ASYMPTOTIC ENTROPY OF RANDOM WALKS ON FREE PRODUCTS

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Abstract. Suppose we are given the free product V of a finite family of finite or countable sets. We consider a transient random walk on the free product arising naturally from a convex combination of random walks on the free factors. We prove the existence of the asymptotic entropy and present three different, equivalent formulas, which are derived by three different techniques. In particular, we will show that the entropy is the rate of escape with respect to the Greenian metric. Moreover, we link asymptotic entropy with the rate of escape and volume growth resulting in two inequalities.

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1. Introduction

Suppose we are given a finite family of finite or countable sets $V_1, \dots, V_r$ with distinguished vertices $o_i \in V_i$ for $i \in \{1, \dots, r\}$. The free product of the sets V_i is given by $V := V_1 * \dots * V_r$, the set of all finite words of the form $x_1 \dots x_n$ such that each letter is an element of $\bigcup_{i=1}^r V_i \setminus \{o_i\}$ and two consecutive letters arise not from the same V_i. We consider a transient Markov chain $(X_n)_{n \in \mathbb{N}_0}$ on V starting at the empty word o, which arises from a convex combination of transition probabilities on the sets V_i. Denote by π_n the distribution of X_n. We are interested in whether the sequence $\mathbb{E}[-\log \pi_n(X_n)]/n$ converges, and if so, to compute this constant. If the limit exists, it is called the *asymptotic entropy.* In this paper, we study this question for random walks on general free products. In particular, we will derive three different formulas for the entropy by using three different techniques.

Let us outline some results about random walks on free products: for free products of finite groups, Mairesse and Mathéus [21] computed an explicit formula for the rate of escape and asymptotic entropy by solving a finite system of polynomial equations. Their result remains valid in the case of free products of infinite groups, but one needs then to solve an infinite system of polynomial equations. Gilch [11] computed two different formulas for the rate of escape with respect to the word length of random walks on free products of graphs by different techniques, and also a third formula for free products of (not necessarily finite) groups. The techniques of [11] are adapted to the present setting. Asymptotic behaviour of return probabilities of random walks on free products has also been studied in many ways; e.g. Gerl and Woess [10], [28], Sawyer [24], Cartwright and Soardi [5], and Lalley [18], Candellero and Gilch [4].

Our proof of existence of the entropy envolves generating functions techniques. The techniques we use for rewriting probability generating functions in terms of functions on the factors of the free product were introduced independently and simultaneously by Cartwright and Soardi [5], Woess [28], Voiculescu [27] and McLaughlin [22]. In particular, we will see that asymptotic entropy is the rate of escape with respect to a distance function in terms of Green functions. While it is well-known by Kingman's subadditive ergodic theorem (see Kingman [17]) that entropy (introduced by Avez [1]) exists for random walks on groups whenever $\mathbb{E}[-\log \pi_1(X_1)] < \infty$, existence for random walks on other structures is not known a priori. We are not able to apply Kingman's theorem in our present setting, since we have no (general) subadditivity and we have only a partial composition law for two elements of the free product. For more details about entropy of random walks on groups we refer to Kaimanovich and Vershik [14] and Derriennic [7].

An important link between drifts and harmonic analysis was obtained by Varopoulos [26]. He proved that for symmetric finite range random walks on groups the existence of non-trivial bounded harmonic functions is equivalent to a non-zero rate of escape. Karlsson and Ledrappier [16] generalized this result to symmetric random walks with finite first moment of the step lengths. This leads to a link between the rate of escape and the entropy of random walks, compare e.g. with Kaimanovich and Vershik [14] and Erschler [8]. Erschler and Kaimanovich [9] asked if drift and entropy of random walks on groups vary continuously on the probability measure, which governs the random walk. We prove real-analyticity of the entropy when varying the probabilty measure of constant support; compare also with the recent work of Ledrappier [19], who simultaneously proved this property for finite-range random walks on free groups.

Apart from the proof of existence of the asymptotic entropy $h = \lim_{n\to\infty} \mathbb{E}[-\log \pi_n(X_n)]/n$ (Theorem 3.7), we will calculate explicit formulas for the entropy (see Theorems 3.7, 3.8, 5.1 and Corollary 4.2) and we will show that the entropy is non-zero. The technique of our proof of existence of the entropy was motivated by Benjamini and Peres [2], where it is shown that for random walks on groups the entropy equals the rate of escape w.r.t. the Greenian distance; compare also with Blachère, Haïssinsky and Mathieu [3]. We are also able to show that, for random walks on free products of graphs, the asymptotic entropy equals just the rate of escape w.r.t. the Greenian distance (see Corollary 3.3 in view of Theorem 3.7). Moreover, we prove convergence in probability and convergence in L_1 (if the non-zero single transition probabilities are bounded away from 0) of the sequence $-\frac{1}{n}\log \pi_n(X_n)$ to h (see Corollary 3.11), and we show also that h can be computed along almost every sample path as the limes inferior of the aforementioned sequence (Corollary 3.9). In the case of random walks on discrete groups, Kingman's subadditive ergodic theorem provides both the almost sure convergence and the convergence in L_1 to the asymptotic entropy; in the case of general free products there is neither a global composition law for elements of the free product nor subadditivity. Thus, in the latter case we have to introduce and investigate new processes. The question of almost sure convergence of $-\frac{1}{n}\log \pi_n(X_n)$ to some constant h, however, remains open. Similar results concerning existence and formulas for the entropy are proved in Gilch and Müller [12] for random walks on directed covers of graphs. The reasoning of our proofs follows the argumentation in [12]: we will show that the entropy equals the rate of escape w.r.t. some special length function, and we deduce the proposed properties analogously. In the present case of free products of graphs, the reasoning is getting more complicated due to the more complex structure of free products in contrast to directed covers, although the main results about existence and convergence types are very similar. We will point out these difficulties and main differences to [12] at the end of Section 3.2. Finally, we will link entropy with the rate of escape and the growth rate of the free product, resulting in two inequalities (Corollary 6.4).

The plan of the paper is as follows: in Section 2 we define the random walk on the free product and the associated generating functions. In Section 3 we prove existence of the asymptotic entropy and give also an explicit formula for it. Another formula is derived in Section 4 with the help of double generating functions and a theorem of Sawyer and Steger [25]. In Section 5 we use another technique to compute a third explicit formula for the entropy of random walks on free products of (not necessarily finite) groups. Section 6 links entropy with the rate of escape and the growth rate of the free product. Sample computations are presented in Section 7.

2. Random Walks on Free Products

2.1. **Free Products and Random Walks.** Let $\mathcal{I} := \{1,\dots,r\} \subseteq \mathbb{N}$, where $r \geq 2$. For each $i \in \mathcal{I}$, consider a random walk with transition matrix P_i on a finite or countable state space V_i. W.l.o.g. we assume that the sets V_i are pairwise disjoint and we exclude the case $r = 2 = |V_1| = |V_2|$ (see below for further explanation). The corresponding single and

n-step transition probabilities are denoted by $p_i(x,y)$ and $p_i^{(n)}(x,y)$, where $x,y\in V_i$. For every $i\in\mathcal{I}$, we select an element o_i of V_i as the "root". To help visualize this, we think of graphs $\mathcal{X}_i$ with vertex sets V_i and roots o_i such that there is an oriented edge $x\to y$ if and only if $p_i(x,y)>0$. Thus, we have a natural graph metric on the set V_i. Furthermore, we shall assume that for every $i\in\mathcal{I}$ and every $x\in V_i$ there is some $n_x\in\mathbb{N}$ such that $p_i^{(n_x)}(o_i,x)>0$. For sake of simplicity we assume $p_i(x,x)=0$ for every $i\in\mathcal{I}$ and $x\in V_i$. Moreover, we assume that the random walks on V_i are *uniformly irreducible*, that is, there are $\varepsilon_0^{(i)}>0$ and $K_i\in\mathbb{N}$ such that for all $x,y\in V_i$

$$p_i(x,y)>0 \quad \Rightarrow \quad p_i^{(k)}(x,y)\geq \varepsilon_0^{(i)} \quad \text{for some } k\leq K_i. \tag{2.1}$$

We set $K:=\max_{i\in\mathcal{I}}K_i$ and $\varepsilon_0:=\min_{i\in\mathcal{I}}\varepsilon_0^{(i)}$. For instance, this property is satisfied for nearest neighbour random walks on Cayley graphs of finitely generated groups, which are governed by probability measures on the groups.

Let $V_i^\times:=V_i\setminus\{o_i\}$ for every $i\in\mathcal{I}$ and let $V_*^\times:=\bigcup_{i\in\mathcal{I}}V_i^\times$. The *free product* is given by

$$\begin{aligned} V &:= V_1 * \ldots * V_r \\ &= \Bigl\{x_1x_2\ldots x_n \Bigm| n\in\mathbb{N}, x_j\in V_*^\times, x_j\in V_k^\times \Rightarrow x_{j+1}\notin V_k^\times \Bigr\}\cup\Bigl\{o\Bigr\}. \end{aligned} \tag{2.2}$$

The elements of V are "words" with letters, also called *blocks*, from the sets $V_i^\times$ such that no two consecutive letters come from the same V_i. The empty word o describes the root of V. If $u=u_1\ldots u_m\in V$ and $v=v_1\ldots v_n\in V$ with $u_m\in V_i$ and $v_1\notin V_i$ then uv stands for their concatenation as words. This is only a partial composition law, which makes defining the asymptotic entropy more complicated than in the case of free products of *groups*. In particular, we set $uo_i:=u$ for all $i\in\mathcal{I}$ and $ou:=u$. Note that $V_i\subseteq V$ and o_i as a word in V is identified with o. The *block length* of a word $u=u_1\ldots u_m$ is given by $\|u\|:=m$. Additionally, we set $\|o\|:=0$. The *type* $\tau(u)$ of u is defined to be i if $u_m\in V_i^\times$; we set $\tau(o):=0$. Finally, $\tilde{u}$ denotes the last letter u_m of u. The set V can again be interpreted as the vertex set of a graph $\mathcal{X}$, which is constructed as follows: take copies of $\mathcal{X}_1,\ldots\mathcal{X}_r$ and glue them together at their roots to one single common root, which becomes o; inductively, at each vertex $v_1\ldots v_k$ with $v_k\in V_i$ attach a copy of every $\mathcal{X}_j$, $j\neq i$, and so on. Thus, we have also a natural graph metric associated to the elements in V.

The next step is the construction of a new Markov chain on the *free product*. For this purpose, we lift P_i to a transition matrix $\bar{P}_i$ on V: if $x\in V$ with $\tau(x)\neq i$ and $v,w\in V_i$, then $\bar{p}_i(xv,xw):=p_i(v,w)$. Otherwise we set $\bar{p}_i(x,y):=0$. We choose $0<\alpha_1,\ldots,\alpha_r\in\mathbb{R}$ with $\sum_{i\in\mathcal{I}}\alpha_i=1$. Then we obtain a new transition matrix on V given by

$$P=\sum_{i\in\mathcal{I}}\alpha_i\bar{P}_i.$$

The random walk on V starting at o, which is governed by P, is described by the sequence of random variables $(X_n)_{n\in\mathbb{N}_0}$. For $x,y\in V$, the associated single and n-step transition probabilities are denoted by $p(x,y)$ and $p^{(n)}(x,y)$. Thus, P governs a nearest neighbour random walk on the graph $\mathcal{X}$, where P arises from a convex combination of the nearest neighbour random walks on the graphs $\mathcal{X}_i$.

Theorem 3.3 in [11] shows existence (including a formula) of a positive number ℓ_0 such that $\ell_0 = \lim_{n\to\infty} \|X_n\|/n$ almost surely. The number ℓ_0 is called the *rate of escape w.r.t. the block length.* Denote by π_n the distribution of X_n. If there is a real number h such that

$$h = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\bigl[-\log \pi_n(X_n)\bigr],$$

then h is called the *asymptotic entropy* of the process $(X_n)_{n\in\mathbb{N}_0}$; we write $\mathbb{N}_0 := \mathbb{N}\setminus\{0\}$. If the sets V_i are groups and the random walks P_i are governed by probability measures μ_i, existence of the asymptotic entropy rate is well-known, and in this case we even have $h = \lim_{n\to\infty} -\frac{1}{n}\log \pi_n(X_n)$ almost surely; see Derriennic [7] and Kaimanovich and Vershik [14]. We prove existence of h in the case of general free products.

2.2. **Generating Functions.** Our main tool will be the usage of generating functions, which we introduce now. The *Green functions* related to P_i and P are given by

$$G_i(x_i,y_i|z) := \sum_{n\geq 0} p_i^{(n)}(x_i,y_i)\, z^n \quad \text{and} \quad G(x,y|z) := \sum_{n\geq 0} p^{(n)}(x,y)\, z^n,$$

where $z\in\mathbb{C}$, $x_i,y_i\in V_i$ and $x,y\in V$. At this point we make the *basic assumption* that the radius of convergence R of $G(\cdot,\cdot|z)$ is strictly bigger than 1. This implies *transience* of our random walk on V. Thus, we may exclude the case $r=2=|V_1|=|V_2|$, because we get recurrence in this case. For instance, if all P_i govern *reversible* Markov chains, then $R>1$; see [29, Theorem 10.3]. Furthermore, it is easy to see that $R>1$ holds also if there is some $i\in\mathcal{I}$ such that $p_i^{(n)}(o_i,o_i)=0$ for all $n\in\mathbb{N}$.

The *first visit generating functions* related to P_i and P are given by

$$\begin{aligned} F_i(x_i,y_i|z) &:= \sum_{n\geq 0} \mathbb{P}\bigl[Y_n^{(i)} = y_i, \forall m\leq n-1: Y_m^{(i)}\neq y_i \mid Y_0^{(i)} = x_i\bigr] z^n \quad \text{and}\\ F(x,y|z) &:= \sum_{n\geq 0} \mathbb{P}\bigl[X_n = y, \forall m\leq n-1: X_m\neq y \mid X_0 = x\bigr] z^n, \end{aligned}$$

where $\bigl(Y_n^{(i)}\bigr)_{n\in\mathbb{N}_0}$ describes a random walk on V_i governed by P_i. The stopping time of the first return to o is defined as $T_o := \inf\{m\geq 1 \mid X_m = o\}$. For $i\in\mathcal{I}$, define

$$\overline{H}_i(z) := \sum_{n\geq 1}\mathbb{P}[T_o = n, X_1\notin V_i^\times]\, z^n \quad \text{and} \quad \xi_i(z) := \frac{\alpha_i z}{1-\overline{H}_i(z)}.$$

We write also $\xi_i := \xi_i(1)$, $\xi_{\min} := \min_{i\in\mathcal{I}}\xi_i$ and $\xi_{\max} := \max_{i\in\mathcal{I}}\xi_i$. Observe that $\xi_i<1$; see [11, Lemma 2.3]. We have $F(x_i,y_i|z) = F_i\bigl(x_i,y_i|\xi_i(z)\bigr)$ for all $x_i,y_i\in V_i$; see Woess [29, Prop. 9.18c]. Thus,

$$\xi_i(z) := \frac{\alpha_i z}{1-\sum_{j\in\mathcal{I}\setminus\{i\}}\sum_{s\in V_j}\alpha_j p_j(o_j,s) z F_j\bigl(s,o_j\big|\xi_j(z)\bigr)}.$$

For $x_i\in V_i$ and $x\in V$, define the stopping times $T_{x_i}^{(i)} := \inf\{m\geq 1 \mid Y_m^{(i)} = x_i\}$ and $T_x := \inf\{m\geq 1\mid X_m = x\}$, which take both values in $\mathbb{N}\cup\{\infty\}$. Then the *last visit* *generating functions* related to P_i and P are defined as

$$\begin{aligned} L_i(x_i,y_i|z) &:= \sum_{n\geq 0} \mathbb{P}\bigl[Y_n^{(i)}=y_i, T_{x_i}^{(i)}>n \mid Y_0^{(i)}=x_i\bigr]\, z^n,\\ L(x,y|z) &:= \sum_{n\geq 0} \mathbb{P}\bigl[X_n=y, T_x>n \mid X_0=x\bigr]\, z^n. \end{aligned}$$

If $x=x_1\dots x_n, y=x_1\dots x_n x_{n+1}\in V$ with $\tau(x_{n+1})=i$ then

$$L(x,y|z)=L_i\bigl(o_i,x_{n+1}\,\big|\,\xi_i(z)\bigr); \tag{2.3}$$

this equation is proved completely analogously to [29, Prop. 9.18c]. If all paths from $x\in V$ to $w\in V$ have to pass through $y\in V$, then

$$L(x,w|z)=L(x,y|z)\cdot L(y,w|z);$$

this can be easily checked by conditioning on the last visit of y when walking from x to w. We have the following important equations, which follow by conditioning on the last visits of x_i and x, the first visits of y_i and y respectively:

$$\begin{aligned} G_i(x_i,y_i|z) &= G_i(x_i,x_i|z)\cdot L_i(x_i,y_i|z)=F_i(x_i,y_i|z)\cdot G_i(y_i,y_i|z),\\ G(x,y|z) &= G(x,x|z)\cdot L(x,y|z)=F(x,y|z)\cdot G(y,y|z). \end{aligned} \tag{2.4}$$

Observe that the generating functions $F(\cdot,\cdot|z)$ and $L(\cdot,\cdot|z)$ have also radii of convergence strictl bigger than 1.

3. The Asymptotic Entropy

3.1. **Rate of Escape w.r.t. specific Length Function.** In this subsection we prove existence of the rate of escape with respect to a specific length function. From this we will deduce existence and a formula for the asymptotic entropy in the upcoming subsection.

We assign to each element $x_i\in V_i$ the "length"

$$l_i(x_i):=-\log L(o,x_i|1)=-\log L_i(o_i,x_i|\xi_i).$$

We extend it to a length function on V by assigning to $v_1\dots v_n\in V$ the length

$$l(v_1\dots v_n):=\sum_{i=1}^n l_{\tau(v_i)}(v_i)=-\sum_{i=1}^n \log L(o,v_i|1)=-\log L(o,v_1\dots v_n|1).$$

Observe that the lengths can also be negative. E.g., this can be interpreted as height differences. The aim of this subsection is to show existence of a number $\ell\in\mathbb{R}$ such that the quotient $l(X_n)/n$ tends to ℓ almost surely as $n\to\infty$. We call ℓ the *rate of escape w.r.t. the length function* $l(\cdot)$.

We follow now the reasoning of [11, Section 3]. Denote by $X_n^{(k)}$ the projection of X_n to the first k letters. We define the *k-th exit time* as

$$\mathbf{e}_k:=\min\bigl\{m\in\mathbb{N}_0 \,\big|\, \forall n\geq m: X_n^{(k)} \text{ is constant}\bigr\}.$$

Moreover, we define $\mathbf{W}_k:=X_{\mathbf{e}_k}$, $\tau_k:=\tau(\mathbf{W}_k)$ and $\mathbf{k}(n):=\max\{k\in\mathbb{N}_0\mid \mathbf{e}_k\leq n\}$. We remark that $\|X_n\|\to\infty$ as $n\to\infty$, and consequently $\mathbf{e}_k<\infty$ almost surely for every

$k\in\mathbb{N}$; see [11, Prop. 2.5]. Recall that $\widetilde{\mathbf{W}}_k$ is just the laster letter of the random word $X_{\mathbf{e}_k}$. The process $(\tau_k)_{k\in\mathbb{N}}$ is Markovian and has transition probabilities

$$\hat{q}(i,j)=\frac{\alpha_j}{\alpha_i}\frac{\xi_i}{\xi_j}\frac{1-\xi_j}{1-\xi_i}\Bigl(\frac{1}{(1-\xi_j)G_j(o_j,o_j|\xi_j)}-1\Bigr)$$

for $i\neq j$ and $\hat{q}(i,i)=0$; see [11, Lemma 3.4]. This process is positive recurrent with invariant probability measure

$$\begin{aligned}\nu(i) &= C^{-1}\cdot\frac{\alpha_i(1-\xi_i)}{\xi_i}\bigl(1-(1-\xi_i)G_i(o_i,o_i|\xi_i)\bigr),\\ \text{where } C &:= \sum_{i\in\mathcal{I}}\frac{\alpha_i(1-\xi_i)}{\xi_i}\bigl(1-(1-\xi_i)G_i(o_i,o_i|\xi_i)\bigr);\end{aligned}$$

see [11, Section 3]. Furthermore, the rate of escape w.r.t. the block length exists almost surely and is given by the almost sure constant limit

$$\ell_0=\lim_{n\to\infty}\frac{\|X_n\|}{n}=\lim_{k\to\infty}\frac{k}{\mathbf{e}_k}=\frac{1}{\sum_{i,j\in\mathcal{I},i\neq j}\nu(i)\,\alpha_j\frac{1-\xi_j}{1-\xi_i}\gamma_{i,j}'(1)}$$

(see [11, Theorem 3.3]), where

$$\gamma_{i,j}(z):=\frac{1}{\alpha_i}\frac{\xi_i(z)}{\xi_j(z)}\Bigl(\frac{1}{\bigl(1-\xi_j(z)\bigr)G_j\bigl(o_j,o_j\big|\xi_j(z)\bigr)}-1\Bigr).$$

Lemma 3.1. *The process $\bigl(\widetilde{\mathbf{W}}_k,\tau_k\bigr)_{k\in\mathbb{N}}$ is Markovian and has transition probabilities*

$$q\bigl((g,i),(h,j)\bigr)=\begin{cases}\frac{\alpha_j}{\alpha_i}\frac{\xi_i}{\xi_j}\frac{1-\xi_j}{1-\xi_i}L_j(o_j,h|\xi_j), & \text{if } i\neq j,\\ 0, & \text{if } i=j.\end{cases}$$

Furthermore, the process is positive recurrent with invariant probability measure

$$\pi(g,i)=\sum_{j\in\mathcal{I}}\nu(j)q\bigl((*,j),(g,i)\bigr).$$

Remark: Observe that the transition probabilities $q\bigl((g,i),(h,j)\bigr)$ of $\bigl(\widetilde{\mathbf{W}}_k,\tau_k\bigr)_{k\in\mathbb{N}}$ do not depend on g. Therefore, we will write sometimes an asterisk instead of g.

Proof. By [11, Section 3], the process $\bigl(\widetilde{\mathbf{W}}_k,\mathbf{e}_k-\mathbf{e}_{k-1},\tau_k\bigr)_{k\in\mathbb{N}}$ is Markovian and has transition probabilities

$$\tilde{q}\bigl((g,m,i),(h,n,j)\bigr)=\begin{cases}\frac{1-\xi_j}{1-\xi_i}\sum_{s\in V_j}k_i^{(n-1)}(s)p(s,h), & \text{if } i\neq j,\\ 0, & \text{if } i=j,\end{cases}$$

where $k_i^{(n)}(s) := \mathbb{P}\bigl[X_n = s, \forall l \leq n : X_l \notin V_i^\times | X_0 = o]$ for $s \in V_*^\times \setminus V_i$. Thus, $\bigl(\widetilde{\mathbf{W}}_k, \tau_k\bigr)_{k\in\mathbb{N}}$ is also Markovian and has the following transition probabilities if $i \neq j$:

$$\begin{aligned} q\bigl((g,i),(h,j)\bigr) &= \sum_{n\geq 1} \tilde{q}\bigl((g,*,i),(h,n,j)\bigr) = \frac{1-\xi_j}{1-\xi_i} \sum_{s\in V_j}\sum_{n\geq 1} k_i^{(n-1)}(s) p(s,h) \\ &= \frac{1-\xi_j}{1-\xi_i} \sum_{s\in V_j} \frac{L_j(o_j,s|\xi_j)}{1-\bar{H}_i(1)} p(s,h) = \frac{\alpha_j}{\alpha_i}\frac{\xi_i}{\xi_j}\frac{1-\xi_j}{1-\xi_i} L_j(o_j,h|\xi_j). \end{aligned}$$

In the third equality we conditioned on the last visit of o before finally walking from o to s and we remark that $h \in V_j^\times$. A straight-forward computation shows that π is the invariant probability measure of $\bigl(\widetilde{\mathbf{W}}_k, \tau_k\bigr)_{k\in\mathbb{N}}$, where we write $\mathcal{A} := \bigl\{(g,i) \,\big|\, i \in \mathcal{I}, g \in V_i^\times\bigr\}$:

$$\begin{aligned} \sum_{(g,i)\in\mathcal{A}} \pi(g,i)\cdot q\bigl((g,i),(h,j)\bigr) &= \sum_{(g,i)\in\mathcal{A}}\sum_{k\in\mathcal{I}} \nu(k)\cdot q\bigl((*,k),(g,i)\bigr)\cdot q\bigl((*,i),(h,j)\bigr) \\ &= \sum_{i\in\mathcal{I}} q\bigl((*,i),(h,j)\bigr) \sum_{k\in\mathcal{I}} \nu(k) \sum_{g\in V_i^\times} q\bigl((*,k),(g,i)\bigr) \\ &= \sum_{i\in\mathcal{I}} q\bigl((*,i),(h,j)\bigr) \sum_{k\in\mathcal{I}} \nu(k)\cdot \hat{q}(k,i) \\ &= \sum_{i\in\mathcal{I}} q\bigl((*,i),(h,j)\bigr)\cdot \nu(i) = \pi(h,j). \end{aligned}$$

□

Now we are able to prove the following:

Proposition 3.2. *There is a number $\ell \in \mathbb{R}$ such that*

$$\ell = \lim_{n\to\infty} \frac{l(X_n)}{n} \quad \textit{almost surely.}$$

Proof. Define $h : \mathcal{A} \to \mathbb{R}$ by $h(g,j) := l(g)$. Then $\sum_{\lambda=1}^k h\bigl(\widetilde{\mathbf{W}}_\lambda, \tau_\lambda\bigr) = \sum_{\lambda=1}^k l\bigl(\widetilde{\mathbf{W}}_\lambda\bigr) = l(\mathbf{W}_k)$. An application of the ergodic theorem for positive recurrent Markov chains yields

$$\frac{l(\mathbf{W}_k)}{k} = \frac{1}{k}\sum_{\lambda=1}^k h\bigl(\widetilde{\mathbf{W}}_\lambda, \tau_\lambda\bigr) \xrightarrow{n\to\infty} C_h := \int h\, d\pi,$$

if the integral on the right hand side exists. We now show that this property holds. Observe that the values $G_j(o_j, g|\xi_j)$ are uniformly bounded from above for all $(g,j) \in \mathcal{A}$:

$$G_j(o_j,g|\xi_j) = \sum_{n\geq 0} p_j^{(n)}(o_j,g)\,\xi_j^n \leq \frac{1}{1-\xi_j} \leq \frac{1}{1-\xi_{\max}}.$$

For $g \in V_*^\times$, denote by $|g|$ the smallest $n \in \mathbb{N}$ such that $p_{\tau(g)}^{(n)}(o_{\tau(g)}, g) > 0$. Uniform irreducibility of the random walk P_i on V_i implies that there are some $\varepsilon_0 > 0$ and $K \in \mathbb{N}$

such that for all $j \in \mathcal{I}$, $x_j, y_j \in V_j$ with $p_j(x_j, y_j) > 0$ we have $p_j^{(k)}(x_j, y_j) \geq \varepsilon_0$ for some $k \leq K$. Thus, for $(g, j) \in \mathcal{A}$ we have

$$G_j(o_j, g|\xi_j) \geq \varepsilon_0^{|g|} \xi_j^{|g| \cdot K} \geq \left(\varepsilon_0\, \xi_{\min}^K\right)^{|g|}.$$

Observe that the inequality $|g| \cdot \left|\log\left(\varepsilon_0\, \xi_{\min}^K\right)\right| < \log 1/(1-\xi_{\max})$ holds if and only if $|g| < \log(1-\xi_{\max})/\log(\varepsilon_0\, \xi_{\min}^K)$. Define the sets

$$M_1 := \left\{ g \in V_*^\times \,\middle|\, |g| \geq \frac{\log(1-\xi_{\max})}{\log(\varepsilon_0\, \xi_{\min}^K)} \right\}, \quad M_2 := \left\{ g \in V_*^\times \,\middle|\, |g| < \frac{\log(1-\xi_{\max})}{\log(\varepsilon_0\, \xi_{\min}^K)} \right\}.$$

Recall Equation (2.4). We can now prove existence of $\int h\, d\pi$:

$$\begin{aligned}
\int |h|\, d\pi &= \sum_{(g,j)\in\mathcal{A}} \left|\log L_j(o_j, g|\xi_j)\right| \cdot \pi(g,j) \\
&\leq \sum_{(g,j)\in\mathcal{A}} \left|\log G_j(o_j, g|\xi_j)\right| \cdot \pi(g,j) + \sum_{(g,j)\in\mathcal{A}} \left|\log G_j(o_j, o_j|\xi_j)\right| \cdot \pi(g,j) \\
&\leq \sum_{(g,j)\in\mathcal{A}: g\in M_1} \left|\log G_j(o_j, g|\xi_j)\right| \cdot \pi(g,j) \\
&\qquad + \sum_{(g,j)\in\mathcal{A}: g\in M_2} \left|\log G_j(o_j, g|\xi_j)\right| \cdot \pi(g,j) + \max_{j\in\mathcal{I}} \log G_j(o_j, o_j|\xi_j) \\
&\leq \sum_{(g,j)\in\mathcal{A}: g\in M_1} \left|\log(\varepsilon_0 \xi_{\min}^K)^{|g|}\right| \cdot \pi(g,j) \\
&\qquad + \sum_{(g,j)\in\mathcal{A}: g\in M_2} \left|\log(1-\xi_{\max})\right| \cdot \pi(g,j) + \max_{j\in\mathcal{I}} \log G_j(o_j, o_j|\xi_j) \\
&\leq \sum_{(g,j)\in\mathcal{A}: g\in M_1} \left|\log(\varepsilon_0 \xi_{\min}^K)\right| \cdot |g| \cdot \pi(g,j) \\
&\qquad + \left|\log(1-\xi_{\max})\right| + \max_{j\in\mathcal{I}} \log G_j(o_j, o_j|\xi_j) < \infty,
\end{aligned}$$

since $\sum_{(g,j)\in\mathcal{A}} |g| \cdot \pi(g,j) < \infty$; see [11, Proof of Prop. 3.2]. From this follows that $l(\mathbf{W}_k)/k$ tends to C_h almost surely. The next step is to show that

$$\frac{l(X_n) - l(\mathbf{W}_{\mathbf{k}(n)})}{n} \xrightarrow{n\to\infty} 0 \quad \text{almost surely.} \tag{3.1}$$

To prove this, assume now that we have the representations $\mathbf{W}_{\mathbf{k}(n)} = g_1 g_2 \dots g_{\mathbf{k}(n)}$ and $X_n = g_1 g_2 \dots g_{\mathbf{k}(n)} \dots g_{\|X_n\|}$. Define $M := \max\bigl\{|\log(\varepsilon_0\, \xi_{\min}^K)|, |\log(1-\xi_{\max})|\bigr\}$. Then:

$$
\begin{aligned}
\bigl|l(X_n) - l(\mathbf{W}_{\mathbf{k}(n)})\bigr| &= \Biggl| - \sum_{i=\mathbf{k}(n)+1}^{\|X_n\|} \log L_{\tau(g_i)}\bigl(o_{\tau(g_i)}, g_i \mid \xi_{\tau(g_i)}\bigr)\Biggr| \\
&\leq \sum_{i=\mathbf{k}(n)+1}^{\|X_n\|} \left| \log \frac{G_{\tau(g_i)}\bigl(o_{\tau(g_i)}, g_i \mid \xi_{\tau(g_i)}\bigr)}{G_{\tau(g_i)}\bigl(o_{\tau(g_i)}, o_{\tau(g_i)} \mid \xi_{\tau(g_i)}\bigr)} \right| \\
&\leq \sum_{i=\mathbf{k}(n)+1: g_i \in M_1}^{\|X_n\|} \bigl| \log G_{\tau(g_i)}\bigl(o_{\tau(g_i)}, g_i \mid \xi_{\tau(g_i)}\bigr)\bigr| \\
&\qquad + \sum_{i=\mathbf{k}(n)+1: g_i \in M_2}^{\|X_n\|} \bigl| \log G_{\tau(g_i)}\bigl(o_{\tau(g_i)}, g_i \mid \xi_{\tau(g_i)}\bigr)\bigr| \\
&\qquad + \bigl(\|X_n\| - \mathbf{k}(n)\bigr) \cdot \bigl|\log(1-\xi_{\max})\bigr| \\
\leq{}& \sum_{i=\mathbf{k}(n)+1: g_i \in M_1}^{\|X_n\|} \bigl|\log(\varepsilon_0\, \xi_{\min}^K)^{|g_i|}\bigr| \\
&+ \sum_{i=\mathbf{k}(n)+1: g_i \in M_2}^{\|X_n\|} \bigl|\log(1-\xi_{\max})\bigr| + \bigl(\|X_n\| - \mathbf{k}(n)\bigr) \cdot \bigl|\log(1-\xi_{\max})\bigr| \\
\leq{}& \sum_{i=\mathbf{k}(n)+1: g_i \in M_1}^{\|X_n\|} |g_i| \cdot M + \sum_{i=\mathbf{k}(n)+1: g_i \in M_2}^{\|X_n\|} M + \bigl(\|X_n\| - \mathbf{k}(n)\bigr) \cdot M \\
\leq{}& 3 \cdot M \cdot (n - \mathbf{e}_{\mathbf{k}(n)}).
\end{aligned}
$$

Dividing the last inequality by n and letting $n \to \infty$ provides analogously to Nagnibeda and Woess [23, Section 5] that $\lim_{n\to\infty}\bigl(l(X_n) - l(\mathbf{W}_{\mathbf{k}(n)})\bigr)/n = 0$ almost surely. Recall also that $k/\mathbf{e}_k \to \ell_0$ and $\mathbf{e}_{\mathbf{k}(n)}/n \to 1$ almost surely; compare [23, Proof of Theorem D] and [11, Prop. 3.2, Thm. 3.3]. Now we can conclude:

$$
\frac{l(X_n)}{n} = \frac{l(X_n) - l(\mathbf{W}_{\mathbf{k}(n)})}{n} + \frac{l(\mathbf{W}_{\mathbf{k}(n)})}{\mathbf{k}(n)} \frac{\mathbf{k}(n)}{\mathbf{e}_{k(n)}} \frac{\mathbf{e}_{\mathbf{k}(n)}}{n} \xrightarrow{n\to\infty} C_h \cdot \ell_0 \quad \text{almost surely.} \tag{3.2}
$$

□

We now compute the constant C_h from the last proposition explicitly:

$$
\begin{aligned}
C_h &= \sum_{(g,j)\in\mathcal{A}} l(g) \cdot \sum_{i\in\mathcal{I}} \nu(i) \cdot q\bigl((*,i),(g,j)\bigr) \\
&= \sum_{\substack{i,j\in\mathcal{I},\\ i\neq j}} \sum_{g\in V_j^\times} -\log L_j(o_j, g|\xi_j)\, \nu(i) \frac{\alpha_j}{\alpha_i} \frac{\xi_i}{\xi_j} \frac{1-\xi_j}{1-\xi_i} L_j(o_j, g|\xi_j). \qquad (3.3)
\end{aligned}
$$

We conclude this subsection with the following observation:

Corollary 3.3. *The rate of escape ℓ is non-negative and it is the rate of escape w.r.t. the Greenian metric, which is given by $d_{Green}(x,y) := -\log F(x,y|1)$. That is,*

$$\ell = \lim_{n\to\infty} -\frac{1}{n}\log F(e,X_n|1) \geq 0.$$

Proof. By (2.4), we get

$$\ell = \lim_{n\to\infty} -\frac{1}{n}\log F(e,X_n|1) - \frac{1}{n}\log G(X_n,X_n|1) + \frac{1}{n}\log G(o,o|1).$$

Since $F(e,X_n|1)\leq 1$ it remains to show that $G(x,x|1)$ is uniformly bounded in $x\in V$: for $v,w\in V$, the *first visit generating function* is defined as

$$U(v,w|z) = \sum_{n\geq 1} \mathbb{P}\bigl[X_n = w, \forall m\in\{1,\dots,n-1\} : X_m \neq w \mid X_0 = v\bigr]\, z^n. \tag{3.4}$$

Therefore,

$$G(x,x|z) = \sum_{n\geq 0} U(x,x|z)^n = \frac{1}{1-U(x,x|z)}.$$

Since $U(x,x|z)<1$ for all $z\in[1,R)$, $U(x,x|0)=0$ and $U(x,x|z)$ is continuous, stricly increasing and strictly convex, we must have $U(x,x|1)\leq \frac{1}{R}$, that is, $1\leq G(x,x|1)\leq \bigl(1-\frac{1}{R}\bigr)^{-1}$. This finishes the proof. □

3.2. **Asymptotic Entropy.** In this subsection we will prove that ℓ equals the asymptotic entropy, and we will give explicit formulas for it. The technique of the proof which we will give was motivated by Benjamini and Peres [2], where it is shown that the asymptotic entropy of random walks on discrete groups equals the rate of escape w.r.t. the Greenian distance. The proof follows the same reasoning as in Gilch and Müller [12].

Recall that we made the assumption that the spectral radius of $(X_n)_{n\in\mathbb{N}_0}$ is strictly smaller than 1, that is, the Green function $G(o,o|z)$ has radius of convergence $R>1$. Moreover, the functions $\xi_i(z)$, $i\in\mathcal{I}$, have radius of convergence bigger than 1. Recall that $\xi_i=\xi_i(1)<1$ for every $i\in\mathcal{I}$. Thus, we can choose $\varrho\in(1,R)$ such that $\xi_i(\varrho)<1$ for all $i\in\mathcal{I}$. We now need the following three technical lemmas:

Lemma 3.4. *For all $m,n\in\mathbb{N}_0$,*

$$p^{(m)}(o,X_n) \leq G(o,o|\varrho)\cdot\Bigl(\frac{1}{1-\max_{i\in\mathcal{I}}\xi_i(\varrho)}\Bigr)^n\cdot\varrho^{-m}.$$

Proof. Denote by $\mathcal{C}_\varrho$ the circle with radius ϱ in the complex plane centered at 0. A straightforward computation shows for $m\in\mathbb{N}_0$:

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_\varrho} z^m\frac{dz}{z} = \begin{cases}1, & \text{if } m=0,\\ 0, & \text{if } m\neq 0.\end{cases}$$

Let be $x = x_1 \dots x_t \in V$. An application of Fubini's Theorem yields

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\mathcal{C}_\varrho} G(o,x|z)\, z^{-m}\frac{dz}{z} &= \frac{1}{2\pi i}\oint_{\mathcal{C}_\varrho}\sum_{k\geq 0} p^{(k)}(o,x) z^k\, z^{-m}\frac{dz}{z}\\ &= \frac{1}{2\pi i}\sum_{k\geq 0} p^{(k)}(o,x)\oint_{\mathcal{C}_\varrho} z^{k-m}\frac{dz}{z} = p^{(m)}(o,x).\end{aligned}$$

Since $G(o,x|z)$ is analytic on $\mathcal{C}_\varrho$, we have $|G(o,x|z)| \leq G(o,x|\varrho)$ for all $|z| = \varrho$. Thus,

$$p^{(m)}(o,x) \leq \frac{1}{2\pi}\cdot \varrho^{-m-1}\cdot G(o,x|\varrho)\cdot 2\pi\varrho = G(o,x|\varrho)\cdot \varrho^{-m}.$$

Iterated applications of equations (2.3) and (2.4) provide

$$G(o,x|\varrho) = G(o,o|\varrho)\prod_{k=1}^{\|x\|} L_{\tau(x_k)}\bigl(o_{\tau(x_k)}, x_k|\xi_i(\varrho)\bigr) \leq G(o,o|\varrho)\Bigl(\frac{1}{1-\max_{i\in\mathcal{I}}\xi_i(\varrho)}\Bigr)^{\|x\|}.$$

Since $\|X_n\| \leq n$, we obtain

$$p^{(m)}(e,X_n) \leq G(o,o|\varrho)\cdot\Bigl(\frac{1}{1-\max_{i\in\mathcal{I}}\xi_i(\varrho)}\Bigr)^n\cdot\varrho^{-m}.$$

□

Lemma 3.5. *Let $(A_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be sequences of strictly positive numbers with $A_n = a_n + b_n$. Assume that $\lim_{n\to\infty} -\frac{1}{n}\log A_n = c \in [0,\infty)$ and that $\lim_{n\to\infty} b_n/q^n = 0$ for all $q \in (0,1)$. Then $\lim_{n\to\infty} -\frac{1}{n}\log a_n = c$.*

Proof. Under the made assumptions it can not be that $\liminf_{n\to\infty} a_n/q^n = 0$ for every $q\in(0,1)$. Indeed, assume that this would hold. Choose any $q>0$. Then there is a subseqence $(a_{n_k})_{k\in\mathbb{N}}$ with $a_{n_k}/q^{n_k}\to 0$. Moreover, there is $N_q\in\mathbb{N}$ such that $a_{n_k}, b_{n_k} < q^{n_k}/2$ for all $k\geq N_q$. But this implies

$$-\frac{1}{n_k}\log(a_{n_k}+b_{n_k}) \geq -\frac{1}{n_k}\log(q^{n_k}) = -\log q.$$

The last inequality holds for every $q>0$, yielding that $\limsup_{n\to\infty} -\frac{1}{n}\log A_n = \infty$, a contradiction.

Thus, there is some $N\in\mathbb{N}$ such that $b_n < a_n$ for all $n\geq N$. We get for all $n\geq N$:

$$\begin{aligned}-\frac{1}{n}\log(a_n+b_n) &\leq -\frac{1}{n}\log(a_n) = -\frac{1}{n}\log\Bigl(\frac{1}{2}a_n+\frac{1}{2}a_n\Bigr)\\ &\leq -\frac{1}{n}\log\Bigl(\frac{1}{2}a_n+\frac{1}{2}b_n\Bigr) \leq -\frac{1}{n}\log\frac{1}{2} - \frac{1}{n}\log(a_n+b_n).\end{aligned}$$

Taking limits yields that $-\frac{1}{n}\log(a_n)$ tends to c, since the leftmost and rightmost side of this inequality chain tend to c. □

For the next lemma recall the definition of K from (2.1).

Lemma 3.6. *For $n \in \mathbb{N}$, consider the function $f_n : V \to \mathbb{R}$ defined by*

$$f_n(x) := \begin{cases} -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o,x), & \text{if } p^{(n)}(o,x) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then there are constants d and D such that $d \leq f_n(x) \leq D$ for all $n \in \mathbb{N}$ and $x \in V$.

Proof. Assume that $p^{(n)}(o,x) > 0$. Recall from the proof of Corollary 3.3 that we have $G(x,x|1) \leq \bigl(1 - \frac{1}{R}\bigr)^{-1}$. Therefore,

$$\sum_{m=0}^{Kn^2} p^{(m)}(o,x) \leq G(o,x|1) \leq F(o,x|1) \cdot G(x,x|1) \leq \frac{1}{1-\frac{1}{R}},$$

that is

$$f_n(x) \geq -\frac{1}{n} \log \frac{1}{1-\frac{1}{R}}.$$

For the upper bound, observe that, by uniform irreducibility, $x \in V$ with $p^{(n)}(o,x) > 0$ can be reached from o in $N_x \leq K \cdot |x| \leq Kn$ steps with a probability of at least $\varepsilon_0^{|x|}$, where $\varepsilon_0 > 0$ from (2.1) is independent from x. Thus, at least one of the summands in $\sum_{m=0}^{Kn^2} p^{(m)}(o,x)$ has a value greater or equal to $\varepsilon_0^{|x|} \geq \varepsilon_0^n$. Thus, $f_n(x) \leq -\log \varepsilon_0$. □

Now we can state and prove our first main result:

Theorem 3.7. *Assume $R > 1$. Then the asymptotic entropy exists and is given by*

$$h = \ell_0 \cdot \sum_{g \in V_*^\times} l(g)\, \pi\bigl(g, \tau(g)\bigr) = \ell.$$

Proof. By (2.4) we can rewrite ℓ as

$$\ell = \lim_{n\to\infty} -\frac{1}{n} \log L(o,X_n|1) = \lim_{n\to\infty} -\frac{1}{n} \log \frac{G(o,X_n|1)}{G(o,o|1)} = \lim_{n\to\infty} -\frac{1}{n} \log G(o,X_n|1).$$

Since

$$G(o,X_n|1) = \sum_{m\geq 0} p^{(m)}(o,X_n) \geq p^{(n)}(o,X_n) = \pi_n(X_n),$$

we have

$$\liminf_{n\to\infty} -\frac{1}{n} \log \pi_n(X_n) \geq \ell. \tag{3.5}$$

The next aim is to prove $\limsup_{n\to\infty} -\frac{1}{n}\mathbb{E}\bigl[\log \pi_n(X_n)\bigr] \leq \ell$. We now apply Lemma 3.5 by setting

$$A_n := \sum_{m\geq 0} p^{(m)}(o,X_n),\ a_n := \sum_{m=0}^{Kn^2} p^{(m)}(o,X_n) \text{ and } b_n := \sum_{m\geq Kn^2+1} p^{(m)}(o,X_n).$$

By Lemma 3.4,

$$b_n \leq \sum_{m\geq Kn^2+1} \frac{G(o,o|\varrho)}{\varrho^m} \cdot \Bigl(\frac{1}{1-\max_{i\in\mathcal{I}} \xi_i(\varrho)}\Bigr)^n = G(o,o|\varrho) \cdot \Bigl(\frac{1}{1-\max_{i\in\mathcal{I}} \xi_i(\varrho)}\Bigr)^n \cdot \frac{\varrho^{-Kn^2-1}}{1-\varrho^{-1}}.$$

Therefore, b_n decays faster than any geometric sequence. Applying Lemma 3.5 yields

$$\ell = \lim_{n\to\infty} -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o, X_n) \quad \text{almost surely.}$$

By Lemma 3.6, we may apply the Dominated Convergence Theorem and get:

$$\begin{aligned}
\ell &= \int \lim_{n\to\infty} -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o, X_n)\, d\mathbb{P} \\
&= \lim_{n\to\infty} \int -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o, X_n)\, d\mathbb{P} \\
&= \lim_{n\to\infty} -\frac{1}{n} \sum_{x\in V} p^{(n)}(o,x) \log \sum_{m=0}^{Kn^2} p^{(m)}(o,x).
\end{aligned}$$

Recall that *Shannon's Inequality* gives

$$-\sum_{x\in V} p^{(n)}(o,x) \log \mu(x) \geq -\sum_{x\in V} p^{(n)}(o,x) \log p^{(n)}(o,x)$$

for every finitely supported probability measure μ on V. We apply now this inequality by setting $\mu(x) := \frac{1}{Kn^2+1} \sum_{m=0}^{Kn^2} p^{(m)}(o,x)$:

$$\begin{aligned}
\ell &\geq \limsup_{n\to\infty} \frac{1}{n} \sum_{x\in V} p^{(n)}(o,x) \log(Kn^2+1) - \frac{1}{n} \sum_{x\in V} p^{(n)}(o,x) \log p^{(n)}(o,x) \\
&= \limsup_{n\to\infty} -\frac{1}{n} \int \log \pi_n(X_n)\, d\mathbb{P}.
\end{aligned}$$

Now we can conclude with Fatou's Lemma:

$$\begin{aligned}
\ell \leq \int \liminf_{n\to\infty} \frac{-\log \pi_n(X_n)}{n} d\mathbb{P} &\leq \liminf_{n\to\infty} \int \frac{-\log \pi_n(X_n)}{n} d\mathbb{P} \\
&\leq \limsup_{n\to\infty} \int \frac{-\log \pi_n(X_n)}{n} d\mathbb{P} \leq \ell. \qquad (3.6)
\end{aligned}$$

Thus, $\lim_{n\to\infty} -\frac{1}{n}\mathbb{E}\bigl[\log \pi_n(X_n)\bigr]$ exists and the limit equals ℓ. The rest follows from (3.2) and (3.3). $\square$

We now give another formula for the asymptotic entropy which shows that it is strictly positive.

Theorem 3.8. *Assume $R > 1$. Then the asymptotic entropy is given by*

$$h = \ell_0 \cdot \sum_{g,h\in V_*^\times} -\pi\bigl(g,\tau(g)\bigr)\, q\bigl((g,\tau(g)),(h,\tau(h))\bigr)\, \log q\bigl((g,\tau(g)),(h,\tau(h))\bigr) > 0.$$

Remarks: Observe that the sum on the right hand side of Theorem 3.8 equals the entropy rate (for positive recurrent Markov chains) of $\bigl(\widetilde{\mathbf{W}}_k,\tau_k\bigr)_{k\in\mathbb{N}}$, which is defined by the almost sure constant limit

$$h_Q := \lim_{n\to\infty} -\frac{1}{n}\log \mu_n\bigl((\widetilde{\mathbf{W}}_1,\tau_1),\dots,(\widetilde{\mathbf{W}}_n,\tau_n)\bigr),$$

where $\mu_n\bigl((g_1,\tau_1),\dots,(g_n,\tau_n)\bigr)$ is the joint distribution of $\bigl((\widetilde{\mathbf{W}}_1,\tau_1),\dots,(\widetilde{\mathbf{W}}_n,\tau_n)\bigr)$. That is, $h=\ell\cdot h_Q$. For more details, we refer e.g. to Cover and Thomas [6, Chapter 4].

At this point it is essential that we have defined the length function $l(\cdot)$ with the help of the functions $L(x,y|z)$ and not by the Greenian metric.

Proof. For a moment let be $x=x_1\dots x_n\in V$. Then:

$$\begin{aligned}
l(x) &= -\log\prod_{j=1}^{n} L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}\bigr)\\
&= -\log\prod_{j=2}^{n}\frac{\alpha_{\tau(x_j)}}{\alpha_{\tau(x_{j-1})}}\frac{\xi_{\tau(x_{j-1})}}{\xi_{\tau(x_j)}}\frac{1-\xi_{\tau(x_j)}}{1-\xi_{\tau(x_{j-1})}}L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}\bigr)\\
&\qquad -\log L_{\tau(x_1)}\bigl(o_{\tau(x_1)},x_1|\xi_{\tau(x_1)}\bigr)+\log\frac{\xi_{\tau(x_1)}\,\alpha_{\tau(x_n)}\,(1-\xi_{\tau(x_n)})}{\alpha_{\tau(x_1)}\,\xi_{\tau(x_n)}\,(1-\xi_{\tau(x_1)})}\\
&= -\log\prod_{j=2}^{n} q\bigl((x_{j-1},\tau(x_{j-1})),(x_j,\tau(x_j))\bigr)\\
&\qquad -\log L_{\tau(x_1)}\bigl(o_{\tau(x_1)},x_1|\xi_{\tau(x_1)}\bigr)+\log\frac{\xi_{\tau(x_1)}\,\alpha_{\tau(x_n)}\,(1-\xi_{\tau(x_n)})}{\alpha_{\tau(x_1)}\,\xi_{\tau(x_n)}\,(1-\xi_{\tau(x_1)})}. \qquad (3.7)
\end{aligned}$$

We now replace x by $X_{\mathbf{e}_k}$ in the last equation: since $l(X_n)/n$ tends to h almost surely, the subsequence $\bigl(l(X_{\mathbf{e}_k})/\mathbf{e}_k\bigr)_{k\in\mathbb{N}}$ converges also to h. Since $\min_{i\in\mathcal{I}}\xi_i>0$ and $\max_{i\in\mathcal{I}}\xi_i<1$, we get

$$\frac{1}{\mathbf{e}_k}\log\frac{\xi_{\tau(x_1)}\,\alpha_{\tau(x_k)}\,(1-\xi_{\tau(x_k)})}{\alpha_{\tau(x_1)}\,\xi_{\tau(x_k)}\,(1-\xi_{\tau(x_1)})}\xrightarrow{k\to\infty}0 \quad \text{almost surely},$$

where $x_1:=X_{\mathbf{e}_1}$ and $x_k:=\widetilde{\mathbf{W}}_k=\widetilde{X}_{\mathbf{e}_k}$. By positive recurrence of $\bigl(\widetilde{\mathbf{W}}_k,\tau_k\bigr)_{k\in\mathbb{N}}$, an application of the ergodic theorem yields

$$\begin{aligned}
&-\frac{1}{k}\log\prod_{j=2}^{k} q\bigl((\widetilde{\mathbf{W}}_{j-1},\tau_{j-1}),(\widetilde{\mathbf{W}}_j,\tau_j)\bigr)\\
\xrightarrow{n\to\infty}\ & h' := -\sum_{\substack{g,h\in V_*^\times;\\ \tau(g)\neq\tau(h)}} \pi\bigl(g,\tau(g)\bigr)\,q\bigl((g,\tau(g)),(h,\tau(h)\bigr)\log q\bigl((g,\tau(g)),(h,\tau(h)\bigr)>0 \text{ a.s.},
\end{aligned}$$

whenever $h'<\infty$. Obviously, for every $x_1\in V_*^\times$

$$\lim_{k\to\infty}-\frac{1}{\mathbf{e}_k}\log L_{\tau(x_1)}\bigl(o_{\tau(x_1)},x_1|\xi_{\tau(x_1)}\bigr)=0 \quad \text{almost surely}.$$

Since $\lim_{k\to\infty} k/\mathbf{e}_k = \ell_0$ we get

$$h = \lim_{k\to\infty} \frac{l\bigl(X_{\mathbf{e}_k}\bigr)}{\mathbf{e}_k} = \lim_{k\to\infty} \frac{l\bigl(X_{\mathbf{e}_k}\bigr)}{k}\frac{k}{\mathbf{e}_k} = h'\cdot \ell_0,$$

whenever $h'<\infty$. In particular, $h>0$ since $\ell_0>0$ by [11, Section 4].

It remains to show that it cannot be that $h'=\infty$. For this purpose, assume now $h'=\infty$. Define for $N\in\mathbb{N}$ the function $h_N : \bigl(V_*^\times\bigr)^2 \to \mathbb{R}$ by

$$h_N(g,h) := N \wedge \bigl(-\log q\bigl((g,\tau(g)),(h,\tau(h))\bigr)\bigr).$$

Then

$$\begin{aligned} & -\frac{1}{k}\sum_{j=2}^{k} \log h_N\bigl(\widetilde{X}_{\mathbf{e}_{j-1}}, \widetilde{X}_{\mathbf{e}_j}\bigr) \\ \xrightarrow{k\to\infty}\ & h'_N := -\sum_{\substack{g,h\in V_*^\times,\\ \tau(g)\neq\tau(h)}} \pi\bigl(g,\tau(g)\bigr)\, q\bigl((g,\tau(g)),(h,\tau(h))\bigr) \log h_N(g,h) \quad \text{almost surely.}\end{aligned}$$

Observe that $h'_N \to \infty$ as $N\to\infty$. Since $h_N(g,h) \leq -\log q\bigl((g,\tau(g)),(h,\tau(h))\bigr)$ and $h'=\infty$ by assumption there is for every $M\in\mathbb{R}$ and almost every trajectory of $\bigl(\widetilde{\mathbf{W}}_k\bigr)_{k\in\mathbb{N}}$ an almost surely finite random time $\mathbf{T_q}\in\mathbb{N}$ such that for all $k\geq \mathbf{T_q}$

$$-\frac{1}{k}\sum_{j=2}^{k} \log q\bigl((\widetilde{\mathbf{W}}_{j-1},\tau_{j-1}),(\widetilde{\mathbf{W}}_j,\tau_j)\bigr) > M. \tag{3.8}$$

On the other hand side there is for every $M>0$, every small $\varepsilon>0$ and almost every trajectory an almost surely finite random time $\mathbf{T_L}$ such that for all $k\geq\mathbf{T_L}$

$$\begin{aligned} & -\frac{1}{\mathbf{e}_k}\sum_{j=1}^{k} \log L_{\tau(X_{\mathbf{e}_j})}\bigl(o_{\tau(X_{\mathbf{e}_j})}, \widetilde{X}_{\mathbf{e}_j} | \xi_{\tau(X_{\mathbf{e}_j})}\bigr) \in (h-\varepsilon, h+\varepsilon) \quad \text{and} \\ & -\frac{1}{\mathbf{e}_k}\sum_{j=2}^{k} \log q\bigl((\widetilde{X}_{\mathbf{e}_{j-1}},\tau_{j-1}),(\widetilde{X}_{\mathbf{e}_j},\tau_j)\bigr) \\ = \ & -\frac{k}{\mathbf{e}_k}\frac{1}{k}\sum_{j=2}^{k} \log q\bigl((\widetilde{X}_{\mathbf{e}_{j-1}},\tau_{j-1}),(\widetilde{X}_{\mathbf{e}_j},\tau_j)\bigr) > \ell_0\cdot M.\end{aligned}$$

Furthermore, since $\min_{i\in\mathcal{I}} \xi_i > 0$ and $\max_{i\in\mathcal{I}}\xi_i<1$ there is an almost surely finite random time $\mathbf{T}_\varepsilon \geq \mathbf{T_L}$ such that for all $k\geq\mathbf{T}_\varepsilon$ and all $x_1 = X_{\mathbf{e}_1}$ and $x_k = \widetilde{X}_{\mathbf{e}_k}$

$$\begin{aligned} & -\frac{1}{\mathbf{e}_k}\log\frac{\xi_{\tau(x_1)}\,\alpha_{\tau(x_k)}\,\bigl(1-\xi_{\tau(x_k)}\bigr)}{\alpha_{\tau(x_1)}\,\xi_{\tau(x_k)}\,\bigl(1-\xi_{\tau(x_1)}\bigr)} \in (-\varepsilon,\varepsilon) \quad \text{and} \\ & \frac{1}{\mathbf{e}_k}\log L_{\tau(x_1)}\bigl(o_{\tau(x_1)}, x_1|\xi_{\tau(x_1)}\bigr) \in (-\varepsilon,\varepsilon).\end{aligned}$$

Choose now $M > (h+3\varepsilon)/\ell_0$. Then we get the desired contradiction, when we substitute in equality (3.7) the vertex x by $X_{\mathbf{e}_k}$ with $k\geq\mathbf{T}_\varepsilon$, divide by $\mathbf{e}_k$ on both sides and see that the left side is in $(h-\varepsilon, h+\varepsilon)$ and the rightmost side is bigger than $h+\varepsilon$. This finishes the proof of Theorem 3.8. $\square$

Corollary 3.9. *Assume $R>1$. Then we have for almost every path of the random walk $(X_n)_{n\in\mathbb{N}_0}$*
$$h = \liminf_{n\to\infty} -\frac{\log \pi_n(X_n)}{n}.$$

Proof. Recall Inequality (3.5). Integrating both sides of this inequality yields together with the inequality chain (3.6) that
$$\int \liminf_{n\to\infty} -\frac{\log \pi_n(X_n)}{n} - h \, d\mathbb{P} = 0,$$
providing that $h = \liminf_{n\to\infty} -\frac{1}{n}\log \pi_n(X_n)$ for almost every realisation of the random walk. $\square$

The following lemma gives some properties concerning general measure theory:

Lemma 3.10. *Let $(Z_n)_{n\in\mathbb{N}_0}$ be a sequence of non-negative random variables and $0<c\in\mathbb{R}$. Suppose that $\liminf_{n\to\infty} Z_n \geq c$ almost surely and $\lim_{n\to\infty}\mathbb{E}[Z_n]=c$. Then the following holds:*

(1) *$Z_n \xrightarrow{\mathbb{P}} c$, that is, Z_n converges in probability to c.*
(2) *If Z_n is uniformly bounded then $Z_n \xrightarrow{L_1} c$, that is, $\int \bigl|Z_n - c\bigr| d\mathbb{P} \to 0$ as $n\to\infty$.*

Proof. First, we prove convergence in probability of $(Z_n)_{n\in\mathbb{N}_0}$. For every $\delta_1>0$, there is some index N_{δ_1} such that for all $n\geq N_{\delta_1}$
$$\int Z_n \, d\mathbb{P} \in (c-\delta_1, c+\delta_1).$$
Furthermore, due to the above made assumptions on $(Z_n)_{n\in\mathbb{N}_0}$ there is for every $\delta_2>0$ some index N_{δ_2} such that for all $n\geq N_{\delta_2}$
$$\mathbb{P}[Z_n > c-\delta_1] > 1-\delta_2. \tag{3.9}$$
Since $c=\lim_{n\to\infty}\int Z_n \, d\mathbb{P}$ it must be that for every arbitrary but fixed $\varepsilon>0$, every $\delta_1<\varepsilon$ and for all n big enough
$$\mathbb{P}\bigl[Z_n > c-\delta_1\bigr]\cdot(c-\delta_1) + \mathbb{P}\bigl[Z_n > c+\varepsilon\bigr]\cdot(\varepsilon+\delta_1) \leq \int Z_n \, d\mathbb{P} \leq c+\delta_1,$$
or equivalently,
$$\mathbb{P}\bigl[Z_n > c+\varepsilon\bigr] \leq \frac{c+\delta_1 - \mathbb{P}\bigl[Z_n > c-\delta_1\bigr]\cdot(c-\delta_1)}{\varepsilon+\delta_1}.$$
Letting $\delta_2\to 0$ we get
$$\limsup_{n\to\infty} \mathbb{P}\bigl[Z_n > c+\varepsilon\bigr] \leq \frac{2\delta_1}{\varepsilon+\delta_1}.$$
Since we can choose δ_1 arbitrarily small we get
$$\mathbb{P}\bigl[Z_n > c+\varepsilon\bigr] \xrightarrow{n\to\infty} 0 \quad \text{for all } \varepsilon>0.$$

This yields convergence in probability of Z_n to c.

In order to prove the second part of the lemma we define for any small $\varepsilon > 0$ and $n \in \mathbb{N}$ the events

$$A_{n,\varepsilon} := \bigl[|Z_n - c| \leq \varepsilon\bigr] \text{ and } B_{n,\varepsilon} := \bigl[|Z_n - c| > \varepsilon\bigr].$$

For arbitrary but fixed $\varepsilon > 0$, convergence in probability of Z_n to c gives an integer $N_\varepsilon \in \mathbb{N}$ such that $\mathbb{P}[B_{n,\varepsilon}] < \varepsilon$ for all $n \geq N_\varepsilon$. Since $0 \leq Z_n \leq M$ is assumed to be uniformly bounded, we get for $n \geq N_\varepsilon$:

$$\int |Z_n - c|\, d\mathbb{P} = \int_{A_{n,\varepsilon}} |Z_n - c|\, d\mathbb{P} + \int_{B_{n,\varepsilon}} |Z_n - c|\, d\mathbb{P} \leq \varepsilon + \varepsilon\,(M + c) \xrightarrow{\varepsilon\to 0} 0.$$

Thus, we have proved the second part of the lemma. □

We can apply the last lemma immediately to our setting:

Corollary 3.11. *Assume $R > 1$. Then we have the following types of convergence:*

(1) *Convergence in probability:*

$$-\frac{1}{n}\log \pi_n(X_n) \xrightarrow{\mathbb{P}} h.$$

(2) *Assume that there is $c_0 > 0$ such that $p(x,y) \geq c_0$ whenever $p(x,y) > 0$. Then:*

$$-\frac{1}{n}\log \pi_n(X_n) \xrightarrow{L_1} h.$$

Proof. Setting $Z_n = -\frac{1}{n}\log \pi_n(X_n)$ and applying Lemma 3.10 yields the claim. Note that the assumption $p(x,y) \geq c_0$ yields $0 \leq \frac{-\log \pi_n(X_n)}{n} \leq -\log c_0$. □

The assumption of the second part of the last corollary is obviously satisfied if we consider free products of *finite* graphs.

The reasoning in our proofs for existence of the entropy and its different properties (in particular, the reasoning in Section 3.2) is very similar to the argumentation in [12]. However, the structure of free products of graphs is more complicated than in the case of directed covers as considered in [12]. We outline the main differences to the reasoning in the aforementionend article. First, in [12] a very similar rate of escape (compare [12, Theorem 3.8] with Proposition 3.2) is considered, which arises from a length function induced by last visit generating functions. While the proof of existence of the rate of escape in [12] is easy to check, we have to make more effort in the case of free products, since $-\log L_i(o_i, x|1)$ is not necessarily bounded for $x \in V_i$. Additionally, one has to study the various ingridients of the proof more carefully, since non-trivial loops are possible in our setting in contrast to random walks on trees. Secondly, in [12] the invariant measure $\pi\bigl(g,\tau(g)\bigr)$ of our proof collapses to $\nu\bigl(\tau(g)\bigr)$, that is, in [12] one has to study the sequence $\bigl(\tau(\mathbf{W}_k)\bigr)_{k\in\mathbb{N}}$, while in our setting we have to study the more complex sequence $\bigl(\widetilde{\mathbf{W}}_k, \tau(\mathbf{W}_k)\bigr)_{k\in\mathbb{N}}$; compare [12, proof of Theorem 3.8] with Lemma 3.1 and Proposition 3.2.

4. A Formula via Double Generating Functions

In this section we derive another formula for the asymptotic entropy. The main tool is the following theorem of Sawyer and Steger [25, Theorem 2.2]:

Theorem 4.1 (Sawyer and Steger)**.** *Let $(Y_n)_{n\in\mathbb{N}_0}$ be a sequence of real-valued random variables such that, for some $\delta>0$,*

$$\mathbb{E}\biggl(\sum_{n\geq 0}\exp(-rY_n-sn)\biggr)=\frac{C(r,s)}{g(r,s)}\qquad \textit{for } 0<r,s<\delta,$$

where $C(r,s)$ and $g(r,s)$ are analytic for $|r|,|s|<\delta$ and $C(0,0)\neq 0$. Denote by g_r' and g_s' the partial derivatives of g with respect to r and s. Then

$$\frac{Y_n}{n}\xrightarrow{n\to\infty}\frac{g_r'(0,0)}{g_s'(0,0)}\qquad \textit{almost surely.}$$

Setting $z=e^{-s}$ and $Y_n:=-\log L(o,X_n|1)$ the expectation in Theorem 4.1 becomes

$$\mathcal{E}(r,z)=\sum_{x\in V}\sum_{n\geq 0}p^{(n)}(o,x)L(o,x|1)^r z^n=\sum_{x\in V}G(o,x|z)L(o,x|1)^r.$$

We define for $i\in\mathcal{I}$, $r,z\in\mathbb{C}$:

$$\begin{aligned}
\mathcal{L}(r,z) &:= 1+\sum_{n\geq 1}\sum_{x_1\dots x_n\in V\setminus\{o\}}\prod_{j=1}^n L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}(z)\bigr)\cdot L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}\bigr)^r,\\
\mathcal{L}_i^+(r,z) &:= \sum_{x\in V_i^\times}L_i\bigl(o_i,x|\xi_i(z)\bigr)L_i(o_i,x|\xi_i)^r.
\end{aligned}$$

Finally, $\mathcal{L}_i(r,z)$ is defined by

$$\mathcal{L}_i^+(r,z)\cdot\Biggl(1+\sum_{n\geq 2}\sum_{\substack{x_2\dots x_n\in V^\times\setminus\{o\},\\ \tau(x_2)\neq i}}\prod_{j=2}^n L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}(z)\bigr)\cdot L_{\tau(x_j)}\bigl(o_{\tau(x_j)},x_j|\xi_{\tau(x_j)}\bigr)^r\Biggr).$$

With these definitions we have $\mathcal{L}(r,z)=1+\sum_{i\in\mathcal{I}}\mathcal{L}_i(r,z)$ and $\mathcal{E}(r,z)=G(o,o|z)\cdot\mathcal{L}(r,z)$. Simple computations analogously to [11, Lemma 4.2, Corollary 4.3] yield

$$\mathcal{E}(r,z)=\frac{G(o,o|z)}{1-\mathcal{L}^*(r,z)},\ \text{ where }\mathcal{L}^*(r,z)=\sum_{i\in\mathcal{I}}\frac{\mathcal{L}_i^+(r,z)}{1+\mathcal{L}_i^+(r,z)}.$$

We now define $C(r,z) := G(o,o|z)$ and $g(r,z) := 1-\mathcal{L}^*(r,z)$ and apply Theorem 4.1 by differentiating $g(r,z)$ and evaluating the derivatives at $(0,1)$:

$$\begin{aligned}
\frac{\partial g(r,z)}{\partial r}\Big|_{r=0,z=1} &= -\sum_{i\in\mathcal{I}} \frac{\sum_{x\in V_i^\times} L_i(o_i,x|\xi_i)\cdot \log L_i(o_i,x|\xi_i)}{\bigl(1+\sum_{x\in V_i^\times} L_i(o_i,x|\xi_i)\bigr)^2}\\
&= -\sum_{i\in\mathcal{I}} G_i(o_i,o_i|\xi_i)\cdot(1-\xi_i)^2\cdot \sum_{x\in V_i^\times} G_i(o_i,x|\xi_i)\log L_i(o_i,x|\xi_i)\\
&= -\sum_{i\in\mathcal{I}} G_i(o_i,o_i|\xi_i)\cdot(1-\xi_i)^2\cdot\\
&\qquad \cdot\biggl(\sum_{x\in V_i} G_i(o_i,x|\xi_i)\log G_i(o_i,x|\xi_i) - \frac{\log G_i(o_i,o_i|\xi_i)}{1-\xi_i}\biggr),\\
\frac{\partial g(r,z)}{\partial s}\Big|_{r=0,s=0} &= \sum_{i\in\mathcal{I}} \frac{\partial}{\partial z}\Bigl(1-\bigl(1-\xi_i(z)\bigr)G_i\bigl(o_i,o_i|\xi_i(z)\bigr)\Bigr)\Big|_{z=1}\\
&= \sum_{i\in\mathcal{I}} \xi_i'(1)\cdot\bigl(G_i(o_i,o_i|\xi_i)-(1-\xi_i)G_i'(o_i,o_i|\xi_i)\bigr).
\end{aligned}$$

Corollary 4.2. *Assume $R>1$. Then the entropy can be rewritten as*

$$h = \frac{\frac{\partial g(r,z)}{\partial r}(0,1)}{\frac{\partial g(r,z)}{\partial s}(0,1)}.$$

□

5. Entropy of Random Walks on Free Products of Groups

In this section let each V_i be a finitely generated group Γ_i with identity $e_i=o_i$. W.l.o.g. we assume that the V_i's are pairwise disjoint. The free product is again a group with concatenation (followed by iterated cancellations and contractions) as group operation. We write $\Gamma_i^\times := \Gamma_i\setminus\{e_i\}$. Suppose we are given a probability measure μ_i on $\Gamma_i\setminus\{e_i\}$ for every $i\in\mathcal{I}$ governing a random walk on Γ_i, that is, $p_i(x,y)=\mu_i(x^{-1}y)$ for all $x,y\in\Gamma_i$. Let $(\alpha_i)_{i\in\mathcal{I}}$ be a family of strictly positive real numbers with $\sum_{i\in\mathcal{I}}\alpha_i=1$. Then the random walk on the free product $\Gamma:=\Gamma_1*\dots*\Gamma_r$ is defined by the transition probabilities $p(x,y)=\mu(x^{-1}y)$, where

$$\mu(w)=\begin{cases}\alpha_i\mu_i(w), & \text{if } w\in\Gamma_i^\times,\\ 0, & \text{otherwise.}\end{cases}$$

Analogously, $\mu^{(n)}$ denotes the n-th convolution power of μ. The random walk on Γ starting at the identity e of Γ is again denoted by the sequence $(X_n)_{n\in\mathbb{N}_0}$. In particular, the radius of convergence of the associated Green function is strictly bigger than 1; see [29, Theorem 10.10, Corollary 12.5]. In the case of free products of groups it is well-known that the entropy exists and can be written as

$$h=\lim_{n\to\infty}\frac{-\log\pi_n(X_n)}{n}=\lim_{n\to\infty}\frac{-\log F(e,X_n|1)}{n};$$

see Derriennic [7], Kaimanovich and Vershik [14] and Blachère, Haïssinsky and Mathieu [3]. For free products of *finite* groups, Mairesse and Mathéus [21] give an explicit formula for h, which remains also valid for free products of countable groups, but in the latter case one needs the solution of an infinite system of polynomial equations. In the following we will derive another formula for the entropy, which holds also for free products of *infinite* groups.

We set $l(g_1 \dots g_n) := -\log F(e, g_1 \dots g_n|1)$. Observe that transitivity yields $F(g, gh|1) = F(e, h|1)$. Thus,

$$l(g_1 \dots g_n) = -\log \prod_{j=1}^{n} F(g_1 \dots g_{j-1}, g_1 \dots g_j|1) = -\sum_{j=1}^{n} \log F(e, g_j|1).$$

First, we rewrite the following expectations as

$$\begin{aligned}
\mathbb{E}l(X_n) &= \sum_{i\in\mathcal{I}} \sum_{g\in\Gamma_i} \alpha_i\mu_i(g) \sum_{h\in\Gamma} l(h)\, \mu^{(n)}(h),\\
\mathbb{E}l(X_{n+1}) &= \sum_{i\in\mathcal{I}} \sum_{g\in\Gamma_i} \alpha_i\mu_i(g) \sum_{h\in\Gamma} l(gh)\, \mu^{(n)}(h).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}l(X_{n+1}) - \mathbb{E}l(X_n) &= \sum_{i\in\mathcal{I}} \sum_{g\in\Gamma_i} \alpha_i\mu_i(g) \int \big(l(gh) - l(h)\big)\, d\mu^{(n)}(h)\\
&= \sum_{i\in\mathcal{I}} \sum_{g\in\Gamma_i} \alpha_i\mu_i(g) \int -\log \frac{F(e, gX_n|1)}{F(e, X_n|1)} d\mu^{(n)}. \qquad (5.1)
\end{aligned}$$

Recall that $\|X_n\| \to \infty$ almost surely. That is, X_n converges almost surely to a random infinite word X_∞ of the form $x_1x_2 \ldots \in \left(\bigcup_{i=1}^r \Gamma_i^\times\right)^{\mathbb{N}}$, where two consecutive letters are not from the same $\Gamma_i^\times$. Denote by $X_\infty^{(1)}$ the first letter of X_∞. Let be $g \in \Gamma_i^\times$. For $n \geq \mathbf{e}_1$, the integrand in (5.1) is constant: if $\tau\big(X_\infty^{(1)}\big) \neq i$ then

$$\log \frac{F(e, gX_n|1)}{F(e, X_n|1)} = \log F(e, g),$$

and if $\tau\big(X_\infty^{(1)}\big) = i$ then

$$\log \frac{F(e, gX_n|1)}{F(e, X_n|1)} = \log \frac{F\big(e, gX_\infty^{(1)}|1\big)}{F\big(e, X_\infty^{(1)}|1\big)}.$$

By [11, Section 5], for $i \in \mathcal{I}$ and $g \in \Gamma_i^\times$,

$$\begin{aligned}
\varrho(i) &:= \mathbb{P}[X_\infty^{(1)} \in \Gamma_i] = 1 - (1-\xi_i)\, G_i(o_i, o_i|\xi_i) \quad \text{and}\\
\mathbb{P}[X_\infty^{(1)} = g] &= F(o_i, g|\xi_i)\,(1-\xi_i)\, G_i(o_i, o_i|\xi_i) = (1-\xi_i)\, G_i(o_i, g|\xi_i).
\end{aligned}$$

Recall that $F(e, g|1) = F_i(o_i, g|\xi_i)$ for each $g \in \Gamma_i$. We get:

Theorem 5.1. *Whenever $h_i := -\sum_{g\in\Gamma_i} \mu_i(g)\log\mu_i(g) < \infty$ for all $i\in\mathcal{I}$, that is, when all random walks on the factors Γ_i have finite single-step entropy, then the asymptotic entropy h of the random walk on Γ is given by*

$$h = -\sum_{i\in\mathcal{I}}\sum_{g\in\Gamma_i} \alpha_i\mu_i(g)\Big[\big(1-\varrho(i)\big)\,\log F_i(o_i,g|\xi_i) + (1-\xi_i)\,G_i(o_i,o_i|\xi_i)\,\mathcal{F}(g)\Big],$$

where

$$\mathcal{F}(g) := \sum_{g'\in\Gamma_i^\times} F_i(o_i,g'|\xi_i)\log\frac{F_i(o_i,gg'|\xi_i)}{F_i(o_i,g'|\xi_i)} \quad \textit{for } g\in\Gamma_i. \tag{5.2}$$

Proof. Consider the sequence $\mathbb{E}l(X_{n+1})-\mathbb{E}l(X_n)$. If this sequence converges, its limit must equal h. By the above made considerations we get

$$\begin{aligned}&\mathbb{E}l(X_{n+1})-\mathbb{E}l(X_n)\\ \xrightarrow{n\to\infty}\ &-\sum_{i\in\mathcal{I}}\sum_{g\in\Gamma_i}\mu(g)\left[(1-\varrho(i))\log F_i(o_i,g|\xi_i)+\sum_{g'\in\Gamma_i^\times}\mathbb{P}[X_\infty^{(1)}=g']\log\frac{F_i(o_i,gg'|\xi_i)}{F_i(o_i,g'|\xi_i)}\right],\end{aligned}$$

if the sum on the right hand side is finite. We have now established the proposed formula, but it remains to verify finiteness of the sum above. This follows from the following observations:

<u>Claim A:</u> $-\sum_{i\in\mathcal{I}}\sum_{g\in\Gamma_i}\alpha_i\mu_i(g)(1-\varrho(i))\log F_i(o_i,g|\xi_i)$ is finite.

Observe that $F_i(o_i,g|\xi_i)\geq\mu_i(g)\xi_i$ for $g\in\mathrm{supp}(\mu_i)$. Thus,

$$0 < -\sum_{g\in\Gamma_i}\mu_i(g)\log F_i(o_i,g|\xi_i) \leq -\sum_{g\in\Gamma_i}\mu_i(g)\log\big(\mu_i(g)\,\xi_i\big) = h_i - \log\xi_i.$$

This proves Claim A.

<u>Claim B:</u> $\sum_{i\in\mathcal{I}}\sum_{g\in\Gamma_i}\alpha_i\mu_i(g)(1-\xi_i)\sum_{g'\in\Gamma_i^\times}G_i(o_i,g'|\xi_i)\left|\log\frac{F_i(o_i,gg'|\xi_i)}{F_i(o_i,g'|\xi_i)}\right|$ is finite.

Observe that $F_i(o_i,gg'|\xi_i)/F_i(o_i,g'|\xi_i) = G_i(o_i,gg'|\xi_i)/G_i(o_i,g'|\xi_i)$. Obviously,

$$\mu_i^{(n)}(g')\,\xi_i^n \leq G_i(o_i,g'|\xi_i) \leq \frac{1}{1-\xi_i} \qquad \text{for every } n\in\mathbb{N} \text{ and } g'\in\Gamma_i.$$

For $g\in\Gamma$ set $N(g):=\bigl\{n\in\mathbb{N}_0 \,\big|\, \mu^{(n)}(g)>0\bigr\}$. Then:

$$\begin{aligned}
0 &< \sum_{g'\in\Gamma_i^\times} \mathbb{P}[X_\infty^{(1)}=g']\cdot \bigl|\log G_i(o_i,g'|\xi_i)\bigr|\\
&= \sum_{g'\in\Gamma_i^\times} (1-\xi_i)\cdot G_i(o_i,g'|\xi_i)\cdot \bigl|\log G_i(o_i,g'|\xi_i)\bigr|\\
&= \sum_{g'\in\Gamma_i^\times} (1-\xi_i)\cdot \sum_{n\in N(g')} \mu_i^{(n)}(g')\cdot \xi_i^n\cdot \bigl|\log G_i(o_i,g'|\xi_i)\bigr|\\
&\leq \sum_{g'\in\Gamma_i^\times} (1-\xi_i)\cdot \sum_{n\in N(g')} \mu_i^{(n)}(g')\cdot \xi_i^n\cdot \max\bigl\{-\log\bigl(\mu_i^{(n)}(g')\,\xi_i^n\bigr),-\log(1-\xi_i)\bigr\}\\
&\leq (1-\xi_i)\cdot \sum_{n\in N(g')} n\,\xi_i^n\cdot \underbrace{\frac{-1}{n}\sum_{g'\in\Gamma_i}\mu_i^{(n)}(g')\log\mu_i^{(n)}(g')}_{(*)} -(1-\xi_i)\,\log\xi_i\sum_{n\geq 1} n\,\xi_i^n\\
&\qquad -(1-\xi_i)\,\log(1-\xi_i)\sum_{n\geq 1}\xi_i^n.
\end{aligned}$$

Recall that $h_i<\infty$ together with Kingman's subadditive ergodic theorem implies existence of a constant $H_i\geq 0$ with

$$\lim_{n\to\infty} -\frac{1}{n}\sum_{g\in\Gamma_i}\mu_i^{(n)}(g)\log\mu_i^{(n)}(g)=H_i. \tag{5.3}$$

Thus, if $n\in\mathbb{N}$ is large enough, the sum $(*)$ is in the interval $(H_i-\varepsilon,H_i+\varepsilon)$ for any arbitrarily small $\varepsilon>0$. That is, the sum $(*)$ is uniformly bounded for all $n\in\mathbb{N}$. From this follows that the rightmost side of the last inequality chain is finite.

Furthermore, we have for each $g\in\Gamma_i$ with $\mu_i^{(n)}(g)>0$:

$$\begin{aligned}
0 &< \sum_{g'\in\Gamma_i^\times} \mathbb{P}[X_\infty^{(1)}=g']\cdot \bigl|\log G_i(o_i,gg'|\xi_i)\bigr|\\
&= \sum_{g'\in\Gamma_i^\times} (1-\xi_i)\cdot G_i(o_i,g'|\xi_i)\cdot \bigl|\log G_i(o_i,gg'|\xi_i)\bigr|\\
&= \sum_{g'\in\Gamma_i^\times} (1-\xi_i)\cdot \sum_{n\in N(g')} \mu_i^{(n)}(g')\cdot \xi_i^n\cdot \bigl|\log G_i(o_i,gg'|\xi_i)\bigr|\\
&\leq \sum_{g'\in\Gamma_i^\times} (1-\xi_i)\cdot \sum_{n\in N(g')} \mu_i^{(n)}(g')\cdot \xi_i^n\cdot \max\bigl\{-\log\bigl(\mu_i(g)\,\mu_i^{(n)}(g')\,\xi_i^{n+1}\bigr),-\log(1-\xi_i)\bigr\}\\
&\leq -(1-\xi_i)\cdot \sum_{n\in N(g')} \xi_i^n\cdot \sum_{g'\in\Gamma_i}\mu_i^{(n)}(g')\log\mu_i^{(n)}(g') -(1-\xi_i)\cdot\log\xi_i\cdot\sum_{n\geq 1}(n+1)\,\xi_i^n\\
&\qquad -\log\mu_i(g)-\log(1-\xi_i).
\end{aligned}$$

If we sum up over all g with $\mu(g) > 0$, we get:

$$\underbrace{-\sum_{i\in\mathcal{I}}\sum_{g\in\Gamma_i}\alpha_i\mu_i(g)(1-\xi_i)\sum_{n\in N(g')}\xi_i^n\sum_{g'\in\Gamma_i}\mu_i^{(n)}(g')\log\mu_i^{(n)}(g')}_{(I)}$$
$$\underbrace{-\sum_{i\in\mathcal{I}}\sum_{g\in\Gamma_i}\alpha_i\mu_i(g)(1-\xi_i)\,\log\xi_i\sum_{n\geq 1}(n+1)\,\xi_i^n}_{(II)}$$
$$\underbrace{-\sum_{i\in\mathcal{I}}\alpha_i\sum_{g\in\Gamma_i}\mu_i(g)\log\mu_i(g)}_{(III)}\underbrace{-\sum_{i\in\mathcal{I}}\alpha_i\log(1-\xi_i)}_{<\infty}.$$

Convergence of (I) follows from (5.3), (II) converges since $\xi_i < 1$ and (III) is convergent by assumption $h_i < \infty$. This finishes the proof of Claim B, and thus the proof of the theorem. □

Erschler and Kaimanovich [9] asked if drift and entropy of random walks on groups depend continuously on the probability measure, which governs the random walk. Ledrappier [19] proves in his recent, simultaneous paper that drift and entropy of finite-range random walks on free groups vary analytically with the probability measure of constant support. By Theorem 5.1, we are even able to show continuity for free products of finitely generated groups, but restricted to nearest neighbour random walks with fixed set of generators.

Corollary 5.2. *Let Γ_i be generated as a semigroup by S_i. Denote by $\mathcal{P}_i$ the set of probability measures μ_i on S_i with $\mu_i(x_i) > 0$ for all $x_i \in S_i$. Furthermore, we write $\mathcal{A} := \{(\alpha_1,\dots,\alpha_r) \mid \alpha_i > 0, \sum_{i\in\mathcal{I}}\alpha_i = 1\}$. Then the entropy function*

$$h : \mathcal{A}\times\mathcal{P}_1\times\cdots\times\mathcal{P}_r \to \mathbb{R} : (\alpha_1,\dots,\alpha_r,\mu_1,\dots,\mu_r) \mapsto h(\alpha_1,\dots,\alpha_r,\mu_1,\dots,\mu_r)$$

is real-analytic.

Proof. The claim follows directly with the formula given in Theorem 5.1: the involved generating functions $F_i(o_i, g|z)$ and $G_i(o_i, o_i|z)$ are analytic when varying the probability measure of constant support, and the values ξ_i can also be rewritten as

$$\xi_i = \sum_{k_1,\dots,k_r,l_{1,1},\dots,l_{r,|S_r|}\geq 1} x(k_1,\dots,k_r,l_{1,1},\dots,l_{r,|S_r|})\prod_{i\in\mathcal{I}}\alpha_i^{k_i}\prod_{j=1}^{|S_i|}\mu_i(x_{i,j})^{l_{i,j}},$$

where $S_i = \{x_{i,1},\dots,x_{i,|S_i|}\}$. This yields the claim. □

Remarks:

(1) Corollary 5.2 holds also for the case of free products of *finite* graphs, if one varies the transition probabilities continously under the assumption that the sets $\{(x_i, y_i) \in V_i \times V_i \mid p_i(x_i, y_i) > 0\}$ remain constant: one has to rewrite ξ_i as power series in terms of (finitely many) $p_i(x_i, y_i)$ and gets analyticity with the formula given in Theorem 3.7.

(2) Analyticity holds also for the drift (w.r.t. the block length and w.r.t. the natural graph metric) of nearest neighbour random walks due to the formulas given in [11, Section 5 and 7].
(3) The formula for entropy and drift given in Mairesse and Mathéus [21] for random walks on free products of *finite* groups depends also analytically on the transition probabilities.

6. Entropy Inequalities

In this section we consider the case of free products of *finite* sets $V_1,\dots,V_r$, where V_i has $|V_i|$ vertices. We want to establish a connection between asymptotic entropy, rate of escape and the volume growth rate of the free product V. For $n\in\mathbb{N}_0$, let $S_0(n)$ be the set of all words of V of block length n. The following lemmas give an answer how fast the free product grows.

Lemma 6.1. *The sphere growth rate w.r.t. the block length is given by*

$$s_0 := \lim_{n\to\infty}\frac{\log|S_0(n)|}{n} = \log\lambda_0,$$

where λ_0 is the Perron-Frobenius eigenvalue of the $r\times r$-matrix $D=(d_{i,j})_{1\leq,i,j\leq r}$ with $d_{i,j}=0$ for $i=j$ and $d_{i,j}=|V_j|-1$ otherwise.

Proof. Denote by $\widehat{D}$ the $r\times r$-diagonal matrix, which has entries $|V_1|-1,|V_2|-1,\dots,|V_r|-1$ on its diagonal. Let $\mathbb{1}$ be the $(r\times 1)$-vector with all entries equal to 1. Thus, we can write

$$|S_0(n)| = \mathbb{1}^T\widehat{D}D^{n-1}\mathbb{1}.$$

Let $0<v_1\leq\mathbb{1}$ and $v_2\geq\mathbb{1}$ be eigenvectors of D w.r.t. the Perron-Frobenius eigenvalue λ_0. Then

$$\begin{aligned}|S_0(n)| &\geq \mathbb{1}^T\widehat{D}D^{n-1}v_1 = C_1\cdot\lambda_0^{n-1},\\ |S_0(n)| &\leq \mathbb{1}^T\widehat{D}D^{n-1}v_2 = C_2\cdot\lambda_0^{n-1},\end{aligned}$$

where C_1,C_2 are some constants independent from n. Thus,

$$\frac{\log|S_0(n)|}{n} = \log|S_0(n)|^{1/n} \xrightarrow{n\to\infty} \log\lambda_0.$$

$\square$

Recall from the Perron-Frobenius theorem that $\lambda_0\geq\sum_{i=1,i\neq j}^r(|\Gamma_i|-1)$ for each $j\in\mathcal{I}$; in particular, $\lambda_0\geq 1$. We also take a look on the natural graph metric and its growth rate. For this purpose, we define

$$S_1(n) := \bigl\{x\in V \,\big|\, p^{(n)}(o,x)>0, \forall m<n: p^{(m)}(o,x)=0\bigr\},$$

that is, the set of all vertices in V which are at distance n to the root o w.r.t. the natural graph metric.

We now construct a new graph, whose adjacency matrix allows us to describe the exponential growth of $S_1(n)$ as $n\to\infty$. For this purpose, we visualize the sets $V_1,\dots,V_r$ as graphs $\mathcal{X}_1,\dots,\mathcal{X}_r$ with vertex sets $V_1,\dots,V_r$ equipped with the following edges: for

$x, y \in V_i$, there is a directed edge from x to y if and only if $p_i(x,y) > 0$. Consider now directed spanning trees $\mathcal{T}_1, \dots, \mathcal{T}_r$ of the graphs $\mathcal{X}_1, \dots, \mathcal{X}_r$ such that the graph distances of vertices in $\mathcal{T}_i$ to the root o_i remain the same as in $\mathcal{X}_i$. We now investigate the free product $\mathcal{T} = \mathcal{T}_1 * \dots * \mathcal{T}_r$, which is again a tree. We make the crucial observation that $\mathcal{T}$ can be seen as the directed cover of a finite directed graph F, where F is defined in the following way:

(1) The vertex set of F is given by $\{o\} \cup \bigcup_{i \in \mathcal{I}} V_i^\times$ with root o.
(2) The edges of F are given as follows: first, we add all edges inherited from one of the trees $\mathcal{T}_1, \dots, \mathcal{T}_r$, where o plays the role of o_i for each $i \in \mathcal{I}$. Secondly, we add for all $i \in \mathcal{I}$ and every $x \in V_i^\times$ an edge from x to each $y \in V_j^\times$, $j \neq i$, whenever there is an edge from o_i to y in $\mathcal{T}_j$.

The tree $\mathcal{T}$ can be seen as a *periodic tree*, which is also called a *tree with finitely many cone types*; for more details we refer to Lyons [20] and Nagnibeda and Woess [23]. Now we are able to state the following lemma:

Lemma 6.2. *The sphere growth rate w.r.t. the natural graph metric defined by*

$$s_1 := \lim_{n \to \infty} \frac{\log |S_1(n)|}{n}$$

exists. Moreover, we have the equation $s_1 = \log \lambda_1$, where λ_1 is the Perron-Frobenius eigenvalue of the adjacency matrix of the graph F.

Proof. Since the graph metric remains invariant under the restriction of V to $\mathcal{T}$ and since it is well-known that the growth rate exists for periodic trees (see Lyons [20, Chapter 3.3]), we have existence of the limit s_1. More precisely, $|S_1(n)|^{1/n}$ tends to the Perron-Frobenius eigenvalue of the adjacency matrix of F as $n \to \infty$. For sake of completeness, we remark that the root of $\mathcal{T}$ plays a special role (as a cone type) but this does not affect the application of the results about directed covers to our case. $\square$

For $i \in \{0,1\}$, we write $B_i(n) = \bigcup_{k=0}^n S_i(k)$. Now we can prove:

Lemma 6.3. *The volume growth w.r.t. the block length, w.r.t. the natural graph metric respectively, is given by*

$$g_0 := \lim_{n \to \infty} \frac{\log |B_0(n)|}{n} = \log \lambda_0, \quad g_1 := \lim_{n \to \infty} \frac{\log |B_1(n)|}{n} = \log \lambda_1 \quad \textit{respectively.}$$

Proof. For ease of better readability, we omit the subindex $i \in \{0,1\}$ in the following, since the proofs for g_0 and g_1 are completely analogous. Choose any small $\varepsilon > 0$. Then there is some K_ε such that for all $k \geq K_\varepsilon$

$$\lambda^k e^{-k\varepsilon} \leq |S(k)| \leq \lambda^k e^{k\varepsilon}.$$

Write $C_\varepsilon = \sum_{i=0}^{K_\varepsilon - 1} |S(i)|$. Then for $n \geq K_\varepsilon$:

$$\begin{aligned} |B(n)|^{1/n} &= \sqrt[n]{\sum_{k=0}^{n} |S(k)|} \leq \sqrt[n]{C_\varepsilon + \sum_{k=K_\varepsilon}^{n} \lambda^k e^{k\varepsilon}} = \lambda e^\varepsilon \sqrt[n]{\frac{C_\varepsilon}{\lambda^n e^{n\varepsilon}} + \sum_{k=K_\varepsilon}^{n} \frac{1}{\lambda^{n-k} e^{(n-k)\varepsilon}}} \\ &\leq \lambda e^\varepsilon \sqrt[n]{\frac{C_\varepsilon}{\lambda^n e^{n\varepsilon}} + (n - K_\varepsilon + 1)} \xrightarrow{n\to\infty} \lambda e^\varepsilon. \end{aligned}$$

In the last inequality we used the fact $\lambda \geq 1$. Since we can choose $\varepsilon > 0$ arbitrarily small, we get $\limsup_{n\to\infty} |B(n)|^{1/n} \leq \lambda$. Analogously:

$$|B(n)|^{1/n} \geq \sqrt[n]{C_\varepsilon + \sum_{k=K_\varepsilon}^{n} \lambda^k e^{-k\varepsilon}} = \lambda \sqrt[n]{\frac{C_\varepsilon}{\lambda^n} + \sum_{k=K_\varepsilon}^{n} \frac{e^{-k\varepsilon}}{\lambda^{n-k}}} \xrightarrow{n\to\infty} \lambda e^{-\varepsilon}.$$

That is, $\lim_{n\to\infty} \frac{1}{n} \log |B(n)| = \log \lambda$. □

For $i \in \{0,1\}$, define $l_i : V \to \mathbb{N}_0$ by $l_0(x) = \|x\|$ and $l_1(x) = \inf\{m \in \mathbb{N}_0 \mid p^{(m)}(o,x) > 0\}$. Then the limits $\ell_i = \lim_{n\to\infty} l_i(X_n)/n$ exist; see [11, Theorem 3.3, Section 7.II]. Now we can establish a connection between entropy, rate of escape and volume growth:

Corollary 6.4. *$h \leq g_0 \cdot \ell_0$ and $h \leq g_1 \cdot \ell_1$.*

Proof. Let be $i \in \{0,1\}$ and $\varepsilon > 0$. Then there is some $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$

$$1-\varepsilon \leq \mathbb{P}\bigl(\bigl\{x \in V \mid -\log \pi_n(x) \geq (h-\varepsilon)n, l_i(x) \leq (\ell_i+\varepsilon)n\bigr\}\bigr) \leq e^{-(h-\varepsilon)n} \cdot \bigl|B_i\bigl((\ell_i+\varepsilon)n\bigr)\bigr|.$$

That is,

$$(h-\varepsilon) + \frac{\log(1-\varepsilon)}{n} \leq (\ell_i + \varepsilon) \cdot \frac{\log\bigl|B_i\bigl((\ell_i+\varepsilon)n\bigr)\bigr|}{(\ell_i+\varepsilon)n}.$$

If we let n tend to infinity and make ε arbitrarily small, we get the claim. □

Finally, we remark that an analogous inequality for random walks on groups was given by Guivarc'h [13], and more generally for space- and time-homogeneous Markov chains by Kaimanovich and Woess [15, Theorem 5.3].

7. Examples

7.1. **Free Product of Finite Graphs.** Consider the graphs $\mathcal{X}_1$ and $\mathcal{X}_2$ with the transition probabilities sketched in Figure 7.1. We set $\alpha_1 = \alpha_2 = 1/2$. For the computation of ℓ_0 we need the following functions:

$$F_1(g_1, o_1|z) = \frac{z^2}{2}\frac{1}{1-z^2/2}, \qquad F_2(h_1, o_2|z) = \frac{z^2}{2}\frac{1}{1-z^3/2},$$

$$\xi_1(z) = \frac{z/2}{1-\frac{z}{2}\frac{\xi_2(z)^2}{2}\frac{1}{1-\xi_2(z)^3/2}}, \qquad \xi_2(z) = \frac{z/2}{1-\frac{z}{2}\frac{\xi_1(z)^2}{2}\frac{1}{1-\xi_1(z)^2/2}}.$$

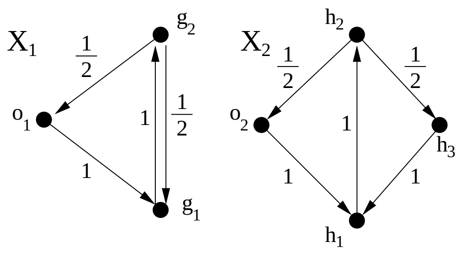

FIGURE 1. Finite graphs $\mathcal{X}_1$ and $\mathcal{X}_2$

Simple computations with the help of [11, Section 3] and MATHEMATICA allow us to determine the rate of escape of the random walk on $\mathcal{X}_1 * \mathcal{X}_2$ as $\ell_0 = 0.41563$. For the computation of the entropy, we need also the following generating functions:

$$L_1(o_1, g_1|z) = \frac{z}{1-z^2/2}, \quad L_1(o_1, g_2|z) = \frac{z^2}{1-z^2/2}, \quad L_2(o_2, h_1|z) = \frac{z}{1-z^3/2},$$
$$L_2(o_2, h_2|z) = \frac{z^2}{1-z^3/2}, \quad L_2(o_2, h_3|z) = \frac{z^3/2}{1-z^3/2}.$$

Thus, we get the asymptotic entropy as $h = 0.32005$.

7.2. $(\mathbb{Z}\times\mathbb{Z}/2) * (\mathbb{Z}\times\mathbb{Z}/2)$**.** Consider the free product $\Gamma = \Gamma_1 * \Gamma_2$ of the *infinite* groups $\Gamma_i = \mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})$ with $\alpha_i = 1/2$ and $\mu_i\bigl((\pm 1, 0)\bigr) = \mu_i\bigl((0,1)\bigr) = 1/3$ for each $i \in \{1,2\}$. We set $a := (1,0)$, $b := (1,1)$, $c := (0,1)$ and $\lambda(x,y) := x$ for $(x,y)\in\Gamma_i$. Define

$$\begin{aligned}
\hat{F}(a|z) &:= \sum_{n\geq 1} \mathbb{P}\bigl[Y_n = a, \forall m < n : \lambda(Y_m) < 1 \bigm| Y_0 = (0,0)\bigr]\, z^n,\\
\hat{F}(b|z) &:= \sum_{n\geq 1} \mathbb{P}\bigl[Y_n = b, \forall m < n : \lambda(Y_m) < 1 \bigm| Y_0 = (0,0)\bigr]\, z^n,
\end{aligned}$$

where $(Y_n)_{n\in\mathbb{N}_0}$ is a random walk on $\mathbb{Z}\times\mathbb{Z}/2$ governed by μ_1. The above functions satisfy the following system of equations:

$$\begin{aligned}
\hat{F}(a|z) &= \frac{z}{3}\Bigl(1 + \hat{F}(b|z) + \hat{F}(a|z)^2 + \hat{F}(b|z)^2\Bigr),\\
\hat{F}(b|z) &= \frac{z}{3}\Bigl(\hat{F}(a|z) + \hat{F}(a|z)\hat{F}(b|z) + \hat{F}(b|z)\hat{F}(a|z)\Bigr).
\end{aligned}$$

From this system we obtain explicit formulas for $\hat{F}(a|z)$ and $\hat{F}(b|z)$. We write $F\bigl(n, j|z\bigr) := F_1\bigl((0,0),(n,j)|z\bigr)$ for $(n,j)\in\mathbb{Z}\times\mathbb{Z}/2$. To compute the entropy rate we have to solve the following system of equations:

$$\begin{aligned}
F(a|z) &= \frac{z}{3}\Bigl(1 + F(b|z) + \hat{F}(a|z)F(a|z) + \hat{F}(b|z)F(b|z)\Bigr),\\
F(b|z) &= \frac{z}{3}\Bigl(F(c|z) + F(a|z) + \hat{F}(a|z)F(b|z) + \hat{F}(b|z)F(a|z)\Bigr),\\
F(c|z) &= \frac{z}{3}\Bigl(1 + 2\,F(b|z)\Bigr).
\end{aligned}$$

Moreover, we need the value $\xi_1(1)=\xi_2(1)=\xi$. This value can be computed analogously to [11, Section 6.2], that is, ξ has to be computed numerically from the equation

$$\frac{\xi}{2-2\xi}=\xi\,G_1(\xi)=\frac{\xi}{1-\frac{2}{3}\xi F(a|\xi)-\frac{1}{3}\xi F(c|\xi)}.$$

Solving this equation with MATHEMATICA yields $\xi=0.55973$. To compute the entropy we have to evaluate the functions $F(g|z)$ at $z=\xi$ for each $g\in\mathbb{Z}\times\mathbb{Z}_2$. For even $n\in\mathbb{N}$, we have the following formulas:

$$\begin{aligned}
F\big((\pm n,0)|\xi\big) &= \sum_{k=0}^{n/2}\binom{n}{2k}\hat{F}(b|\xi)^{2k}\hat{F}(a|\xi)^{n-2k}+\\
&\quad \sum_{k=0}^{n/2-1}\binom{n}{2k+1}\hat{F}(b|\xi)^{2k+1}\hat{F}(a|\xi)^{n-2k-1}F(c|\xi),\\
F\big((\pm n,1)|\xi\big) &= \sum_{k=0}^{n/2-1}\binom{n}{2k+1}\hat{F}(b|\xi)^{2k+1}\hat{F}(a|\xi)^{n-2k-1}+\\
&\quad \sum_{k=0}^{n/2}\binom{n}{2k}\hat{F}(b|\xi)^{2k}\hat{F}(a|\xi)^{n-2k}F(c|\xi).
\end{aligned}$$

For odd $n\in\mathbb{N}$,

$$\begin{aligned}
F\big((\pm n,0)|\xi\big) &= \sum_{k=0}^{(n-1)/2}\binom{n}{2k}\hat{F}(b|\xi)^{2k}\hat{F}(a|\xi)^{n-2k}+\\
&\quad \sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}\hat{F}(b|\xi)^{2k+1}\hat{F}(a|\xi)^{n-2k-1}F(c|\xi),\\
F\big((\pm n,1)|\xi\big) &= \sum_{k=0}^{(n-1)/2}\binom{n}{2k+1}\hat{F}(b|\xi)^{2k+1}\hat{F}(a|\xi)^{n-2k-1}+\\
&\quad \sum_{k=0}^{(n-1)/2}\binom{n}{2k}\hat{F}(b|\xi)^{2k}\hat{F}(a|\xi)^{n-2k}F(c|\xi).
\end{aligned}$$

Moreover, we define $\hat{F}:=\mathbb{P}[\exists n\in\mathbb{N}:\lambda(X_n)=1]$. This probability can be computed by conditioning on the first step and solving

$$\hat{F}=\frac{\xi}{3}\big(1+\hat{F}+\hat{F}^2\big),$$

that is, $\hat{F}=0.24291$. Observe that we get the following estimations:

$$\begin{aligned}
F_1(o,g|\xi) &\leq \hat{F}^{|\lambda(g)|} && \text{for } g\in\mathbb{Z}\times\mathbb{Z}_2,\\
F_1(o,g|\xi) &\geq \hat{F}^{|\lambda(g)|-1}\cdot\min\big\{F_1(o_1,a|\xi),F_1(o_1,b|\xi)\big\} && \text{for } g\in\big(\mathbb{Z}\times\mathbb{Z}_2\big)\setminus\{(0,0),c\}.
\end{aligned}$$

These bounds allow us to cap the sum over all $g' \in \Gamma_i^\times$ in (5.2) and to estimate the tails of these sums. Thus, we can compute the entropy rate numerically as $h = 1.14985$.

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