004524. URL <http://dx.doi.org/10.1142/S0218196708004524>
- [16] Madras, N., Wu, C.C.: Self-avoiding walks on hyperbolic graphs. *Combin. Probab. Comput.* **14**(4), 523–548 (2005). DOI 10.1017/S0963548305006772. URL <http://dx.doi.org/10.1017/S0963548305006772>
- [17] McLaughlin, J.: Random walks and convolution operators on free products. Ph.D. thesis, New York Univ. (1986)
- [18] Špakulová, I.: Critical percolation of virtually free groups and other tree-like graphs. *Ann. Probab.* **37**(6), 2262–2296 (2009). DOI 10.1214/09-AOP458. URL <http://dx.doi.org/10.1214/09-AOP458>
- [19] Swierczak, E., Guttmann, A.J.: Self-avoiding walks and polygons on non-Euclidean lattices. *J. Phys. A* **29**(23), 7485–7500 (1996). DOI 10.1088/0305-4470/29/23/015. URL <http://dx.doi.org/10.1088/0305-4470/29/23/015>
- [20] Voiculescu, D.: Addition of certain non-commuting random variables. *J. Funct. Anal.* **66**, 323–346 (1986)
- [21] Woess, W.: Nearest neighbour random walks on free products of discrete groups. *Boll. Un. Mat. Ital.* **5-B**, 961–982 (1986)
- [22] Woess, W.: Random walks on infinite graphs and groups, *Cambridge Tracts in Mathematics*, vol. 138. Cambridge University Press, Cambridge (2000). DOI 10.1017/CBO9780511470967. URL <http://dx.doi.org/10.1017/CBO9780511470967>

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