

REGULARITY OF THE DRIFT AND ENTROPY OF RANDOM WALKS ON GROUPS

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Random walks on a group G model many natural phenomena. A random walk is defined by a probability measure p on G . We are interested in global asymptotic properties of the random walks and in particular in the linear drift and the asymptotic entropy. If the geometry of the group is rich, then these numbers are both positive and the way of dependence on p is some global property of G . In this note, we review recent results about the regularity of the drift and the entropy in some examples.

1. ENTROPY AND LINEAR DRIFT

We recall in this section the main notations for the objects under consideration associated to a group G and a probability measure p on G . Background on random walks can be found in the survey papers [KV] and [V] and in the book by W. Woess ([W]).

Let G be a finitely generated group and S a symmetric finite generator. For $g \in G$, let $|g|$ denote the smallest $n \in \mathbb{N}$ such that g can be written as $g = s_1 \cdots s_n$, where $s_1, \dots, s_n \in S$. We denote by $d(g, h) := |g^{-1}h|$ the left invariant associated metric. Let p be a probability measure on G with support B . Unless otherwise specified, we always assume that B is finite and that $\bigcup_{n \in \mathbb{N}} B^n = G$. We denote by $\mathcal{P}(B)$ the set of probability measures with support B . The set $\mathcal{P}(B)$ is naturally identified with an open subset of the probabilities on B , which is a contractible open polygonal bounded convex domain in $\mathbb{R}^{|B|-1}$. We form, with $p^{(0)}$ being the Dirac measure at the identity e ,

$$p^{(n)}(g) = [p^{(n-1)} \star p](g) = \sum_{h \in G} p^{(n-1)}(gh^{-1})p(h),$$

where $g \in G$. The spectral radius is given by $\varrho(p) = \limsup_{n \rightarrow \infty} p^{(n)}(e)^{1/n}$. Define the entropy $H_{n,p}$ and the drift $L_{n,p}$ of $p^{(n)}$ by:

$$H_{n,p} := - \sum_{g \in G} p^{(n)}(g) \ln p^{(n)}(g), \quad L_{n,p} := \sum_{g \in G} |g| p^{(n)}(g),$$

and the average entropy h_p and the linear drift ℓ_p by

$$h_p := \lim_{n \rightarrow \infty} \frac{1}{n} H_{n,p}, \quad \ell_p := \lim_{n \rightarrow \infty} \frac{1}{n} L_{n,p}.$$

Both limits exist by subadditivity and Fekete's Lemma. The linear drift makes sense as soon as $\sum_{g \in G} |g| p(g) < +\infty$, the entropy under the slightly weaker condition $H_{1,p} < +\infty$. The entropy h_p was introduced by Avez ([Av]) and is related to bounded solutions of the equation on G of the form $f(g) = \sum_{h \in G} f(gh)p(h)$ (see

e.g. [KV]). In particular, $h_p = 0$ if and only if the only bounded solutions are the constant functions ([KV], [De2]). The general relation is ([Gu])

$$(1) \quad h_p \leq \ell_p v,$$

where $v := \lim_{n \rightarrow \infty} \frac{1}{n} \ln(\#\{g \in G; |g| \leq n\})$ is the *volume entropy* of G . In particular, if $\ell_p = 0$ then $h_p = 0$.

We say that p is symmetric if $B = B^{-1}$ and $p(g) = p(g^{-1})$ for all $g \in B$. We call p centered if $\sum_{g \in B} \chi(g)p(g) = 0$ for all group morphisms $\chi : G \rightarrow \mathbb{R}$. Clearly, symmetric probabilities are centered. If p is centered and $h_p = 0$, then $\ell_p = 0$ ([Va], [Ma1]). If p is not centered, we may have $h_p = 0$ and $\ell_p \neq 0$, for instance on \mathbb{Z} . If this is the case, there is a group morphism $\chi : G \rightarrow \mathbb{R}$ such that $\ell_p = \sum_{g \in B} \chi(g)p(g)$ ([KL], see also [EKn] for finite versions of this result).

Both h_p and ℓ_p describe asymptotic properties of the *random walk* directed by p . Let $(\Omega, P) = (G^{\mathbb{N}}, p^{\otimes \mathbb{N}})$ be the infinite product space such that $\omega = (\omega_1, \omega_2, \dots) \in G^{\mathbb{N}}$ is realized by a sequence of i.i.d. random variables with values in G and distribution p . We form the right random walk by $X_n(\omega) := \omega_1 \omega_2 \cdots \omega_n$. The probability $p^{(n)}$ is the distribution of X_n , and an application of Kingman's subadditive ergodic theorem ([Ki]) gives that, for P -a.e. ω ,

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} |X_n| = \ell_p \quad \text{and} \quad \lim_{n \rightarrow \infty} -\frac{1}{n} \ln(p^{(n)}(X_n)) = h_p.$$

The random walk is said to be recurrent if, for P -a.e. ω , there is a positive $n \in \mathbb{N}$ with $X_n(\omega) = e$. In this case there is an infinite number of integers n with $X_n = e$ and, by (2), $\ell_p = 0$. Hence, $h_p = 0$. From here on, we assume that the random walk is transient, i.e. $|X_n| \rightarrow \infty$ for P -a.e. ω . The *Green function* $G(g, h)$, $g, h \in G$, is defined by

$$G(g, h) := \sum_{n \geq 0} p^{(n)}(g^{-1}h).$$

By decomposing of the first visit to h and using transitivity of the random walk we get

$$G(g, h) = F(g, h)G(h, h) = F(g, h)G(e, e),$$

where $F(g, h)$ is the probability of reaching h starting from g . If p is symmetric, then the (left invariant) Green distance is defined by $d_G(g, h) := -\ln F(g, h)$. The drift $\ell_{p, G}$ for that distance coincides with the entropy h_p ([BP], Proposition 6.2, [BHM]) and the volume entropy is 1, so that there is equality in (1) for that distance ([BHM]).

We now turn to another representation of the drift and entropy. Let X be a compact space. X is called a G -space if the group G acts by continuous transformations on X . This action extends naturally to probability measures on X . We say that the measure ν on X is stationary if $\sum_{g \in G} (g_* \nu)p(g) = \nu$. The *entropy* of a stationary measure ν is defined by

$$(3) \quad h_p(X, \nu) := - \sum_{g \in G} \left(\int_X \ln \frac{dg_*^{-1} \nu}{d\nu}(\xi) d\nu(\xi) \right) p(g).$$

The entropy h_p and the linear drift ℓ_p are given by variational formulas over stationary measures (see [KV] for the entropy, [KL] for the linear drift):

$$(4) \quad h_p = \max\{h_p(X, \nu); X \text{ } G\text{-space and } \nu \text{ stationary on } X\},$$

$$(5) \quad \ell_p = \max\left\{\sum_{g \in G} \left(\int_{\overline{G}} \xi(g^{-1}) d\nu(\xi)\right) p(g); \nu \text{ stationary on } \overline{G}\right\},$$

where \overline{G} is the Busemann compactification of G , the elements of which are horo-functions ξ on G . A pair (X, ν) , where X is a G -space and ν a p -stationary measure is called a *boundary* if, for P -a.e. ω , $(X_n(\omega))_* \nu$ converge towards a Dirac measure. It is called a *Poisson boundary* if it is a boundary and it realizes the maximum in formula (4).

From the definition of ℓ_p and h_p , one sees that the mappings $p \mapsto \ell_p$ and $p \mapsto h_p$ are uppersemicontinuous on $\mathcal{P}(B)$. Erschler ([Er]) raised the question of continuity of these functions and gave examples where these mappings are not continuous on the closure of $\mathcal{P}(B)$. The question of continuity in general on the interior of $\mathcal{P}(B)$ is open. In the rest of the paper, we discuss several examples where one can prove stronger regularity results.

2. NEAREST NEIGHBOUR RANDOM WALKS ON A FREE GROUP

In the case when the group G is a *free group* with d generators, $d \geq 2$, and p is supported by these generators, explicit computations can be made (see [DM]).

Let G be the free group with set of generators $S = \{\pm i; i = 1, \dots, d\}$, where $-i = i^{-1}$ for $i \in S$. Let $\mathcal{P}(S)$ be the set of probability measures on G with support S . Since $d \geq 2$, as n goes to infinity, the reduced word representing $X_n(\omega)$ converges towards an infinite reduced word $X_\infty(\omega) = s_1(\omega)s_2(\omega) \cdots$ with $s_j(\omega) \neq -s_{j+1}(\omega)$. Denote by G_∞ the space of infinite reduced words. The stationary measure is unique: it is the distribution ν of $X_\infty(\omega)$. Then (G_∞, ν) is both the Poisson boundary and the Busemann boundary of G . Let $q_i = P(\{\omega; s_1(\omega) = i\}) = \nu([i])$, where $[i]$ consists of all infinite words in G_∞ starting with letter $i \in S$. We have $\sum_{i \in S} q_i = 1$. Let $i_1 \dots i_k$ be a reduced word in G . Then ν is uniquely determined by the values $\nu([i_1 \dots i_k]) = F(e, i_1 \dots i_k)(1 - q_{-i_k})$, where $F(e, i_1 \dots i_k)$ is the probability of hitting $i_1 \dots i_k$ when starting at the identity e . Formula (5) writes:

$$\ell_p = 1 - 2 \sum_{i \in S} p_i q_{-i}.$$

In order to write the formula for the entropy, we introduce $z_i := F(e, i)$ for $i \in S$. The density $\frac{dg_*^{-1}\nu}{d\nu}(\xi)$ gives the minimal positive harmonic function with pole at $\xi = i_1 i_2 \dots \in G_\infty$. The Green function satisfies the following multiplicative structure:

$$G(e, i_1 \dots i_k) = F(e, i_1)G(i_1, i_1 \dots i_k) = F(e, i_1)G(e, i_2 \dots i_k).$$

This yields together with [L1, Theorem 2.10]

$$\frac{di_*^{-1}\nu}{d\nu}(\xi) = \lim_{k \rightarrow \infty} \frac{G(-i, i_1 \dots i_k)}{G(e, i_1 \dots i_k)} = \begin{cases} z_i, & \text{if } i_1 \neq -i, \\ z_{-i}^{-1}, & \text{if } i_1 = -i. \end{cases}$$

Formula (4) writes:

$$h_p = \sum_{i \in S} p_i [q_{-i} \ln z_{-i} - (1 - q_{-i}) \ln z_i].$$

We can express the q_i in terms of the p_i , and vice versa, thanks to the *traffic equations*: using the Markov property, we can write:

$$z_i = p_i + z_i \sum_{j \in S \setminus \{i\}} p_j z_{-j} \quad \text{and} \quad q_i = z_i(1 - q_{-i}).$$

Setting $Y := \sum_{j \in S} p_j z_{-j}$, we get

$$p_i = \frac{z_i(1 - Y)}{1 - z_i z_{-i}} \quad \text{and} \quad z_i = \frac{q_i}{1 - q_{-i}},$$

so that we find:

$$\ell_p = 1 - \frac{2}{A} \sum_{i \in S} \frac{q_i q_{-i} (1 - q_i)}{1 - q_i - q_{-i}}, \quad \text{where} \quad A = (1 - Y)^{-1} = 1 + \sum_{i \in S} \frac{q_i q_{-i}}{1 - q_i - q_{-i}},$$

which writes:

$$\ell_p = \frac{B}{A}, \quad \text{where} \quad B := 1 - \sum_{i \in S} \frac{q_i q_{-i} (1 - 2q_i)}{1 - q_i - q_{-i}}.$$

Hence, in terms of the $q_i, i \in S, p_i$ and ℓ_p are rational, and the expression of h_p involves rational functions and $\ln q_i, \ln(1 - q_i)$.

Proposition 2.1. *The mappings $p \mapsto \ell_p$ and $p \mapsto h_p$ are real analytic on $\mathcal{P}(S)$.*

Proof. Since all formulas are explicit in terms of the q_i , we only have to check that the q_i are real analytic functions on $\mathcal{P}(S)$. First, we can write z_i as a power series in terms of the p_i 's and the additional variable $z \in \mathbb{C}$, namely as

$$z_i(z) = \sum_{(n_1, \dots, n_{2d}) \in \mathbb{N}^{2d}} c(n_1, \dots, n_{2d}) p_1^{n_1} p_{-1}^{n_2} p_2^{n_3} p_{-2}^{n_4} \cdots p_d^{n_{2d-1}} p_{-d}^{n_{2d}} z^{n_1 + \dots + n_{2d}},$$

where $c(n_1, \dots, n_{2d}) \geq 0$. Since the spectral radius is strictly smaller than 1 (see e.g. [W, Cor. 12.5]), the power series $G(e, i|z) = \sum_{n \geq 0} p^{(n)}(i) z^n$ has radius of convergence strictly bigger than 1 and satisfies $G(e, i|z) \geq z_i(z)$ for all real $z > 0$. That is, for each $p \in \mathcal{P}(S)$, $z_i(z)$ has radius of convergence $R_i > 1$. Choose now any $\delta > 0$ with $1 + \delta < R_i$. Then

$$\begin{aligned} z_i &= z_i(1) \leq z_i(1 + \delta) \\ &= \sum_{(n_1, \dots, n_{2d}) \in \mathbb{N}^{2d}} c(n_1, \dots, n_{2d}) ((1 + \delta) p_1)^{n_1} \cdots ((1 + \delta) p_{-d})^{n_{2d}} < \infty. \end{aligned}$$

In other words, $z_i = z_i(1)$ is real analytic in a neighbourhood of any $p \in \mathcal{P}(S)$. The equations $q_i = z_i(1 - q_{-i}), q_{-i} = z_{-i}(1 - q_i)$ give

$$q_i = \frac{z_i(1 - z_{-i})}{1 - z_i z_{-i}},$$

and this finishes the proof. \square

Observe that, for $d = 1$, the group G is \mathbb{Z} , $S = \{\pm 1\}$ and $p \mapsto \ell_p = |p_1 - p_{-1}|$ is not a real analytic function on $\mathcal{P}(\pm 1)$.

The formulas are even simpler when the probability p is symmetric. Let $\mathcal{P}_\sigma(S)$ be the set of symmetric probability measures on S ; elements of $\mathcal{P}_\sigma(S)$ are described by d positive numbers $\{p_1, \dots, p_d\}$ such that $\sum_{i=1}^d p_i = 1/2$. If $p \in \mathcal{P}_\sigma(S)$, $q_i = q_{-i}$ and we have:

$$\begin{aligned} \ell_p &= \frac{B}{A} \text{ with } A = 1 + 2 \sum_{i=1}^d \frac{q_i^2}{1 - 2q_i} \text{ and } B = 1 - 2 \sum_{i=1}^d q_i^2, \\ h_p &= -\frac{2}{A} \sum_{i=1}^d q_i(1 - q_i) \ln \frac{q_i}{1 - q_i}, \text{ whereas} \\ p_i &= \frac{q_i(1 - q_i)}{A(1 - 2q_i)} \text{ for } i = 1, \dots, d. \end{aligned}$$

Proposition 2.2. *The functions $p \mapsto \ell_p$ and $p \mapsto h_p$ reach their maxima on $\mathcal{P}_\sigma(S)$ at the constant vector $p_0 = (1/2d, \dots, 1/2d)$ and*

$$\ell_{p_0} = 1 - \frac{1}{d}, \quad h_{p_0} = \left(1 - \frac{1}{d}\right) \ln(2d - 1).$$

Proof. By symmetry, the constant vector p_0 is a critical point for ℓ_p . At p_0 , $q_i = 1/2d$ by symmetry and $\ell_{p_0} = 1 - 1/d$, $h_{p_0} = (1 - 1/d) \ln(2d - 1)$ by the formulas above (observe that these expressions are also valid for $d = 1$: the only point of $\mathcal{P}_\sigma(\pm 1)$ is $(1/2, 1/2)$, for which $\ell = 0 = 1 - 1/d$ and $h = 0 = (1 - 1/d) \ln(2d - 1)$). Moreover, the volume entropy of G is $\ln(2d - 1)$. By (1), the result for ℓ_p implies that $(1 - 1/d) \ln(2d - 1)$ is the maximal value that the entropy might take on $\mathcal{P}_\sigma(S)$. Since this is the entropy h_{p_0} , p_0 achieves the maximum of the entropy as well.

We are going to prove that the function $(q_1, \dots, q_d) \mapsto B/A$ has a unique critical point on the set $\{(q_1, \dots, q_d); q_j > 0, \sum_{j=1}^d q_j = 1/2\}$. Observe that the formula for ℓ_p is continuous on the domain $0 \leq q_i \leq 1/2$ and that the value of ℓ_p at the boundary of the domain $\{(q_1, \dots, q_d); q_j > 0, \sum_{j=1}^d q_j = 1/2\}$ is the one computed with only the non-zero q_i 's on a free group with a smaller set of generators. Since at the constant vector p_0 , $\ell_{p_0} = 1 - 1/d$, it follows, by induction on the dimension, that the critical point p_0 is a maximum. The proof for $d = 2$ is the same as in the general case: there is only one critical point by the argument below and the limit of the expression for ℓ_p at $(0, 1/2), (1/2, 0)$ is 0.

Using a Lagrange multiplier, we are looking for the critical points of the function $F(q, \lambda) = \ell_p - \lambda(\sum_{j=1}^d q_j - 1/2)$ satisfying $0 \leq q_j \leq 1/2$ for $j = 1, \dots, d$. Setting as above

$$A = 1 + 2 \sum_{i=1}^d \frac{q_i^2}{1 - 2q_i} \quad \text{and} \quad B = 1 - 2 \sum_{i=1}^d q_i^2,$$

all equations $\frac{\partial F}{\partial q_i} = 0$ depend only on A, B, λ and q_i .

Indeed, they write $G(A, B, \lambda, q_i) = 0$, where:

$$G(A, B, \lambda, q) = 16Aq^3 + 4q^2(\lambda A^2 - 4A - B) + 4q(-\lambda A^2 + A + B) + \lambda A^2.$$

If, for fixed A, B, λ , the equation $G(A, B, \lambda, q_i) = 0$ has only one solution $q \in [0, 1/2)$, then, for these values of A, B, λ , the only possible critical point of F is

$q_j = 1/2d$ for all j . Then, unless $A = \frac{2d-1}{2d-2}$, $B = \frac{2d-1}{2d}$, there is no critical point for F with those values of A, B .

To summarize, we only have to verify that the equation $G(A, B, \lambda, q_i) = 0$ has at most one solution $q \in [0, 1/2)$ for all A, B, λ with $0 < B < 1 < A$.

The function $q \mapsto G(A, B, \lambda, q)$ is a third degree polynomial with positive highest coefficient, $1/2$ is a critical point and $G(A, B, \lambda, 1/2) = B > 0$. Therefore, there is at most one solution $q \in [0, 1/2)$. \square

It is likely that p_0 gives also the maximum of the entropy on the whole $\mathcal{P}(S)$, but we do not have a proof of that fact. We also conjecture that the mapping $p \mapsto \ell_p$ is a concave function; calculating the drift for small $d \in \mathbb{N}$ supports and confirms this conjecture, but we do not have a proof for general d .

3. FREE PRODUCTS, ARTIN DIHEDRAL GROUPS AND BRAID GROUPS

The computations in Section 2 have been known for fifty years (even if Proposition 2.2 seems to be formally new). There are very few other examples where it is possible to describe geometrically the Poisson boundary and the Busemann boundary, and it is even rarer to be able to give useful formulas for the stationary measure. In this section, we review the examples we are aware of.

One important concept of constructing new groups from given ones is the free product of groups. The crucial point is that free products have a tree-like structure. More precisely, suppose we are given finitely generated groups G_1, \dots, G_r equipped with finitely supported probability measures p_1, \dots, p_r . The identity of G_i is denoted by e_i , and w.l.o.g. we assume that these groups are pairwise disjoint and we exclude the case $r = 2 = |G_1| = |G_2|$ (this case leads to recurrent random walks in our setting). The free product $G_1 * \dots * G_r$ is given by

$$G = *_{i=1}^r G_i = \{x_1 x_2 \dots x_n \mid x_j \in \bigcup_{i=1}^r G_i \setminus \{e_i\}, x_j \in G_k \Rightarrow x_{j+1} \notin G_k\} \cup \{e\},$$

the set of finite words over the alphabet $\bigcup_{i=1}^r G_i \setminus \{e_i\}$ such that two consecutive letters do *not* come from the same group G_k , where e describes the empty word. A group operation on G is given by concatenation of words with possible contractions and cancellations in the middle such that one gets a reduced word as above. For $x = x_1 \dots x_n \in G$, define the *block length* of x as $\|x\| := n$.

A random walk on G is constructed in a natural way as follows: we lift p_i to a probability measure \bar{p}_i on G : if $x = x_1 \dots x_n \in G$ with $x_n \notin G_i$ and $v, w \in G_i$, then $\bar{p}_i(xv, xw) := p_i(v, w)$. Otherwise we set $\bar{p}_i(x, y) := 0$. Choose $0 < \alpha_1, \dots, \alpha_r \in \mathbb{R}$ with $\sum_{i=1}^r \alpha_i = 1$. Then we obtain a new probability measure on G defined by

$$p = \sum_{i=1}^r \alpha_i \bar{p}_i$$

with $B = \text{supp}(p) = \bigcup_{i=1}^r \text{supp}(p_i)$. We consider random walks $(X_n)_{n \in \mathbb{N}_0}$ on G starting at e , which are governed by p . For $i \in \{1, \dots, r\}$, denote by ξ_i the probability of hitting the set $G_i \setminus \{e_i\}$ when starting at e . The spectral radius $\rho(p)$ is strictly less than 1 due to the non-amenability of G . Let ∂G_i be the Martin boundary of G_i with respect to p_i , and denote by G_∞ the set of infinite words $x_1 x_2 \dots$

such that $x_i \in G_k$ implies $x_{i+1} \notin G_k$. Then the Martin boundary of G is given by

$$\partial G = G_\infty \cup \bigcup_{i=1}^r \{x\xi; x = x_1 \dots x_n \in G, x_n \notin G_i, \xi \in \partial G_i\};$$

see e.g. [W, Proposition 26.21]. The random walk on G converges almost surely to an infinite word in G_∞ , and the limit distribution ν is determined by

$$\nu(\{x_1 x_2 \dots \in G_\infty; x_1 = y_1, \dots, x_n = y_n\}) = F(e, y_1 \dots y_n)(1 - (1 - \xi_i)G_i(\xi_i)),$$

where $n \in \mathbb{N}$, $y_1 \dots y_n \in G$ with $y_n \notin G_i$, $F(e, y_1 \dots y_n)$ being the probability of hitting $y_1 \dots y_n$ and $G_i(z) = \sum_{n \geq 0} p_i^{(n)}(e_i)z^n$ with $z \in \mathbb{C}$; see e.g. [Gi1].

The next propositions summarize results about regularity of drift and entropy. Explicit formulas can be found in the cited sources.

Proposition 3.1 ([Gi1]). *The drift w.r.t. the block length $\ell_B = \lim_{n \rightarrow \infty} \frac{1}{n} \|X_n\|$ exists and varies real analytically in $p \in \mathcal{P}(B)$.*

Proof. In [Gi1, Equ. (9)] a formula for ℓ_B is given:

$$\ell_B = \sum_{i=1}^r \alpha_i \frac{1 - \xi_i}{\xi_i} (1 - (1 - \xi_i) G_i(\xi_i)).$$

Let be $d = |B| - 1$, and write $p = (q_1, \dots, q_d) \in \mathcal{P}(B)$. Analogously to the proof of Proposition 2.1 one can write ξ_i as a power series (evaluated at $z = 1$) in the form

$$\xi_i(z) = \sum_{(n_1, \dots, n_d) \in \mathbb{N}^d} c(n_1, \dots, n_d) q_1^{n_1} q_2^{n_2} \dots q_d^{n_d} z^{n_1 + \dots + n_d}, \quad z \in \mathbb{C}.$$

Since $\varrho(p) < 1$ the Green functions $G(g|z) = \sum_{n \geq 0} p^{(n)}(g)z^n$, $g \in G$, have radii of convergence $R = 1/\varrho(p) > 1$ and dominate $\xi_i(z)$ for real $z > 0$. Hence, $\xi_i(z)$ has radius of convergence bigger than 1, which in turn – following the same argumentation as in Proposition 2.1 – yields real analyticity of $\xi_i = \xi_i(1)$ in a neighbourhood of any $p \in \mathcal{P}(B)$. Furthermore, $G_i(z)$ can be expanded in the same form as $\xi_i(z)$ and, for each real positive $z_0 < 1$, the mapping $p \mapsto G_i(z_0)$ is real analytic. Since $\xi_i < 1$ (see e.g. [Gi1, Lemma 2.3]) the mapping $p \mapsto G_i(\xi_i)$ is also real analytic as a composition of real analytic functions. This yields the proposed statement. \square

Proposition 3.2 ([Gi1]). *Let p govern a nearest neighbour random walk on G , that is, the length $|g|$ is computed with respect to the generator B . Then the drift function $p \mapsto \ell_p = \lim_{n \rightarrow \infty} \frac{1}{n} |X_n|$ is real analytic.*

Proof. By the formula for ℓ given in [Gi1, Section 7] we just have to check that the mapping

$$p \mapsto \tilde{G}_j(y, z) := \sum_{m, n \geq 0} \sum_{x \in G_j: |x|=m} p_j^{(n)}(e_j, x) y^n z^m$$

is real analytic for all $y \in (0, 1)$ and $z = 1$. For a moment fix $y < 1$ and choose $\delta > 0$ small enough such that $y(1 + 2\delta)^2 < 1$. Since $p_j^{(n)}(e_j, x) > 0$, $x \in G_j$ with $|x| = m$, implies $n \geq m$, we get

$$\tilde{G}_j(y, (1 + 2\delta)^2) = \sum_{m, n \geq 0} \sum_{x \in G_j: |x|=m} p_j^{(n)}(e_j, x) (y(1 + \delta)^2)^n \leq \frac{1}{1 - y(1 + 2\delta)^2} < \infty.$$

This yields $\frac{\partial}{\partial z} \tilde{G}(y, (1 + \delta)^2) < \infty$. Since each term $p_j^{(n)}(e_j, x)$ can be written as a polynomial

$$\sum_{\substack{(n_1, \dots, n_d) \in \mathbb{N}^d \\ n_1 + \dots + n_d = n}} c(n_1, \dots, n_d) q_1^{n_1} \cdot \dots \cdot q_d^{n_d},$$

where $p = (q_1, \dots, q_d) \in \mathcal{P}(B)$ and $c(n_1, \dots, n_d) \geq 0$, we rewrite $\frac{\partial}{\partial z} \tilde{G}(y, (1 + \delta)^2)$ as

$$\frac{1}{1 + \delta} \sum_{m, n \geq 0} \sum_{x \in G_j: |x|=m} m(p_j^{(n)}(e_j, x)(1 + \delta)^n) (\xi_j(1 + \delta))^n.$$

That is, $\sum_{m, n \geq 0} \sum_{x \in G_j: |x|=m} m p_j^{(n)}(e_j, x) \xi_j^n$ is real analytic in $\mathcal{P}(B)$ as a composition of real analytic functions, and this yields the claim. \square

Let us mention that – in contrast to Proposition 2.2 – simple random walk is not necessarily the fastest random walk. Namely, it can be verified with the help of MATHEMATICA that – with p_i describing the simple random walk on G_i – the simple random walk on $(\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (that is, $\alpha_1 = 2/3, \alpha_2 = 1/3$) is slower than the random walk on this free product with the parameters $\alpha_1 = \alpha_2 = 1/2$.

Furthermore, we have the following regularity result:

Proposition 3.3 ([Gi3]). *Assume that $h_i := -\sum_{g \in G_i} p_i(g) \ln p_i(g) < \infty$ for all $i \in \{1, \dots, r\}$, that is, all random walks on the factors G_i have finite single-step entropy. Then the mapping $p \mapsto h_p$ is real analytic.*

We now turn to another class of groups whose Cayley graphs have a tree-like structure. A group G is called *virtually free* if it has a free subgroup of finite index. At this point we assume that G has a free subgroup with at least $d \geq 2$ generators; otherwise, G is a finite extension of \mathbb{Z} where we either get recurrent random walks or non-regularity points on $\mathbb{P}(B)$. It is well-known that virtually free groups can be constructed from a finite number of finite groups by iterated amalgamation and HNN extensions. Each element of G can be written as $x_1 \dots x_n h$, where $x_i \in \{\pm i; i = 1, \dots, d\}$ and h being one of finitely many representatives for the different cosets. Suppose we are given a weight or length function $l(\pm i) \in \mathbb{R}$ for $i \in \{1, \dots, d\}$. Then a natural length function on G is defined by $l(x_1 \dots x_n h) = \sum_{j=1}^n l(x_j)$. We have the following result:

Proposition 3.4 ([Gi2]). *Let G be a virtually free group. Let p govern a finite range random walk on G . Then the mapping $p \mapsto \lim_{n \rightarrow \infty} l(X_n)/n$ is real analytic.*

Proof. Random walks on virtually free groups can be interpreted as a random walk on a regular language in the sense of [Gi2]. The claim follows from the formula for $\lim_{n \rightarrow \infty} l(X_n)/n$, the drift with respect to the length function l , given in [Gi2, Theorem 2.4]. Due to non-amenability of G we have again $\rho(p) < 1$. The rest follows analogously as in the proofs of Propositions 2.1 and 3.1. \square

For the case l being the natural word length the last proposition is also covered by Corollary 4.2.

At this point we want to mention the article [MM2], which uses similar techniques to establish statements about the drift of random walks on the braid group B_3 and on Artin groups of dihedral type. Traffic equations are established, whose unique solutions lead to formulas for the drift. For random walks on these groups there might occur transitions (when varying the probability measures of constant

support), where one has no regularity. An explicit example for a non-differentiability point is given on the braid group B_3 . However, [MM2] gives explicit formulas for the drift in terms of the solutions of the traffic equations splitted up into different branches. By methods similar to the above, one can show that the drift is real analytic on each branch. Indeed, solutions of the traffic equations can be written as converging power series as in the proofs of Propositions 3.1 and 3.2.

4. HYPERBOLIC GROUPS

A geodesic metric space is called *hyperbolic* if geodesic triangles are thin: there is $\delta \geq 0$ such that each side of a geodesic triangle is contained in a δ -neighbourhood of the union of the other two sides. A finitely generated group is called hyperbolic if the Cayley graph defined by some finite symmetric generator is hyperbolic. This property does not depend on the set of generators. Free groups are hyperbolic, as are fundamental groups of compact manifolds of negative curvature, and small cancellation groups. See e.g. [GH] for the main geometric properties of hyperbolic groups. The geometric boundary of a hyperbolic space is the space of equivalence classes of geodesic rays, where two geodesic rays are equivalent if they are at a bounded Hausdorff distance. The geometric boundary ∂G of the Cayley graph of a hyperbolic group G is a compact G -space. It is endowed with the Gromov metric (see [GH]). The mapping $\Phi : G \rightarrow \mathbb{Z}^G$, $\Phi(g)(h) = |h^{-1}g| - |g|$ is an isometry such that $\Phi(G)$ is relatively compact for the product topology. The Busemann compactification \bar{G} is the closure of $\Phi(G)$ in \mathbb{Z}^G . There is an equivariant homomorphism $\pi : \bar{G} \setminus G \rightarrow \partial G$ (see e.g. [WW]). The homomorphism π is finite-to-one (see e.g. [CP]). Following [Bj], we say that G with the generator S satisfies (BA) if the homomorphism π is one-to-one. In this case, we write, for $\xi \in \partial G, h \in G$, $\xi(h)$ for the value at h of the sequence $\pi^{-1}\xi \in \mathbb{Z}^G$. Free groups and surface groups with their natural generators satisfy (BA). It is an open problem whether any hyperbolic group admits a symmetric generator with the property (BA).

Let p be a probability on G with finite support. Then, there is a unique p -stationary probability measure ν_p on ∂G and $(\partial G, \nu_p)$ is a Poisson boundary for (G, p) ([An], [K] Theorem 7.6). If (BA) is satisfied, the measure ν_p is the unique stationary probability measure on the Busemann compactification and formulas (4) and (5) write:

$$(6) \quad h_p = - \sum_{g \in B} \left(\int_{\partial G} \ln \frac{dg_*^{-1} \nu_p}{d\nu_p}(\xi) d\nu_p(\xi) \right) p(g), \quad \ell_p = \sum_{g \in B} \left(\int_{\partial G} \xi(g^{-1}) d\nu_p(\xi) \right) p(g).$$

Proposition 4.1. *Assume that (G, S) is a non-elementary hyperbolic group and satisfies (BA). Let $p \in \mathcal{P}(B)$, α be small enough, and let f be an α -Hölder continuous function on ∂G . Then the mapping $p \mapsto \int_{\partial G} f(\xi) d\nu_p(\xi)$ is real analytic on a neighbourhood of p in $\mathcal{P}(B)$.*

Proof. Let \mathcal{K}_α be the space of α -Hölder continuous functions on ∂G . The space \mathcal{K}_α is a Banach space with norm $\|f\|_\alpha$, where

$$\|f\|_\alpha = \max_{\xi \in \partial G} |f(\xi)| + \sup_{\xi, \eta \in \partial G: \xi \neq \eta} \frac{|f(\xi) - f(\eta)|}{(d(\xi, \eta))^\alpha}.$$

For $p \in \mathcal{P}(B)$, let \mathcal{Q}_p be the operator on \mathcal{K}_α defined by

$$\mathcal{Q}_p f(\xi) = \sum_{g \in B} f(g^{-1}\xi) p(g).$$

Clearly, the mapping $p \mapsto \mathcal{Q}_p$ is real analytic from $\mathcal{P}(B)$ into $\mathcal{L}(\mathcal{K}_\alpha)$. If G is not elementary and satisfies (BA), it can be shown (see [Bj], Lemma 4) that, for α small enough, $f \mapsto \int f d\nu_p$ is an isolated eigenvector for the transposed operator \mathcal{Q}_p^* on the dual space \mathcal{K}_α^* . The proposition follows by a perturbation lemma. \square

Corollary 4.2. *Assume that (G, S) is a non-elementary hyperbolic group and satisfies (BA). Then the mapping $p \mapsto \ell_p$ is real analytic.*

Indeed, the function $\xi(g^{-1})$ in formula (6) belongs to \mathcal{K}_α for all α . Corollary 4.2 is due to [L2] in the case of the free group. P. Mathieu ([Ma2]) proved the C^1 regularity and gave a formula for $\nabla_p \nu_p$ and $\nabla_p \ell_p$ in the symmetric case, to be compared with formulas for linear response of dynamical systems (cf. [R]).

The formula (6) for the entropy is valid in general, even without the (BA) hypothesis, but observe that the integrand $\varphi_p(g, \xi) := -\ln \frac{dg_*^{-1} \nu_p}{d\nu_p}(\xi)$ is itself a function of p . To study this function, we use the description by A. Ancona ([An]) of the Martin boundary of a random walk with finite support on a hyperbolic group. Recall that $F_p(g, h)$ is the probability of reaching h starting from g in dependence of p .

Proposition 4.3 ([An]). *Assume that G is hyperbolic and that p has finite support. Then*

$$\varphi_p(g, \xi) = \lim_{h \rightarrow \xi} \ln \frac{F_p(e, h)}{F_p(g^{-1}, h)} \quad \text{for all } g \in G, \xi \in \partial G.$$

A consequence of the proof of Proposition 4.3 is that, for all $g \in G$, for α small enough $\varphi_p(g, \xi) \in \mathcal{K}_\alpha$ (see [INO]). In the case of free groups, Proposition 4.3 goes back to Derriennic ([De1]) and using his arguments one can prove:

Proposition 4.4 ([L2]). *If G is a free group and p has finite support B , there is α small enough that, for all $g \in B$, the mapping $p \mapsto \varphi_p$ is real analytic from a neighbourhood of p in $\mathcal{P}(B)$ into \mathcal{K}_α .*

Corollary 4.5 ([L2]). *If G is a free group and p has finite support B , the mapping $p \mapsto h_p$ is real analytic on $\mathcal{P}(B)$.*

For cocompact Fuchsian groups there is the following recent result:

Proposition 4.6 ([HMM]). *Let G be a cocompact Fuchsian group with planar presentation. Then the mapping $p \mapsto \ell_p$ is real analytic.*

For a general hyperbolic group, we have a weaker result:

Proposition 4.7 ([L3]). *If G is a hyperbolic group and p has finite support B , there is α small enough such that, for all $g \in B$, the mapping $p \mapsto \varphi_p$ is Lipschitz continuous from a neighbourhood of p in $\mathcal{P}(B)$ into \mathcal{K}_α .*

Corollary 4.8 ([L3]). *If G is a hyperbolic group and p has finite support B , the mappings $p \mapsto h_p$, $p \mapsto \ell_p$ are Lipschitz continuous on $\mathcal{P}(B)$.*

[Gi4] proves also continuity of the mapping $p \mapsto h_p$ for random walks on regular languages, which adapt, for instance, to the case of virtually free groups.

The best results of regularity to-date are due to P. Mathieu:

Proposition 4.9 ([Ma2]). *If G is a hyperbolic group and B is finite and symmetric, the mapping $p \mapsto h_p$ is C^1 on the set $\mathcal{P}_\sigma(B)$ of symmetric probability measures on B .*

Let $\lambda \mapsto p_\lambda, \lambda \in [-\varepsilon, +\varepsilon]$ be a smooth curve in $\mathcal{P}(B)$. We write:

$$\begin{aligned} & \lim_{\lambda \rightarrow 0} \frac{h(p_\lambda) - h(p_0)}{\lambda} \\ = & \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \sum_{g \in B} \left(\int_{\partial G} \varphi_{p_\lambda}(g, \xi) d\nu_{p_\lambda}(\xi) p_\lambda(g) - \int_{\partial G} \varphi_{p_0}(g, \xi) d\nu_{p_0}(\xi) p_0(g) \right) \\ = & \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \sum_{g \in B} \left(\int_{\partial G} (\varphi_{p_\lambda}(g, \xi) - \varphi_{p_0}(g, \xi)) d\nu_{p_\lambda}(\xi) \right) p_\lambda(g) + \\ & + \sum_{g \in B} \left(\int_{\partial G} \varphi_{p_0}(g, \xi) \left[\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (d\nu_{p_\lambda} - d\nu_{p_0}) \right](\xi) \right) p_\lambda(g) + \\ & + \sum_{g \in B} \left(\int_{\partial G} \varphi_{p_0}(g, \xi) d\nu_{p_0}(\xi) \right) \left[\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (p_\lambda(g) - p_0(g)) \right]. \end{aligned}$$

The third line converges by definition. To prove that the second line converges, P. Mathieu observes that the Green metric $-\ln F_{p_0}(g, h)$ on G satisfies (BA) and a form of hyperbolicity that allows him to extend Proposition 4.1. More precisely, he shows directly the differentiability of $\lambda \mapsto \int f(\xi) d\nu_{p_\lambda}(\xi)$, for $f \in \mathcal{K}_\alpha$, and gives a formula for the derivative. For the first line, P. Mathieu shows a general result for any non-amenable group G and p_λ with finite support. In our case, his result writes:

Proposition 4.10 ([Ma2]). *Let $\lambda \mapsto p_\lambda, \lambda \in [-\varepsilon, +\varepsilon]$ be a smooth curve in $\mathcal{P}(B)$. Then,*

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \sum_{g \in B} \left(\int (\varphi_{p_\lambda}(g, \xi) - \varphi_{p_0}(g, \xi)) d\nu_{p_\lambda}(\xi) \right) p_\lambda(g) = 0.$$

It is likely that the function $p \mapsto h_p$ has more regularity on $\mathcal{P}(B)$, but this is an open problem.

Another natural extension is towards more general families of probability measures on G . Proposition 2.1 is valid for p varying in finite dimensional affine subsets of $\{p; \sum_{g \in G} e^{\gamma|g|} p(g) < +\infty\}$ for some $\gamma > 0$ (see [L1]). The other properties rest on Harnack inequality at infinity (see [An]), which has been proven only for probability measures with finite support on hyperbolic groups. Finally, let $\mathcal{P}^1(G)$ be the set of probabilities on G satisfying $\sum_{g \in G} |g| p(g) < +\infty$ endowed with the topology of convergence on the functions which grow slower than $C|g|$ at infinity. The first observation on this topic of regularity of the entropy is the fact that, if G is hyperbolic, $p \mapsto h_p$ is continuous on $\mathcal{P}^1(G)$ ([EKc]).

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