Asymptotic Word Length of Random Walks on HNN Extensions

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Abstract. In this article we consider transient random walks on HNN extensions of finitely generated groups. We prove that the rate of escape w.r.t. some generalised word length exists. Moreover, a central limit theorem with respect to the generalised word length is derived. Finally, we show that the rate of escape, which can be regarded as a function in the finitely many parameters which describe the random walk, behaves as a real-analytic function in terms of probability measures of constant support.

1. Introduction

Consider a finitely generated group G_0, which contains two isomorphic, finite subgroups A, B with isomorphism $\varphi : A \to B$. Let $S_0 \subseteq G_0$ be a finite set which generates G_0 as a semigroup, and let t be an additional symbol/letter not contained in G_0. The *HNN extension of* G_0 *with respect to* (A, B, φ) is given by the set G of all finite words over the alphabet $G_0 \cup \{t, t^{-1}\}$, where two words $w_1, w_2 \in G$ are identified as the same element of G if one can transform w_1 to w_2 by applying the relations inherited from G_0 or applying one of the following rules:

$$\forall a \in A : at = t\varphi(a) \ \text{ and } \ \forall b \in B : bt^{-1} = t^{-1}\varphi^{-1}(b).$$

A natural group operation on G is given by concatenation of words with possible cancellations of letters in the middle; the empty word e is the group identity. This group construction was introduced by Higman, Neumann and Neumann (see Higman et al. (1949)), whose initials lead to the abbreviation HNN. As we will see later, we can write each $g \in G$ in a unique normal form over some alphabet $\mathcal{A} \subset G_0 \cup \{t, t^{-1}\}$. We denote by $\|g\|$ the word length of $g \in G$ over the alphabet $\mathcal{A}$.

Consider now a group-invariant, transient random walk $(X_n)_{n\in\mathbb{N}_0}$ on G governed by probability measure μ with $\mathrm{supp}(\mu) = S_0 \cup \{t, t^{-1}\}$. One important random walk invariant is the *rate of escape w.r.t. the word length* given by the almost sure constant limit $\mathfrak{l} = \lim_{n\to\infty} \|X_n\|/n$, which exists due to Kingman's

2000 *Mathematics Subject Classification.* 60J10, 20E06.

Key words and phrases. random walk, drift, rate of escape, HNN extension, central limit theorem, analyticity.

subadditive ergodic theorem (see Kingman (1968)). The starting point of this article was the question whether $\mathfrak{l}$ – regarded as a function in the finitely many parameters $\mu(g)$, $g \in S_0 \cup \{t, t^{-1}\}$ – varies real-analytically in terms of probability measures of constant support. We will study this question in a more generalised setting. For this purpose, let the function $\ell : G_0 \cup \{t, t^{-1}\} \to [0, \infty)$ represent a "word length/weight". We can naturally extend ℓ to a length function on G as follows: if $g = g_1 \dots g_n \in G$ has the above mentioned normal form representation over the alphabet $\mathcal{A}$ then we set

$$\ell(g) = \ell(g_1 \dots g_n) := \sum_{i=1}^{n} \ell(g_i),$$

The *asymptotic word length w.r.t. the length function* ℓ is given by

$$\lambda_\ell = \lim_{n\to\infty} \ell(X_n)/n,$$

provided the limit exists. We will also call λ_ℓ the *rate of escape* or *drift w.r.t.* ℓ. For arbitrary length functions ℓ, existence of the rate of escape w.r.t. ℓ is not guaranteed a-priori and can *not* be deduced from Kingman's subadditive ergodic theorem in general; see Remark 2.6. This article addresses to typical, related questions like existence of the rate of escape λ_ℓ (including formulas), a central limit theorem for λ_ℓ and its real-analytic behaviour in terms of probability measures of constant support. In the following let me explain the importance of these questions for random walks on HNN extensions from three different points of view, namely from the view of random walks on regular languages, from the view of group theory and from the view of analyticity of random walk invariants.

Due to the unique representation of each $g \in G$ over the (possibly infinite) alphabet $\mathcal{A}$ we may consider $(X_n)_{n\in\mathbb{N}_0}$ as a random walk on a regular language, where at each instant of time only a bounded number of letters at the end of the current word may be modified, removed or added. This class of random walks have been studied in large variety, but mostly for regular languages over *finite* alphabets. Amongst others, Malyshev (1995, 1996), Gairat et al. (1995), and Lalley (2000) investigated random walks on regular languages over finite alphabets. In particular, Malyshev proved limit theorems concerning existence of the stationary distribution and the rate of escape w.r.t. the word length. Gilch (2008) proved existence of the rate of escape w.r.t. general length functions for random walks on regular languages. All the articles above study regular languages generated by *finite* alphabets. Straight-forward adaptions of the proofs concerning the questions under consideration in the present article are not possible. This article extends results concerning existence of the drift from the finite case to the *infinite* case in the setting of HNN extensions. Studying the rate of escape w.r.t. ℓ deserves its own right, since the transient random walks studied in this article converge almost surely to some infinite random word ω over the alphabet $\mathcal{A}$ in the sense that the length of the common prefix of X_n and ω increases as $n \to \infty$. As an application from information theory one may, e.g., consider X_n as the state of a stack (a last-in first-out queue used in many fundamental algorithms of computer science) at time n, and each stabilised letter at the beginning of X_n produces some final "cost". Hence, the rate of escape w.r.t. ℓ describes the average asymptotic cost.

Let me now outline the importance of the questions under consideration from a group theoretical point of view. The importance of HNN extensions is due to

Stallings' Splitting Theorem (see Stallings (1971)): a finitely generated group Γ has more than one (geometric) end if and only if Γ admits a non-trivial decomposition as a free product by amalgamation or an HNN extension over a finite subgroup. Let me summarize some results about random walks on free products, which are amalgams over the trivial subgroup. For free products of finite groups, Mairesse and Mathéus (2007a) computed an explicit formula for the rate of escape and the asymptotic entropy by solving a finite system of polynomial "traffic equations". In Gilch (2011) different formulas for the rate of escape with respect to the word length of random walks on free products of graphs by three different techniques were computed. The main tool in Gilch (2011) was a heavy use of generating function techniques, which will also play a crucial role in the present article. Asymptotic behaviour of return probabilities of random walks on free products has also been studied in many ways; e.g., see Gerl and Woess (1986), Woess (1986), Sawyer (1978), Cartwright and Soardi (1986), Lalley (1993), and Candellero and Gilch (2009). Random walks on amalgams have been studied in Cartwright and Soardi (1986) and Gilch (2008), where a formula for the rate of escape has been established for amalgams of finite groups. While random walks on free products have been studied in many ways due to their tree-like structure and random walks on amalgams at least to some extent, random walks on HNN extensions, in general, have experienced much less attention. Woess (1989) proved that irreducible random walks with finite range on HNN extensions converge almost surely to infinite words over the alphabet $\mathcal{A}$ and that the set of infinite words together with the hitting distribution form the Poisson boundary. Further valuable contributions have been done by Kaimanovich (1991) and by Cuno and Sava-Huss (2018), who studied the Poisson-Furstenberg boundary of random walks on Baumslag-Solitar groups, which form a special class of HNN extensions. The present article shall encourage further study of random walks on HNN extensions.

Another main goal of this article is to derive a central limit theorem related to the rate of escape λ_ℓ. If $(Z_n)_{n\in\mathbb{N}_0}$ is a random walk on $\mathbb{Z}^d$ satisfying some second moment condition, then the classical central limit theorem states that $(Z_n - n\cdot v)/n^{1/2}$ converges in distribution to $\mathcal{N}(0,\sigma^2)$, where v is the rate of escape w.r.t. the natural distance on the lattice and σ^2 is the asymptotic variance. A natural question going back to Bellman (1954) and Furstenberg and Kesten (1960) is whether this law can be generalized to random walks on finitely generated groups w.r.t. some word metrics. However, a central limit theorem can not be stated in the general setting: Björklund (2010) used results of Dyubina (1999) and Erschler (2001) to construct a counterexample. Nonetheless, in several situations central limit theorems have been established; e.g., Sawyer and Steger (1987), Lalley (1993) and Ledrappier (2001) proved central limit theorems for free groups, Nagnibeda and Woess (2002) for trees with finitely many cone types, and Björklund (2010) for hyperbolic groups with respect to the Greenian metric.

The third main goal of this article will be to show that λ_ℓ varies real-analytically in terms of probability measures of constant support. The question of analyticity goes back to Kaimanovich and Erschler who asked whether drift and entropy of random walks on groups vary continuously (or even analytically) when the support of single step transitions is kept constantly; for counterexamples, see Remark 7.1. This question has been studied in great variety, amongst others, by Ledrappier (2012, 2013), Mathieu (2015) and Gilch (2007, 2011, 2016). Haïssinsky et al. (2018) proved analyticity of the drift for random walks on surface groups and also established a central limit theorem for the word length. The survey article of Gilch and Ledrappier (2013) collects several results on analyticity of drift and entropy of random walks on groups. Last but not least, the excellent work of Gouëzel (2017) shows that the rate of escape w.r.t. some word distance, the asymptotic variance and the asymptotic entropy vary real-analytically for random walks on hyperbolic groups. However, HNN extensions do not necessarily have to be hyperbolic, which makes it interesting to study the question of analyticity of the rate of escape w.r.t. the word length for random walks on HNN extensions.

Finally, let me mention that another random walk's speed invariant is given by the rate of escape w.r.t. the natural graph metric of the underlying Cayley graph of G w.r.t. the generating set $S_0 \cup \{t, t^{-1}\}$, which exists due to Kingman's subadditive ergodic theorem; see Kingman (1968), Derriennic (1980) and Guivarc'h (1980). We remark that, in general, the rate of escape w.r.t. the natural graph metric can *not* necessarily be described via a length function using *stabilising* normal forms of elements of G; this is due to the quite subtle behaviour of shortest paths in the Cayley graph, which needs a different approach and goes beyond of the scope of this article; for a discussion on these problems, see Remark 5.9.

The plan of this article is as follows: in Section 2 we give an introduction to random walks on HNN extensions, summarize some basic properties and present the main results of this article. In Section 3 we introduce our main tool, namely generating functions. Section 4 describes a boundary (see Proposition 4.2) towards which our random walk converges. In Section 5 we introduce a special Markov chain (see Proposition 5.1) which allows us to track the random walk's path to infinity. This construction finally enables us to derive a formula for the rate of escape w.r.t. the natural word length $\mathfrak{l}$ (see Corollary 5.5) and existence and formulas for the drift λ_ℓ for general length functions ℓ (see Theorems 2.7 and 5.6). A central limit theorem (see Theorem 2.8) associated with the word length w.r.t. ℓ is derived in Section 6 and analyticity of the drift and the asymptotic variance is then proven in Section 7, see Theorems 2.9 and 2.10. Some proofs are outsourced into Appendix A in order to allow a better reading flow.

2. HNN Extensions and Random Walks

In this section we recall the definition of HNN extensions, summarise some essential properties, and introduce a natural class of random walks on them. In particular, we introduce length functions on HNN extensions in dependence of some normal form representation of the elements.

2.1. *HNN Extensions of Groups.* Let $G_0 = \langle S_0 \mid R_0 \rangle$ be a finitely generated group with finite set of generators $S_0 \subseteq G_0$, relations R_0 and identity e_0. Let A, B be finite, isomorphic subgroups of G_0 and $\varphi : A \to B$ be an isomorphism. Moreover, let t be a symbol (called *stable letter*), which is not an element of G_0. Then the *HNN extension* of G_0 over A, B w.r.t. φ is given by

$$G := G_0 *_\varphi := \left\langle S_0, t, t^{-1} \,\middle|\, R_0, at = t\varphi(a) \text{ for } a \in A \right\rangle.$$

That is, G consists of all finite words over the alphabet $S_0 \cup \{t, t^{-1}\}$, where any two words which can be deduced from each other with the above relations represent the same element of G_0. The empty word is denoted by e. A natural group operation on G is given by concatenation of words with possible contractions or cancellations in the middle, where e is then the group identity. The definition of G implies that $G_0 *_\varphi$ is infinite, since $t^n \in G$ for all $n \in \mathbb{N}$. Note that the relation $at = t\varphi(a)$ implies

$$bt^{-1} = t^{-1}\varphi^{-1}(b) \quad \text{for all } b \in B.$$

This group structure was introduced by Higman, Neumann and Neumann, whose initials lead to the abbreviation HNN; see Higman et al. (1949). For further details and explanations of HNN extensions, we refer, e.g., to Lyndon and Schupp (1977).

In order to help visualize the concept of HNN extensions, we may think of the Cayley graph of G w.r.t. the generating set $S_0 \cup \{t, t^{-1}\}$. This graph is constructed as follows: initially, take the Cayley graph $\mathcal{X}_0$ of G_0 with respect to the generating set S_0. At each $a \in A$ we attach an additional edge leading to $at = t\varphi(a)$; at those endpoints we attach another copy of $\mathcal{X}_0$, in which we identify B with the already existing vertices $t\varphi(a)$, $a \in A$. This construction is now performed for every coset $g_0 A$, $g_0 \in G_0$; analogously, we attach new edges from each $b \in B$ to new vertices $bt^{-1} = t^{-1}\varphi^{-1}(b)$, attach then a new copy of $\mathcal{X}_0$ to those endpoints, which are identified with A in the new copy. This construction is then iterated with each coset and each new attached copy of $\mathcal{X}_0$.

Example 2.1. Consider the base group

$$G_0 = \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z}) = \langle a, b \mid a^2 = b^2 = e_0, ab = ba \rangle$$

with subgroups $A = \{e_0, a\}, B = \{e_0, b\}$ and isomorphism $\varphi : A \to B$ defined by $\varphi(e_0) = e_0, \varphi(a) = b$. The Cayley graph of the HNN extension is drawn in Figure 2.1.

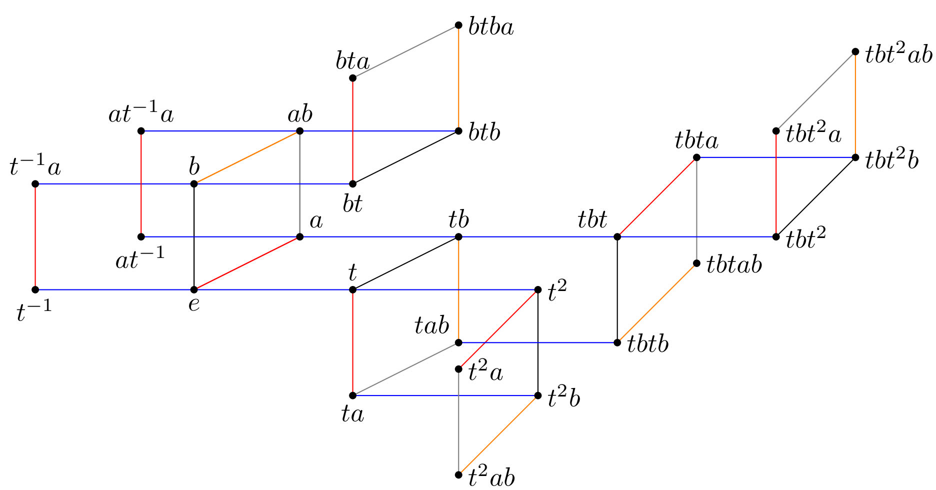

Figure 2.1. Part of the Cayley graph of the HNN extension in Example 2.1.

A normal form of the elements of $G_0 *_\varphi$ can be obtained as follows: let X be a set of representatives of the left cosets of G_0/A and Y be a set of representatives of the left cosets of $G_0/\varphi(A) = G_0/B$. We assume w.l.o.g. that $e_0 \in X, Y$. Observe that

$$t^{-1}e_0 t = t^{-1}t\varphi(e_0) = e_0 \quad \text{and} \quad te_0t^{-1} = e_0.$$

We get the following normal form expression of each element of G:

Lemma 2.2. *Each element $g \in G_0 *_\varphi$ has a unique representation of the form*

$$g = g_1 t_1 g_2 t_2 \dots g_n t_n g_{n+1}, \tag{2.1}$$

which satisfies:

- *$n \in \mathbb{N}_0$, $g_{n+1} \in G_0$, $t_i \in \{t, t^{-1}\}$ for $i \in \{1, \dots, n\}$,*
- *$g_i \in X$, if $t_i = t$, and $g_i \in Y$, if $t_i = t^{-1}$,*
- *no consecutive subsequences of the form te_0t^{-1} or $t^{-1}e_0t$.*

Proof: We prove the claim by induction on the number of letters $t^{\pm 1}$ in *any* given word over the alphabet $S_0 \cup \{t^{\pm 1}\}$. First, any $g \in G_0$ is already in the proposed form (2.1). Now consider the case of given $g = s_1 \dots s_d t^\varepsilon s_{d+1} \dots s_{d+e}$ with $d, e \in \mathbb{N}_0$, $\varepsilon \in \{-1, 1\}$ and $s_i \in S_0$ for $1 \leq i \leq d+e$. If $\varepsilon = 1$, we rewrite $s_1 \dots s_d = g_1 a_1$ with $g_1 \in X$ and $a_1 \in A$. Then:

$$g = s_1 \dots s_d t s_{d+1} \dots s_{d+e} = g_1 a_1 t s_{d+1} \dots s_{d+e} = g_1 t \underbrace{\varphi(a_1) s_{d+1} \dots s_{d+e}}_{=: g_2 \in G_0},$$

which yields the proposed form. In the case $\varepsilon = -1$, we recall that $bt^{-1} = t^{-1}\varphi^{-1}(b)$ for all $b \in B$. We now write $s_1 \dots s_d = g_1 b_1$ with $g_1 \in Y$ and $b_1 \in B$ and obtain the proposed form (2.1):

$$g = s_1 \dots s_d t^{-1} s_{d+1} \dots s_{d+e} = g_1 b_1 t^{-1} s_{d+1} \dots s_{d+e} = g_1 t^{-1} \underbrace{\varphi^{-1}(b_1) s_{d+1} \dots s_{d+e}}_{=: g_2 \in G_0}.$$

In particular, the number of letters $t^{\pm 1}$ did *not* increase and the representation is obviously unique.

The induction step follows the same reasoning: consider any word over the alphabet $S_0 \cup \{t, t^{-1}\}$ of the form

$$g = \underbrace{s_1^{(1)} \dots s_{m_1}^{(1)} t_1 s_1^{(2)} \dots s_{m_2}^{(2)} t_2 \dots s_{m_{n-1}}^{(n-1)} t_{n-1}}_{=: g'} s_1^{(n)} \dots s_{m_n}^{(n)} t_n \underbrace{s_1^{(n+1)} \dots s_{m_{n+1}}^{(n+1)}}_{=: h},$$

where $n \geq 2$, $m_1 \dots, m_{n+1} \in \mathbb{N}_0$ and $s_1^{(1)}, \dots, s_{m_{n+1}}^{(n+1)} \in S_0$. By induction assumption we can rewrite g' in the form (2.1), say

$$g' = g_1' t_1' g_2' t_2' \dots g_k' t_k' g_{k+1}' \quad \text{with } k \leq n-1.$$

We now consider the case $t_k' = t$ and $t_n = t$. Rewrite

$$g_{k+1}' s_1^{(n)} \dots s_{m_n}^{(n)} = g_n a_n$$

with $g_n \in X$ and $a_n \in A$. Then:

$$g = g_1' t_1' g_2' t_2' \dots g_k' t_k' g_n a_n t_n h = g_1' t_1' g_2' t_2' \dots g_k' t_k' g_n t \varphi(a_n) h = g_1' t_1' g_2' t_2' \dots g_k' t_k' g_n t h',$$

where $h' := \varphi(a_n) h \in G_0$, that is, we have established the required form (2.1). If $t_n = t^{-1}$, rewrite

$$g_{k+1}' s_1^{(n)} \dots s_{m_n}^{(n)} = g_n b_n$$

with $g_n \in Y$ and $b_n \in B$. Then:

$$\begin{aligned} g &= g_1' t_1' g_2' t_2' \dots g_k' t_k' g_n b_n t^{-1} h \\ &= g_1' t_1' g_2' t_2' \dots g_k' t_k' g_n t^{-1} \varphi^{-1}(b_n) h = g_1' t_1' g_2' t_2' \dots g_k' t_k' g_n t^{-1} h', \end{aligned}$$

where $h' := \varphi^{-1}(b_n)h \in G_0$. If $g_n \neq e_0$, we have established the proposed form (2.1). In the case $g_n = e_0$, $t_k' e_0 t^{-1}$ cancels out, that is, $g = g_1' t_1' g_2' t_2' \dots g_k' h'$, which is in the form (2.1). The case $t_k' = t^{-1}$ follows by symmetry. Uniqueness of the representations follows immediately from the uniqueness of representatives of the cosets. This proves the claim. □

We will refer to the expression in (2.1) as *normal form* of the elements of G and we write $\|g\|$ for the word length of $g \in G$ w.r.t. the normal form. Sometimes we will omit the letter e_0 when using normal forms; e.g., instead of writing $e_0 t e_0 t$ we just write t^2. In this setting we may omit counting the letter e_0 and get the analogous word length. Since this will *not* cause any problems below, we will omit a case distinction whether e_0 is counted or not. Furthermore, we define $[g_1 t_1 g_2 t_2 \dots g_n t_n g_{n+1}] := g_1 t_1 g_2 t_2 \dots g_n t_n$.

Example 2.3. We revisit Example 2.1. In this case we set $X = \{e_0, b\}$, $Y = \{e_0, a\}$ and obtain, e.g., the following normal forms:

$$abt^{-1} = at^{-1}\varphi^{-1}(b) = at^{-1}a, \quad tbt = \varphi^{-1}(b)tt = att = at^2.$$

Note in Figure 2.1 the "rotation" of the different coloured cosets when pushed along blue t-edges.

As a final remark observe that G is amenable if and only if $G_0 = A = B$: if $A \subsetneq G_0$ then the removal of $A \cup B$ from the Cayley graph of G splits the remaining graph into at least three connected components (e.g., $t, t^{-1}, g_0 t$ with $g_0 \in G_0 \setminus A$ are in different components), yielding non-amenability of G (e.g., see Woess (2000, Thm. 10.10)); if $G_0 = A = B$ then the Cayley graph of G has linear growth, yielding amenability of G (e.g., see Woess (2000, Thm 12.2)).

2.2. *Random Walks on HNN Extensions.* We now introduce a natural class of random walks on HNN extensions arising from random walks on the base group G_0. Let μ_0 be a finitely supported probability measure on G_0 whose support generates G_0 as a semi-group. W.l.o.g. we assume that $\mathrm{supp}(\mu_0) = S_0$. Furthermore, let be $\alpha, p \in (0,1)$. Then

$$\mu := \alpha \cdot \mu_0 + (1-\alpha) \cdot \big(p \cdot \delta_t + (1-p) \cdot \delta_{t^{-1}}\big) \tag{2.2}$$

is a probability measure on G with $\langle \mathrm{supp}(\mu) \rangle = G$. Let $(\zeta_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of random variables with distribution μ. A random walk $(X_n)_{n \in \mathbb{N}_0}$ on $G = G_0 *_\varphi$ is then given by

$$X_0 = e, \quad \forall n \geq 1 : X_n = \zeta_1 \zeta_2 \dots \zeta_n.$$

For $x, y \in G$, we denote by $p(x,y) := \mu(x^{-1}y)$ the single-step transition probabilities of $(X_n)_{n \in \mathbb{N}_0}$ and by $p^{(n)}(x,y) := \mu^{(n)}(x^{-1}y)$ the corresponding n-step transition probabilities, where $\mu^{(n)}$ is the n-fold convolution power of μ. We abbreviate $\mathbb{P}_x[\,\cdot\,] := \mathbb{P}[\,\cdot\,|X_0 = x]$. Analogously, we set $p_0^{(n)}(x_0, y_0) := \mu_0^{(n)}(x_0^{-1} y_0)$ for $x_0, y_0 \in G_0$ and $n \in \mathbb{N}$. We have the following characterisation for the recurrence/transience behaviour of random walks on HNN extensions:

Lemma 2.4. *The random walk on G is recurrent if and only if $A = B = G_0$ and $p = \frac{1}{2}$.*

Proof: Assume that $A = B = G_0$ and $p = \frac{1}{2}$. Then every normal form has the form $t^n g_0$ or $t^{-n} g_0$ with $n \in \mathbb{N}_0$ and $g_0 \in G_0$. Define $\psi : G \to \mathbb{Z}$ by $\psi(t^n g_0) := n$, $\psi(t^{-n} g_0) := -n$ respectively. Then $(X_n)_{n\in\mathbb{N}_0}$ is recurrent if and only if $\big(\psi(X_n)\big)_{n\in\mathbb{N}_0}$ is recurrent. But $\big(\psi(X_n)\big)_{n\in\mathbb{N}_0}$ is just a delayed simple random walk on $\mathbb{Z}$, which is obviously recurrent.

If we assume $p \neq \frac{1}{2}$ but $A = B = G_0$, then we get transience of $(X_n)_{n\in\mathbb{N}_0}$.

Assume now that $A \subsetneq G_0$, that is $|X|, |Y| \geq 2$. Then G is non-amenable, which yields together with Woess (2000, Cor.12.5) that the spectral radius given by $\limsup_{n\to\infty} p^{(n)}(e,e)^{1/n}$ is strictly smaller than 1, that is, the random walk on G is transient. □

Consider the Cayley graph of G w.r.t. the generating set $S_0 \cup \{t, t^{-1}\}$, which induces a natural metric $d(\cdot,\cdot)$. We have:

Lemma 2.5. *For nearest neighbour random walks on G, the rate of escape w.r.t. the natural graph metric*

$$\mathfrak{s} = \lim_{n\to\infty} \frac{d(e, X_n)}{n}$$

exists. Moreover, we have $\mathfrak{s} > 0$ if and only if $(X_n)_{n\in\mathbb{N}_0}$ is transient.

Proof: Existence is well-known due to Kingman's subadditive ergodic theorem, see Kingman (1968). Obviously, $\mathfrak{s} > 0$ implies transience. Vice versa, by Lemma 2.4, transience is equivalent to $A \subset G_0$ or $G_0 = A = B$ with $p \neq \frac{1}{2}$. If $A \subsetneq G_0$ then G is non-amenable and we obtain a spectral radius strictly smaller than 1, see Woess (2000, Cor. 12.5). This yields $\mathfrak{s} > 0$; see Woess (2000, Thm. 8.14). If $G_0 = A = B$ and $p \neq \frac{1}{2}$ then we can project the random walk onto $\mathbb{Z}$ (see proof of Lemma 2.4), which gives $\mathfrak{s} = (1-\alpha)|2p-1| > 0$. □

If G_0 is finite and $\mathrm{supp}(\mu_0) = G_0$, then one can regard $(X_n)_{n\in\mathbb{N}_0}$ as a random walk on a regular language over a finite alphabet, for which existence and analyticity of $\lim_{n\to\infty} \|X_n\|/n$ follows from the formulas in Gilch (2008). If G is hyperbolic then analyticity of $\mathfrak{s}$ and the asymptotic entropy follows from the work of Gouëzel (2017). Note that, in general, HNN extensions need not to be hyperbolic.

Since we are interested in transient random walks, we exclude from now on the case that both $A = B = G_0$ and $p = \frac{1}{2}$ hold.

2.3. *Generalised Length Functions on G.* Let $\ell : G_0 \cup \{t, t^{-1}\} \to [0,\infty)$ be a function, which plays the role of a *generalised length* or *weight function* for each letter. For $g = g_1 t_1 g_1 t_2 \dots g_n t_n g_{n+1}$ in normal form as in (2.1), we extend ℓ to a "length function" on G via

$$\ell(g_1 t_1 g_1 t_2 \dots g_n t_n g_{n+1}) := \sum_{k=1}^{n} \Big(\ell(g_k) + \ell(t_k)\Big) + \ell(g_{n+1}).$$

Note that the natural word length is obtained by setting $\ell(\cdot) = 1$. If there is a non-negative constant number λ_ℓ such that

$$\lambda_\ell = \lim_{n\to\infty} \frac{\ell(X_n)}{n} \quad \text{almost surely,}$$

then λ_ℓ is called the *rate of escape* (or *drift* or *asymptotic word length*) *w.r.t. the length function* ℓ. One aim of this paper is to show existence of this limit in the transient case under the following growth assumption on ℓ, which will be needed as an integrability condition later. We say that ℓ is of *polynomial growth* if there are some $\kappa \in \mathbb{N}$ and $C > 0$ such that $\ell(g_0) \leq C \cdot |g_0|^\kappa$ for all $g_0 \in G_0$, where

$$\begin{aligned} |g_0| &= \min\{m \in \mathbb{N} \mid \exists s_1, \dots, s_m \in G_0 : g = s_1 \dots s_m\} \\ &= \min\{m \in \mathbb{N}_0 \mid p^{(m)}(e, g) > 0\}. \end{aligned}$$

Remark 2.6. While existence of the rate of escape w.r.t. the natural graph metric given by the almost sure constant limit $\mathfrak{s} = \lim_{n\to\infty} d(e, X_n)/n$ is well-known due to Kingman's subadditive ergodic theorem, existence of λ_ℓ is not given a-priori for arbitrary length functions ℓ: e.g., if $g_1, g_2, g_3 \in G_0$ with $g_3 = g_1^{-1} g_2$, $\ell(g_1) = \ell(g_3) = 1$ and $\ell(g_2) = 3$, then

$$\ell(g_1) + \ell(g_1^{-1} g_2) < \ell(g_2);$$

that is, subadditivity does not necessarily hold, and therefore Kingman's subadditive ergodic theorem is not applicable.

As an application of generalized length functions we can construct an upper bound for the asymptotic entropy, see Corollary 5.8. We note that, in general, the natural graph metric can *not* be expressed via length functions. We refer to Remark 5.9 for further discussion on the obstacles when studying rate of escape w.r.t. the natural graph metric $\mathfrak{s}$.

2.4. *Main Results.* We summarize the main results of this article. The first main result shows existence of the rate of escape w.r.t. length functions ℓ of polynomial growth. As we will see in Section 4, the prefixes of X_n of increasing length stabilize (that is, the prefixes of increasing length are not changed any more after some finite time). We denote by $\mathbf{e}_1$, $\mathbf{e}_2$ respectively, the random time from which on the first two letters of X_n, the first four letters of X_n respectively, stabilize. The involved expectations (denoted by $\mathbb{E}_\pi[\cdot]$) in the following theorem are taken w.r.t. some invariant probability distribution π, see (5.1) in Section 5 for more details.

Theorem 2.7. *Let* $(X_n)_{n\in\mathbb{N}_0}$ *be a transient random walk on* G *governed by* μ *as defined in (2.2), and let* ℓ *be a length function of polynomial growth. Then there exists a positive constant* λ_ℓ *such that*

$$\lambda_\ell = \lim_{n\to\infty} \frac{\ell(X_n)}{n} = \frac{\mathbb{E}_\pi\big[\ell([X_{\mathbf{e}_2}]) - \ell([X_{\mathbf{e}_1}])\big]}{\mathbb{E}_\pi[\mathbf{e}_2 - \mathbf{e}_1]} > 0 \quad \textit{almost surely.} \tag{2.3}$$

In particular, the formula holds also for the rate of escape w.r.t. the natural word length (that is, if $\ell(g_0) = \ell(t^{\pm 1}) = 1$ for all $g_0 \in G_0$). We remark that Haïssinsky et al. (2018) derived a similar formula for random walks on hyperbolic surface groups. The proof of this theorem is given in Section 5, where the main steps are as follows: we construct a positive-recurrent Markov chain which is derived from the random times when new pairs of letters in the prefix of X_n stabilize, see Proposition 5.1. This Markov chain traces the random walk's path to infinity. The crucial point here is that these random times are no stopping times which destroys the Markov property of $(X_n)_{n\in\mathbb{N}_0}$ when conditioning on these random times. Having shown some necessary integrability property in Lemma 5.3, we are able to derive a formula for the rate of escape w.r.t. the natural word length (see Proposition 5.4

and Corollary 5.5), from which we can finally deduce existence of λ_ℓ in Theorem 5.6 and the formula in Theorem 2.7. Let me remark that the theorem generalizes the result of Gilch (2008) for infinite G_0.

The next main result is a central limit theorem for the word length w.r.t. ℓ. For this purpose, we use the Markov chain introduced in Proposition 5.1 for the definition of regeneration times $(T_n)_{n\in\mathbb{N}_0}$ (defined in (6.1)), which are special random times (no stopping times!) at which the random walk $(X_n)_{n\in\mathbb{N}_0}$ stabilizes further letters in its prefix in a specific way.

Theorem 2.8. *Let $(X_n)_{n\in\mathbb{N}_0}$ be a transient random walk on G governed by μ as defined in (2.2), and let ℓ be a length function of polynomial growth. Then the rate of escape w.r.t. ℓ satisfies*

$$\frac{\ell(X_n)-n\cdot\lambda_\ell}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,\sigma^2),$$

where $\sigma^2 = \dfrac{\mathbb{E}\Bigl[\bigl(\ell(X_{T_1})-\ell(X_{T_0})-(T_1-T_0)\lambda_\ell\bigr)^2\Bigr]}{\mathbb{E}[T_1-T_0]}$.

The proof is given in Section 6. The idea of the proof is to cut the trajectory of $(X_n)_{n\in\mathbb{N}_0}$ into i.i.d. subsequences. Lemmas 6.1 and 6.2 show that the time increments between two consecutive regeneration times have exponential moments. From this follows then the proposed central limit theorem.

The third main result demonstrates that λ_ℓ varies real-analytically in terms of probability measures of constant support. Let $S_0=\{s_1,\dots,s_d\}$ generate G_0 as a semigroup and denote by

$$\mathcal{P}_0(S_0)=\Bigl\{(p_1,\dots,p_d)\,\Big|\,\forall i\in\{1,\dots,d\}: p_i>0, \sum_{j=1}^d p_j=1\Bigr\}$$

the set of all probability measures μ_0 on S_0, where $\mu_0(s_i)=p_i$ for $i\in\{1,\dots,d\}$. Hence, we may regard λ_ℓ as a mapping $(\mu_0,\alpha,p)\mapsto\lambda_\ell(\mu_0,\alpha,p)$.

Theorem 2.9. *Let $S_0\subseteq G_0$ be finite and generating G_0 as a semi-group, and consider transient random walks on G governed by probability measures of the form $\mu=\alpha\mu_0+(1-\alpha)\bigl(p\delta_t+(1-p)\delta_{t^{-1}}\bigr)$ with $\mathrm{supp}(\mu_0)=S_0$. Furthermore, let ℓ be a length function of at most polynomial growth. Then the mapping*

$$\lambda_\ell:\mathcal{P}_0(S_0)\times(0,1)\times(0,1)\to\mathbb{R}:\mu=(\mu_0,\alpha,p)\mapsto\lambda_\ell(\mu_0,\alpha,p)$$

is real-analytic.

For the proof of the theorem in Section 7 we will use the formula for λ_ℓ given in 6.3. We show in Lemmas 7.2 and 7.3 that both nominator and denominator can be rewritten as multivariate power series in terms of μ_0,α,p with sufficiently large radii of convergence. In the same way we obtain our last main result:

Theorem 2.10. *The asymptotic variance σ^2 from Theorem 2.8 varies real-analytically when considered as a multivariate power series, that is, the mapping*

$$(\mu_0,\alpha,p)\mapsto\sigma^2=\sigma^2(\mu_0,\alpha,p)$$

varies real-analytically.

Concerning the rate of escape $\mathfrak{s}$ w.r.t. the natural graph metric, we obtain a special case if $A=B$ is normal in G_0:

Corollary 2.11. *Assume that $A = B \trianglelefteq G_0$ and $\varphi = \mathrm{id}_A$. Then Theorems 2.8, 2.9 and 2.10 hold also, if $\ell(g) = d(e,g)$, $g \in G$, is the distance of g to e w.r.t. the natural graph metric in the Cayley graph of G.*

Proof: It is easy to show that G/A is isomorphic to the free product $(G_0/A) * \mathbb{Z}$. In this case one can project the random walk $(X_n)_{n\in\mathbb{N}_0}$ onto G/A, for which a formula for the rate of escape w.r.t. the natural graph metric is given in Gilch (2007). If $\ell : G_0 \to \mathbb{N}$ describes the distance of the elements of G_0 to e_0 in G_0 w.r.t. the natural graph metric and if we set $\ell(t^{\pm 1}) := 1$, then the extension of $\ell(\cdot)$ to G describes the distance of any $g \in G$ to e w.r.t. the natural graph metric in the associated Cayley graph of G. □

3. Generating Functions

In this section we introduce several important probability generating functions, which are power series with some probabilities of interest as coefficients. These generating functions will play a technical key role in our proofs.

For $x, y \in G$ and $z \in \mathbb{C}$, the *Green function* is defined as

$$G(x,y|z) := \sum_{n\geq 0} p^{(n)}(x,y)\, z^n.$$

For any $M \subseteq G_0$, we write $tM := \{tm \mid m \in M\}$ and $t^{-1}M := \{t^{-1}m \mid m \in M\}$. For $a \in A$, $b \in B$, define the *generating functions w.r.t. the first visit of* G_0 when starting at tb, or at $t^{-1}a$ respectively,

$$\begin{aligned}
\eta(tb|z) &:= \sum_{n\geq 1} \mathbb{P}_{tb}\bigl[X_n \in G_0, X_{n-1} \in tB, \forall m \in \{1,\dots,n-2\} : X_m \notin G_0\bigr] z^n,\\
\eta(t^{-1}a|z) &:= \sum_{n\geq 1} \mathbb{P}_{t^{-1}a}\left[\begin{array}{c} X_n \in G_0, X_{n-1} \in t^{-1}A, \\ \forall m \in \{1,\dots,n-2\} : X_m \notin G_0 \end{array}\right] z^n.
\end{aligned}$$

Furthermore, we define

$$\begin{aligned}
\xi(tb|z) &:= 1 - \eta(tb|z),\\
\xi(t^{-1}a|z) &:= 1 - \eta(t^{-1}a|z).
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\xi(tb) &:= \xi(tb|1) = \mathbb{P}_{tb}[\forall n \in \mathbb{N} : X_n \notin G_0] = \mathbb{P}_{tb}[\forall n \in \mathbb{N} : X_n \notin A],\\
\xi(t^{-1}a) &:= \xi(t^{-1}a|1) = \mathbb{P}_{t^{-1}a}[\forall n \in \mathbb{N} : X_n \notin G_0] = \mathbb{P}_{t^{-1}a}[\forall n \in \mathbb{N} : X_n \notin B].
\end{aligned}$$

Observe that all paths from tb to G_0 have to pass through A: in order to walk from any tg, where $g \in G_0$, to $g_0 \in G_0$ one has to eliminate the t-letter, which is only possible if $g \in B$; in this case

$$tgt^{-1} = tt^{-1}\varphi^{-1}(g) = \varphi^{-1}(g) \in A.$$

Analogously, each path from $t^{-1}a$ to G_0 has to pass through B.

Lemma 3.1. *Assume $A, B \subsetneq G_0$. Then we have for all $a \in A$ and $b \in B$:*

$$\xi(tb) > 0 \quad \textit{and} \quad \xi(t^{-1}a) > 0.$$

Proof: Since the random walk $(X_n)_{n\in\mathbb{N}_0}$ on G is assumed to be transient and A and B are finite, we have

$$\mathbb{P}[A \text{ is visited infinitely often}] = \mathbb{P}[B \text{ is visited infinitely often}] = 0. \tag{3.1}$$

Assume now for a moment that $\xi(tb) = 0$ and $\xi(t^{-1}a) = 0$ for all $a \in A, b \in B$. This implies that $\eta(tb|1) = \eta(t^{-1}a|1) = 1$ for all $a \in A$ and $b \in B$. Hence, for all $x \in X$, $y \in Y$, we have

$$\mathbb{P}_{xtb}[\exists n \in \mathbb{N} : X_n \in G_0] = \eta(tb) = 1 = \eta(t^{-1}a) = \mathbb{P}_{yt^{-1}a}[\exists n \in \mathbb{N} : X_n \in G_0];$$

that is, every time when the random leaves G_0 to some point xtb or $yt^{-1}a$, it returns almost surely to G_0. This gives together with vertex-transitivity of the random walk:

$$\mathbb{P}[G_0 \text{ is visited infinitely often}] = \mathbb{P}_t[tG_0 \text{ is visited infinitely often}] = 1.$$

This in turn yields that

$$\begin{aligned} \mathbb{P}[tG_0 \text{ is visited infinitely often}] &\geq \mathbb{P}[X_1 = t, tG_0 \text{ is visited infinitely often}] \\ &= (1-\alpha)\cdot p\cdot \mathbb{P}_t[tG_0 \text{ is visited inf. often}] > 0. \end{aligned}$$

Therefore, the event that both G and tG_0 are visited infinitely often has positive probability. Since every path from tG_0 to G_0 has to pass through A, the event that A is visited infinitely often has also positive probability, which now gives a contradiction to the transience behaviour in (3.1).

Assume now that $\xi(tb_0) > 0$ for some $b_0 \in B$ and let be $b \in B$. Then, due to irreducibility of μ_0 there is some $n_0 \in \mathbb{N}$ with $p_0^{(n_0)}(b, b_0) = \mu_0^{(n_0)}(b^{-1}b_0) > 0$. This yields:

$$\begin{aligned} \xi(tb) &\geq \mathbb{P}_{tb}[X_1, \dots, X_{n_0-1} \in tG_0, X_{n_0} = tb_0, \forall n \geq 1 : X_n \notin G_0] \\ &\geq \alpha^{n_0} p_0^{(n_0)}(b, b_0)\xi(tb_0) > 0. \end{aligned}$$

Choose now any $x \in X \setminus \{e_0\}$ (observe that $A \subsetneq G_0$ implies $|X| \geq 2$) and let be $a \in A$. Then there is some $n_1 \in \mathbb{N}$ with $p_0^{(n_1)}\bigl(a, x\varphi^{-1}(b_0)\bigr) > 0$. We bound $\xi(t^{-1}a)$ by paths which start at $t^{-1}a$, go directly to $t^{-1}x$, then to $t^{-1}xt$ without any further modification of the first three letters afterwards:

$$\begin{aligned} \xi(t^{-1}a) &\geq \mathbb{P}_{t^{-1}a}\left[\substack{X_1,\dots,X_{n_1-1}\in t^{-1}G_0, X_{n_1}=t^{-1}x\varphi^{-1}(b_0),\\ X_{n_1+1}=t^{-1}x\varphi^{-1}(b_0)t, \forall n\geq n_1+1: X_n\notin t^{-1}G_0}\right] \\ &\geq \alpha^{n_1} p_0^{(n_1)}\bigl(a, x\varphi^{-1}(b_0)\bigr)\cdot (1-\alpha)\cdot p\cdot \xi(tb_0) > 0. \end{aligned}$$

Here, recall that $\varphi^{-1}(b_0)t = tb_0$. This finishes the proof. □

An analogous statement is obtained in the remaining case for transient random walks.

Lemma 3.2. *Consider the case $A = B = G_0$ and $p \neq \frac{1}{2}$. Let be $a, b \in G_0$. Then $\xi(tb) > 0$ and $\xi(t^{-1}a) = 0$, if $p > \frac{1}{2}$, and $\xi(tb) = 0$ and $\xi(t^{-1}a) > 0$, if $p < \frac{1}{2}$.*

Proof: In the case $p > \frac{1}{2}$ the stochastic process $\bigl(\psi(X_n)\bigr)_{n\in\mathbb{N}_0}$ from the proof of Lemma 2.4 tends to $+\infty$ almost surely, yielding $\xi(tb) > 0$ and $\xi(t^{-1}a) = 0$ for all $a, b \in G_0 = A = B$. The case $p < \frac{1}{2}$ follows by symmetry. □

The following property will be essential in the proofs of the upcoming sections.

Lemma 3.3. *The common radius of convergence R of $G(g_1,g_1|z)$, $g_1,g_2\in G$, is strictly bigger than 1. Moreover, the generating functions $\eta(\cdot|z)$ and $\xi(\cdot|z)$ have also radii of convergence of at least R.*

Proof: First, we remark that all Green functions must have the same radius of convergence R due to irreducibility of the underlying random walk. Since we consider only transient random walks, Lemma 2.4 implies that either $A,B\subsetneq G_0$ or $p\neq\frac{1}{2}$ must hold.

If $A,B\subsetneq G_0$ then G is non-amenable, implying that the spectral radius satisfies $\varrho=\limsup_{n\to\infty}p^{(n)}(e,e)^{1/n}<1$; see, e.g., Woess (2000, Cor. 12.5). This in turn implies $R=\varrho^{-1}>1$.

The proof of the fact that $G(e,e|z)$ has also in the case $p\neq\frac{1}{2}$ a radius of convergence strictly bigger than 1 is outsourced to Lemma A.1 in the Appendix.

It remains to consider $\eta(\cdot|z)$ and $\xi(\cdot|z)$. For $b\in B$, choose $n_b\in\mathbb{N}$ with $\mu_0^{(n_b)}\bigl(\varphi^{-1}(b)\bigr)>0$, which is possible due to irreducibilty of μ_0. Then for real $z>0$:

$$\begin{aligned}\sum_{a\in A}G(e,a|z) &\geq \sum_{n\geq n_b+2}\mathbb{P}\left[\begin{smallmatrix}X_{n_b}=\varphi^{-1}(b),\forall m\in\{1,\dots,n_b-1\}X_m\in G_0,\\ X_{n_b+1}=\varphi^{-1}(b)t,X_n\in G_0\end{smallmatrix}\right]\cdot z^n\\ &= \alpha^{n_b}\cdot\mu_0^{(n_b)}\bigl(\varphi^{-1}(b)\bigr)\cdot(1-\alpha)\cdot p\cdot z^{n_b+1}\cdot\eta(tb|z),\end{aligned}$$

where the right hand sides describes all paths, where one walks in n_b steps inside G_0 to $\varphi^{-1}(b)$, then walks to $\varphi^{-1}(b)t=tb$ and returns afterwards to the set A. The above inequality implies that $\eta(tb|z)$ has also radius of convergence of at least R for all $b\in B$; analogously for $\eta(t^{-1}a|z)$. The same holds for $\xi(tb|z)$ and $\xi(t^{-1}a|z)$ by definition. □

In the proofs later the following lemma will be a convenient tool:

Lemma 3.4. *The generating function*

$$\mathcal{K}(z):=\sum_{g_0\in G_0}G(e,g_0|z)=\sum_{g_0\in G_0}\sum_{n\geq 0}p^{(n)}(e,g_0)z^n$$

has radius of convergence strictly bigger than 1. In particular, $\mathcal{K}(z)$ is arbitrarily often differentiable at $z=1$.

Proof: For $n\in\mathbb{N}$, define

$$\zeta_n:=\mathbb{P}\bigl[X_n\in G_0,\forall m\in\{1,\dots,n-1\}:X_m\notin G_0\bigr],$$

the probability of starting in e and returning to G_0 at time n without making any steps within G_0 until time n. Recall that this implies $X_n\in A\cup B$. Set

$$\mathcal{G}_0(z):=\sum_{n\geq 0}\zeta_n\cdot z^n,\quad z\in\mathbb{C}.$$

We decompose every path from $e=e_0$ to any $g_0\in G_0$ by the number m of steps performed w.r.t. μ_0: set $\mathbf{s}(0):=0$ and define

$$\mathbf{s}(k):=\inf\{n>\mathbf{s}(k-1)\mid X_{n-1},X_n\in G_0\}\text{ for }k\geq 1.$$

In other words, at times $\mathbf{s}_k$ the random walk makes a step within G_0. For all $n \in \mathbb{N}$, we can write

$$\sum_{g_0 \in G_0} p^{(n)}(e,g) = \sum_{g_0 \in G_0} \sum_{m=0}^{n} \mathbb{P}\big[\mathbf{s}(m) \leq n, \mathbf{s}(m+1) > n, X_n = g_0\big]$$
$$= \zeta_n + \sum_{g_0 \in G_0} \sum_{m=1}^{n} \sum_{\substack{t_1,\ldots,t_m \in \mathbb{N}:\\ t_1 < t_2 < \ldots < t_m \leq n}} \mathbb{P}\left[\begin{array}{c} \mathbf{s}(1) = t_1, \ldots, \mathbf{s}(m) = t_m, \\ \mathbf{s}(m+1) > n, X_n = g_0 \end{array}\right]$$
$$= \zeta_n + \sum_{m=1}^{n} \sum_{\substack{t_1,\ldots,t_m \in \mathbb{N}:\\ t_1 < t_2 < \ldots < t_m \leq n}} \big(\zeta_{t_1 - 1} \cdot \alpha\big) \cdot \big(\zeta_{(t_2 - t_1) - 1} \cdot \alpha\big) \cdot \ldots \cdot \big(\zeta_{(t_m - t_{m-1}) - 1} \cdot \alpha\big).$$

This allows us to rewrite $\mathcal{K}(z)$ for $z \in \mathbb{C}$ in the interior of the domain of convergence:

$$\mathcal{K}(z) := \sum_{g_0 \in G_0} \sum_{n \geq 0} p^{(n)}(e, g_0) z^n = \mathcal{G}_0(z) \cdot \sum_{m \geq 0} \big(\mathcal{G}_0(z) \cdot \alpha \cdot z\big)^m.$$

Observe that, for real $z > 0$, we have

$$\mathcal{G}_0(z) = \sum_{n \geq 0} \zeta_n\, z^n \leq \sum_{n \geq 0} \mathbb{P}[X_n \in A \cup B]\, z^n = \sum_{h \in A \cup B} G(e, h|z).$$

Since $A \cup B$ is finite and the generating functions $G(e,h|z)$, $h \in A \cup B$, have common radius of convergence strictly bigger than 1 due to Lemma 3.3, $\mathcal{G}_0(z)$ has also radius of convergence strictly bigger than 1.

Consider now

$$q(z) := \mathcal{G}_0(z) \cdot \alpha \cdot z.$$

Observe that starting at e_0 (or equivalently due to transitivity, starting at any $g_0 \in G_0$) the probability of returning to G_0 followed directly by a step performed w.r.t to μ_0 is given by $q(1)$, that is,

$$\mathbb{P}[\mathbf{s}(1) < \infty] = \mathbb{P}\big[\exists m \in \mathbb{N}_0 : X_m, X_{m+1} \in G_0\big] = q(1).$$

Since $\mathcal{G}_0(1) = \mathbb{P}[\exists n \in \mathbb{N} : X_n \in G_0]$ we have $q(1) = \mathcal{G}_0(1) \cdot \alpha \leq \alpha < 1$. Moreover, $q(z)$ has radius of convergence $R(q) > 1$. Since $q(z)$ as a power series is continuous, we can choose $\rho \in \big(1, R(q)\big)$ with $q(\rho) < 1$. Then:

$$\mathcal{K}(\rho) = \mathcal{G}_0(\rho) \cdot \sum_{m \geq 0} q(\rho)^m = \frac{\mathcal{G}_0(\rho)}{1 - q(\rho)} < \infty.$$

Hence, $\mathcal{K}(z)$ has radius of convergence of at least $\rho > 1$. □

4. Boundary of the Random Walk

In this section we describe a natural boundary of the random walk on G. Define

$$\mathcal{B} := \left\{ g_1 t_1 g_2 t_2 \ldots \;\middle|\; \begin{array}{c} g_1, g_2, \dots \in X \cup Y,\ t_1, t_2, \dots \in \{t, t^{-1}\}, \\ t_i = t \Rightarrow g_i \in X, t_i = t^{-1} \Rightarrow g_i \in Y, g_i = e_0 \Rightarrow t_{i-1} t_i \neq e \end{array} \right\} \subset \big(X \cup Y \cup \{t, t^{-1}\}\big)^{\mathbb{N}},$$

the set of infinite words in normal form. Woess (1989) showed that an irreducible random walk with finite range on an HNN extension with $A \subsetneq G_0$ converges to a random infinite word in $\mathcal{B}$. Nonetheless, we give a precise mathematical related statement and a general proof (which covers also the case $A = G_0$) of this convergence towards $\mathcal{B}$, because the proofs are short and help the reader to get a better understanding of the structure of HNN extensions.

The *t-length* of a word $g = g_1 t_1 g_2 t_2 \dots g_n t_n g_{n+1}$ in normal form in the sense of (2.1) is defined as

$$|g|_t := n. \tag{4.1}$$

We make the first observation that each copy of G_0 is visited finitely often only:

Lemma 4.1. *Let be $g_1 t_1 \dots g_k t_k \in G_0$ in normal form. Then the set $g_1 t_1 \dots g_k t_k G_0$ is visited finitely often almost surely.*

Proof: First, we consider the case $A, B \subsetneq G_0$. Let be $n_1, n_2, \dots \in \mathbb{N}$ the instants of time at which the random walk $(X_n)_{n\in\mathbb{N}_0}$ visits the set $g_1 t_1 \dots g_k t_k G_0$. Suppose that the random walk is at $g = g_1 t_1 \dots g_k t_k g_{k+1}^{(j)}$, $g_{k+1}^{(j)} \in G_0$, at some time n_j. Then the probability of walking from g to $g t_k$ with no further revisit of $g_1 t_1 \dots g_k t_k G_0$ is at least

$$(1-\alpha) \cdot \min\{p, 1-p\} \cdot \min_{h\in H} \xi(t_k h) > 0,$$

where $H = A$ if $t_k = t^{-1}$, and $H = B$ if $t_k = t$; here, we used Lemma 3.1. Therefore, a geometric distribution argument shows that there are almost surely only finitely many indices $m \in \mathbb{N}$ with $X_m \in g_1 t_1 \dots g_k t_k G_0$. This proves the claim in the case $A, B \subsetneq G_0$.

In the case $A = B = G_0$ and $p \neq \frac{1}{2}$ the claim follows directly from transience of the projections $\big(\psi(X_n)\big)_{n\in\mathbb{N}_0}$ in the proof of Lemma 2.4 and finiteness of A and B. □

The last lemma motivates the definition of the *exit times* $\mathbf{e}_k$, $k \in \mathbb{N}$, as

$$\mathbf{e}_k := \min\big\{m \in \mathbb{N}_0 \mid \forall n \geq m : |X_n|_t \geq k\big\}.$$

Let be $g_\infty = g_1 t_1 g_2 t_2 \dots \in \mathcal{B}$ and denote by $X_n \wedge g_\infty$ the common prefix of X_n and g_∞, that is, if $X_n = g_1' t_1' g_2' t_2' \dots g_k' t_k' g_{k+1}'$, then

$$X_n \wedge g_\infty = g_1 t_1 \dots g_l t_l,$$

where $l = \max\{i \in \mathbb{N} \mid g_1 t_1 \dots g_i t_i = g_1' t_1' \dots g_i' t_i\}$. We say that a realisation $(x_0, x_1, \dots) \in G^{\mathbb{N}_0}$ of $(X_n)_{n\in\mathbb{N}_0}$ converges to g_∞ if $\lim_{n\to\infty} |x_n \wedge g_\infty| = \infty$. Now we are able to show that $\mathcal{B}$ is a natural boundary of the random walk towards which $(X_n)_{n\in\mathbb{N}_0}$ converges.

Proposition 4.2. *For all $k \in \mathbb{N}$, $\mathbf{e}_k < \infty$ almost surely. Moreover, the random walk $(X_n)_{n\in\mathbb{N}_0}$ converges almost surely to some $\mathcal{B}$-valued random variable X_∞.*

Proof: It is sufficient to prove that, for all $m \in \mathbb{N}$, there is some index N_m such that we have $|X_n|_t \geq m$ for all $n \geq N_m$. We prove this claim by induction. By Lemma 4.1, the set G_0 is almost surely visited finitely often, that is, there is some minimal, almost surely finite random time $\mathbf{e}_1$ such that $|X_n|_t \geq 1$ for all $n \geq \mathbf{e}_1$. In particular, the first two letters of X_n are stabilized and will *not* change for $n \geq \mathbf{e}_1$.

Assume now that there is some finite random time $\mathbf{e}_m$ such that $|X_n|_t \geq m$ for all $n \geq \mathbf{e}_m$. This implies that the prefix of X_n of t-length m is constant, that is, there is some word $g = g_1 t_1 \dots g_m t_m$ such that X_n starts with g for all $n \geq \mathbf{e}_m$. Once again by Lemma 4.1, the set gG_0 is almost surely visited finitely often only, that is, there is some almost surely finite random time $\mathbf{e}_{m+1} \in \mathbb{N}$ such that $|X_n|_t \geq m + 1$

for all $n \geq \mathbf{e}_{m+1}$. But this means that there are $g_{m+1} \in X \cup Y$, $t_{m+1} \in \{t, t^{-1}\}$ such that X_n starts with $g g_{m+1} t_{m+1}$ for all $n \geq \mathbf{e}_{m+1}$. This finishes the proof. $\square$

In Woess (1989) it is shown that $(\mathcal{B}, \nu)$ is a model for the Poisson boundary, where ν is the hitting probability of $\mathcal{B}$, that is, for measureable $B \subset \mathcal{B}$, $\nu(B)$ is the probability that $(X_n)_{n \in \mathbb{N}_0}$ converges to some element in B.

5. Existence of the Rate of Escape w.r.t. ℓ

In this section we derive existence of the rate of escape w.r.t. the length function ℓ by introducing a new Markov chain which tracks the random walk's way towards the boundary $\mathcal{B}$; compare with Gilch (2007, 2008, 2011).

Recall the definition of the *exit times* $\mathbf{e}_k$, $k \in \mathbb{N}$, from the last section. By Proposition 4.2, $\mathbf{e}_k < \infty$ almost surely for all $k \in \mathbb{N}$. The *increments* are defined as

$$\mathbf{i}_k := \mathbf{e}_k - \mathbf{e}_{k-1}.$$

Furthermore, if $X_{\mathbf{e}_k} = g_1 t_1 \dots g_k t_k h$, where $h \in B$, if $t_k = t$, and $h \in A$, if $t_k = t^{-1}$, then we set

$$\mathbf{W}_k := g_k t_k h.$$

Set

$$\begin{aligned} \mathcal{D} &:= \{gth \mid g \in X, h \in B\} \cup \{gt^{-1}h \mid g \in Y, h \in A\}, \\ \mathbb{D} &:= \left\{ (gt'h, n) \in \mathcal{D} \times \mathbb{N} \,\middle|\, \exists g_1 t_1 h_1 \in \mathcal{D} : \mathbb{P}\left[\begin{array}{c} X_{\mathbf{e}_1} = g_1 t_1 h_1, \\ X_{\mathbf{e}_2} = g_1 t_1 g t' h, \mathbf{i}_2 = n \end{array} \right] > 0 \right\}. \end{aligned}$$

Since the events $[X_{\mathbf{e}_k} = n]$, $n \in \mathbb{N}$, depend on the future after $\mathbf{e}_k$, the exit times are no stopping times. Hence, conditioning the random walk $(X_n)_{n \in \mathbb{N}_0}$ on exit times destroys the Markov property. However, we make the following crucial observation:

Proposition 5.1. $(\mathbf{W}_k, \mathbf{i}_k)_{k \in \mathbb{N}}$ *is an irreducible, aperiodic Markov chain on* $\mathbb{D}$ *with transition probabilities*

$$\begin{aligned} &\mathbb{P}[\mathbf{W}_{k+1} = w_2 t_2 h_2, \mathbf{i}_{k+1} = n \mid \mathbf{W}_k = w_1 t_1 h_1, \mathbf{i}_k = m] \\ &= \begin{cases} \frac{\xi(t_2 h_2)}{\xi(t_1 h_1)} \cdot \mathbb{P}_{t_1 h_1}\left[\substack{X_n = t_1 w_2 t_2 h_2, |X_{n-1}|_t = 1, \\ \forall n' < n : |X_{n'}|_t \geq 1} \right], & \text{if } t_1 w_2 t_2 \neq e, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

where $(w_1 t_1 h_1, m), (w_2 t_2 h_2, n) \in \mathbb{D}$.

Proof: Let be $(w_1 t_1 h_1, n_1), \dots, (w_{k+1} t_{k+1} h_{k+1}, n_{k+1}) \in \mathbb{D}$ such that this sequence satisfies $\mathbb{P}[\forall j \in \{1, \dots, k\} : \mathbf{W}_j = w_j t_j h_j, \mathbf{i}_j = n_j] > 0$. In particular, the words $w_1 t_1 \dots w_j t_j h_j$, $j \leq k+1$, are in normal form in the sense of (2.1) Then:

$$\begin{aligned} &\mathbb{P}\big[\mathbf{W}_1 = w_1 t_1 h_1, \mathbf{i}_1 = n_1, \dots, \mathbf{W}_k = w_k t_k h_k, \mathbf{i}_k = n_k\big] \\ =\ & \mathbb{P}\left[\substack{X_{\mathbf{e}_1} = w_1 t_1 h_1, \mathbf{i}_1 = n_1, X_{\mathbf{e}_2} = w_1 t_1 w_2 t_2 h_2, \mathbf{i}_2 = n_2, \\ \dots, X_{\mathbf{e}_k} = w_1 t_1 \dots w_k t_k h_k, \mathbf{i}_k = n_k} \right] \\ =\ & \mathbb{P}\left[\substack{\forall j \in \{1, \dots, k\} \forall m \in \{0, \dots, n_j\} : \\ |X_{n_1 + \dots + n_{j-1} + m}|_t \geq j-1, |X_{n_1 + \dots + n_j - 1}|_t = j-1, X_{n_1 + \dots + n_j} = w_1 t_1 \dots w_j t_j h_j} \right] \\ & \cdot \mathbb{P}_{w_1 t_1 \dots w_k t_k h_k}\big[\forall n \geq 1 : |X_{n_1 + \dots + n_k + n}|_t \geq k\big] \\ =\ & \mathbb{P}\left[\substack{\forall j \in \{1, \dots, k\} \forall m \in \{0, \dots, n_j\} : \\ |X_{n_1 + \dots + n_{j-1} + m}|_t \geq j-1, |X_{n_1 + \dots + n_j - 1}|_t = j-1, X_{n_1 + \dots + n_j} = w_1 t_1 \dots w_j t_j h_j} \right] \cdot \xi(t_k h_k). \end{aligned}$$

The last equation is due to transitivity (group invariance) of our random walk $(X_n)_{n\in\mathbb{N}_0}$. Analogously,

$$\begin{aligned}
&\mathbb{P}[\mathbf{W}_1 = w_1t_1h_1, \mathbf{i}_1 = n_1, \ldots, \mathbf{W}_{k+1} = w_{k+1}t_{k+1}h_{k+1}, \mathbf{i}_{k+1} = n_{k+1}] \\
= \; & \mathbb{P}\left[\substack{X_{\mathbf{e}_1}=w_1t_1h_1,\mathbf{i}_1=n_1,X_{\mathbf{e}_2}=w_1t_1w_2t_2h_2,\mathbf{i}_2=n_2,\ldots,\\ X_{\mathbf{e}_{k+1}}=w_1t_1\ldots w_{k+1}t_{k+1}h_{k+1},\mathbf{i}_{k+1}=n_{k+1}}\right] \\
= \; & \mathbb{P}\left[\substack{\forall j\in\{1,\ldots,k\}\forall m\in\{0,\ldots,n_j\}:\\ |X_{n_1+\cdots+n_{j-1}+m}|_t\geq j-1,|X_{n_1+\cdots+n_{j}-1}|_t=j-1,X_{n_1+\cdots+n_j}=w_1t_1\ldots w_jt_jh_j}\right] \\
& \cdot\mathbb{P}_{w_1t_1\ldots w_kt_kh_k}\left[\substack{\forall m\in\{0,\ldots,n_{k+1}\}:|X_{n_1+\cdots+n_k+m}|_t\geq k,|X_{n_1+\cdots+n_{k+1}-1}|_t=k,\\ X_{n_1+\cdots+n_{k+1}}=w_1t_1\ldots w_{k+1}t_{k+1}h_{k+1}}\right] \\
& \cdot\mathbb{P}_{w_1t_1\ldots w_{k+1}t_{k+1}h_{k+1}}\left[\forall n\geq 1: |X_{n_1+\cdots+n_{k+1}+n}|_t \geq k+1\right] \\
= \; & \mathbb{P}\left[\substack{\forall j\in\{1,\ldots,k\}\forall m\in\{0,\ldots,n_j\}:\\ |X_{n_1+\cdots+n_{j-1}+m}|_t\geq j-1,|X_{n_1+\cdots+n_{j}-1}|_t=j-1,X_{n_1+\cdots+n_j}=w_1t_1\ldots w_jt_jh_j}\right] \\
& \cdot\mathbb{P}_{w_1t_1\ldots w_kt_kh_k}\left[\substack{\forall m\in\{0,\ldots,n_{k+1}\}:|X_{n_1+\cdots+n_k+m}|_t\geq k,\\ |X_{n_1+\cdots+n_{k+1}-1}|_t=k,\\ X_{n_1+\cdots+n_{k+1}}=w_1t_1\ldots w_{k+1}t_{k+1}h_{k+1}}\right]\cdot \xi(t_{k+1}h_{k+1}).
\end{aligned}$$

Hence, transitivity of the random walk yields yields once again:

$$\begin{aligned}
&\mathbb{P}\left[\mathbf{W}_{k+1} = w_{k+1}t_{k+1}h_{k+1}, \mathbf{i}_{k+1} = n_{k+1} \,\middle|\, \substack{\mathbf{W}_1=w_1t_1h_1,\mathbf{i}_1=n_1,\ldots,\\ \mathbf{W}_k=w_kt_kh_k,\mathbf{i}_k=n_k}\right] \\
= \; & \frac{\xi(t_{k+1}h_{k+1})}{\xi(t_kh_k)}\mathbb{P}_{t_kh_k}\left[\begin{array}{c} X_{n_{k+1}} = t_kw_{k+1}t_{k+1}h_{k+1}, |X_{n_{k+1}-1}|_t = 1, \\ \forall n < n_{k+1}: |X_n|_t \geq 1 \end{array}\right].
\end{aligned}$$

From the formula above follows that $\mathrm{supp}(\mathbf{W}_k,\mathbf{i}_k)=\mathbb{D}$ for $k\geq 2$: indeed, for any $(g_1t_1h_1,n_1)\in\mathbb{D}$, there exists some $g_0t_0h_0\in\mathcal{D}$ with $g_0\neq e_0$ such that

$$\mathbb{P}\bigl[X_{\mathbf{e}_1} = g_0t_0h_0, X_{\mathbf{e}_2} = g_0t_0g_1t_1h_1, \mathbf{i}_2 = n_1\bigr] > 0,$$

yielding

$$\mathbb{P}\left[\begin{array}{c} X_{\mathbf{e}_1} = t_0, X_{\mathbf{e}_2} = t_0^2, \ldots, X_{\mathbf{e}_{k-2}} = t_0^{k-2}, \\ X_{\mathbf{e}_{k-1}} = t_0^{k-2}g_0t_0h_0, X_{\mathbf{e}_k} = t_0^{k-2}g_0t_0g_1t_1h_1, \mathbf{i}_2 = n_1 \end{array}\right] > 0,$$

that is, $(g_1t_1h_,n_1)\in\mathrm{supp}(\mathbf{W}_k,\mathbf{i}_k)$.

For irreducibility and aperiodicity, it suffices to show that any $(g_1t_1h_1,n_1)\in\mathbb{D}$ can be reached from any other $(g_0t_0h_0,n_0)\in\mathbb{D}$ in two steps. First, we consider the case $t_1=t$. Let be $g_1t_1h_1=xtb$ with $x\in X$ and $b\in B$ and choose $\bar{g}_0\bar{t}_0\bar{h}_0,\bar{g}_1\bar{t}_1\bar{h}_1\in\mathcal{D}$ with $\bar{g}_1\neq e_0$ such that

$$\begin{aligned}
\mathbb{P}[X_{\mathbf{e}_1} = \bar{g}_0\bar{t}_0\bar{h}_0, X_{\mathbf{e}_2} = \bar{g}_0\bar{t}_0g_0t_0h_0, \mathbf{i}_2 = n_0] &> 0 \text{ and} \\
\mathbb{P}[X_{\mathbf{e}_1} = \bar{g}_1\bar{t}_1\bar{h}_1, X_{\mathbf{e}_2} = \bar{g}_1\bar{t}_1xtb, \mathbf{i}_2 = n_1] &> 0;
\end{aligned}$$

compare with definition of $\mathbb{D}$ and recall transitivity of the random walk. Take any $m\in\mathbb{N}$ such that $\mu_0^{(m)}\bigl(h_0^{-1}\bar{g}_1\varphi^\delta(\bar{h}_1)\bigr)>0$, where $\delta:=1$, if $\bar{t}_1=t^{-1}$, and $\delta:=-1$, if $\bar{t}_1=t$; then for all $k\geq 2$:

$$\mathbb{P}\left[\begin{array}{c} X_{\mathbf{e}_1} = \bar{t}_0, X_{\mathbf{e}_2} = \bar{t}_0^2, \ldots, X_{\mathbf{e}_{k-2}} = \bar{t}_0^{k-2}, \\ X_{\mathbf{e}_{k-1}} = \bar{t}_0^{k-2}\bar{g}_0\bar{t}_0\bar{h}_0, X_{\mathbf{e}_k} = \bar{t}_0^{k-2}\bar{g}_0\bar{t}_0g_0t_0h_0, \mathbf{i}_k = n_0, \\ X_{\mathbf{e}_{k+1}} = \bar{t}_0^{k-2}\bar{g}_0\bar{t}_0g_0t_0\bar{g}_1\bar{t}_1\bar{h}_1, \mathbf{i}_{k+1} = m+1, \\ X_{\mathbf{e}_{k+2}} = \bar{t}_0^{k-2}\bar{g}_0\bar{t}_0g_0t_0\bar{g}_1\bar{t}_1xtb, \mathbf{i}_{k+2} = n_1 \end{array}\right] > 0.$$

Hence, we have proven that each element of $\mathbb{D}$ can be reached in two steps from any other state if $t_1=t$. The case $t_1=t^{-1}$ is shown analogously. This finishes the proof. □

Observe that, for all $(w_1t_1h_1,m),(w_2t_2h_2,n)\in\mathbb{D}$, the transition probabilities of $(\mathbf{W}_k,\mathbf{i}_k)_{k\in\mathbb{N}}$ in Lemma 5.1

$$\begin{aligned}
&q\big((w_1t_1h_1,m),(w_2t_2h_2,n)\big)\\
:=\ &\begin{cases}\mathbb{P}\big[\mathbf{W}_{k+1}=w_2t_2h_2,\mathbf{i}_{k+1}=n\mid \mathbf{W}_k=w_1t_1h_1,\mathbf{i}_k=m\big], & \text{if } t_1w_2t_2\neq e\\ 0, & \text{otherwise}\end{cases}
\end{aligned}$$

depend only on t_1h_1, $w_2t_2h_2$ and n, but *not* on w_1 and m. If $\mathbf{W}_k=w_kt_kh_k$ then set

$$\mathbf{h}_k:=t_kh_k$$

and define

$$\mathcal{D}_0:=\{th\mid h\in B\}\cup\{t^{-1}h\mid h\in A\}.$$

Note that $\mathbf{h}_k$ can take only finitely many different values. It is easy to see that $(\mathbf{h}_k)_{k\in\mathbb{N}}$ forms an irreducible Markov chain on $\mathcal{D}_0$ with transition probabilities

$$q_{\mathbf{h}}(t_1h_1,t_2h_2)=\begin{cases}\displaystyle\sum_{x\in X,n\in\mathbb{N}}q\big((e_0th_1,m),(xth_2,n)\big), & \text{if } t_1=t_2=t,\\ \displaystyle\sum_{y\in Y\setminus\{e_0\},n\in\mathbb{N}}q\big((e_0th_1,m),(yt^{-1}h_2,n)\big), & \text{if } t_1=t_2^{-1}=t,\\ \displaystyle\sum_{y\in Y,n\in\mathbb{N}}q\big((e_0t^{-1}h_1,m),(yt^{-1}h_2,n)\big), & \text{if } t_1=t_2=t^{-1},\\ \displaystyle\sum_{x\in X\setminus\{e_0\},n\in\mathbb{N}}q\big((e_0t^{-1}h_1,m),(xth_2,n)\big), & \text{if } t_1=t_2^{-1}=t^{-1},\end{cases}$$

where the quantities on the left do not depend on m as long as $(e_0t_1h_1,m)\in\mathbb{D}$. Due to the finite state space of $(\mathbf{h}_k)_{k\in\mathbb{N}}$, this process is positive recurrent and possesses an invariant probability measure $\nu_{\mathbf{h}}$. For $(w_1t_1h_1,n)\in\mathbb{D}$, set

$$\pi(w_1t_1h_1,n):=\sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)q\big((e_0t_0h_0,m),(w_1t_1h_1,n)\big). \tag{5.1}$$

Lemma 5.2. *π is an invariant probability measure of $(\mathbf{W}_k,\mathbf{i}_k)_{k\in\mathbb{N}}$. In particular, $(\mathbf{W}_k,\mathbf{i}_k)_{k\in\mathbb{N}}$ is a positive recurrent Markov chain on $\mathbb{D}$.*

Proof: Let be $(w_1t_1h_1,n)\in\mathbb{D}$. Then:

$$\begin{aligned}
&\sum_{(w_0t_0h_0,m)\in\mathbb{D}}\pi(w_0t_0h_0,m)q\big((w_0t_0h_0,m),(w_1t_1h_1,n)\big)\\
=\ &\sum_{(w_0t_0h_0,m)\in\mathbb{D}}\sum_{t'h'\in\mathcal{D}_0}\nu_{\mathbf{h}}(t'h')q\big((t'h',m'),(w_0t_0h_0,m)\big)q\big((w_0t_0h_0,m),(w_1t_1h_1,n)\big)
\end{aligned}$$

$$= \underbrace{\sum_{h_0\in B}\sum_{t'h'\in\mathcal{D}_0}\nu_{\mathbf{h}}(t'h')\underbrace{\sum_{\substack{x\in X,\\ m\in\mathbb{N}}} q\big((e_0t'h',m'),(xth_0,m)\big)}_{=q_{\mathbf{h}}(t'h',th_0)}}_{=\nu_{\mathbf{h}}(th_0)}\underbrace{q\big((xth_0,m),(w_1t_1h_1,n)\big)}_{=q\left((e_0th_0,m_0),(w_1t_1h_1,n)\right)}$$

$$+ \underbrace{\sum_{h_0\in A}\sum_{t'h'\in\mathcal{D}_0}\nu_{\mathbf{h}}(t'h')\underbrace{\sum_{\substack{y\in Y,\\ m\in\mathbb{N}}} q\big((e_0t'h',m'),(yt^{-1}h_0,m)\big)}_{=q_{\mathbf{h}}(t'h',t^{-1}h_0)}}_{=\nu_{\mathbf{h}}(t^{-1}h_0)}\underbrace{q\big((yt^{-1}h_0,m),(w_1t_1h_1,n)\big)}_{=q\left((e_0t^{-1}h_0,m_0),(w_1t_1h_1,n)\right)}$$

$$= \sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)q\big((e_0t_0h_0,m),(w_1t_1h_1,n)\big) = \pi(w_1t_1h_1,n).$$

Above we have chosen $m\in\mathbb{N}$ such that $(e_0t^{\pm 1}h_0,m)\in\mathbb{D}$; the exact value of m, however, does not play any role. □

Now we can prove:

Lemma 5.3. *For all $s\in\mathbb{N}$,*

$$\Lambda_s := \sum_{(w_1t_1h_1,m)\in\mathbb{D}} m^s\cdot\pi(w_1t_1h_1,m) < \infty.$$

Proof: We prove finiteness only in the case $s=1$. Set $H(t):=A$ and $H(t^{-1}):=B$. Rewriting the above sum yields:

$$\begin{aligned}
&\sum_{(w_1t_1h_1,m)\in\mathbb{D}} m\cdot\pi(w_1t_1h_1,m)\\
=\;&\sum_{(w_1t_1h_1,m)\in\mathbb{D}}\sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)\cdot q\big((e_0t_0h_0,m_0),(w_1t_1h_1,m)\big)\cdot m\\
=\;&\sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)\sum_{(w_1t_1h_1,m)\in\mathbb{D}} q\big((e_0t_0h_0,m_0),(w_1t_1h_1,m)\big)\cdot m\\
=\;&\sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)\sum_{\substack{(w_1t_1h_1,m)\in\mathbb{D}:\\ t_0w_1t_1\neq e}} m\cdot\frac{\xi(t_1h_1)}{\xi(t_0h_0)}\mathbb{P}_{t_0h_0}\left[\substack{\forall m\leq n: X_m\notin H(t_0), X_{m-1}\in t_0G_0,\\ X_m=t_0w_1t_1h_1}\right]\\
\leq\;&\sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)\sum_{m\in\mathbb{N}} m\cdot\frac{\max_{t_1h_1\in\mathcal{D}_0}\xi(t_1h_1)}{\xi(t_0h_0)}\underbrace{\mathbb{P}_{t_0h_0}[X_{m-1}\in t_0G_0]}_{=\mathbb{P}[X_{m-1}\in G_0]}\\
\leq\;&\sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)\frac{\max_{t_1h_1\in\mathcal{D}_0}\xi(t_1h_1)}{\xi(t_0h_0)}\sum_{m\geq 1} m\cdot\mathbb{P}[X_{m-1}\in G_0]\\
\leq\;&\sum_{t_0h_0\in\mathcal{D}_0}\nu_{\mathbf{h}}(t_0h_0)\frac{\max_{t_1h_1\in\mathcal{D}_0}\xi(t_1h_1)}{\xi(t_0h_0)}\cdot\frac{\partial}{\partial z}\big[z\cdot\mathcal{K}(z)\big]\Big|_{z=1}<\infty,
\end{aligned}$$

due to Lemma 3.4. In the case $s>1$, the reasoning is analogously, where we use the fact that $\mathcal{K}(z)$ is arbitrarily often differentiable at $z=1$. □

We set $\Lambda := \Lambda_1$. The last lemma leads to our first results, where we follow a reasoning, which was similarly used also, e.g., in Nagnibeda and Woess (2002) and Gilch (2007, 2008, 2011).

Proposition 5.4. *The rate of escape w.r.t. the t-length exists and satisfies*

$$\lim_{n\to\infty} \frac{|X_n|_t}{n} = \frac{1}{\Lambda} \quad \textit{almost surely.}$$

Proof: First, observe that the ergodic theorem for positiv recurrent Markov chains together with Lemma 5.3 yields

$$\frac{\mathbf{e}_k}{k} = \frac{1}{k}\sum_{l=1}^{k} \mathbf{i}_l \xrightarrow{k\to\infty} \Lambda \quad \text{almost surely.}$$

Define $\mathbf{k}(n) := \max\{k \in \mathbb{N} \mid \mathbf{e}_k \leq n\}$. Then we obtain almost surely:

$$1 \leq \frac{n}{\mathbf{e}_{\mathbf{k}(n)}} \leq \frac{\mathbf{e}_{\mathbf{k}(n)+1}}{\mathbf{e}_{\mathbf{k}(n)}} = \frac{\mathbf{e}_{\mathbf{k}(n)+1}}{\mathbf{k}(n)+1}\frac{\mathbf{k}(n)+1}{\mathbf{e}_{\mathbf{k}(n)}} \xrightarrow{n\to\infty} 1,$$

hence

$$\lim_{n\to\infty} \frac{\mathbf{e}_{\mathbf{k}(n)}}{n} = 1 \text{ almost surely.}$$

This yields:

$$0 \leq \frac{|X_n|_t - |X_{\mathbf{e}_{\mathbf{k}(n)}}|_t}{n} \leq \frac{n - \mathbf{e}_{\mathbf{k}(n)}}{n} = 1 - \frac{\mathbf{e}_{\mathbf{k}(n)}}{n} \xrightarrow{n\to\infty} 0 \text{ almost surely.}$$

Finally, we obtain:

$$\frac{|X_n|_t}{n} = \underbrace{\frac{|X_n|_t - |X_{\mathbf{e}_{\mathbf{k}(n)}}|_t}{n}}_{\to 0} + \underbrace{\frac{|X_{\mathbf{e}_{\mathbf{k}(n)}}|_t}{\mathbf{k}(n)}}_{=1}\underbrace{\frac{\mathbf{k}(n)}{\mathbf{e}_{\mathbf{k}(n)}}}_{\to\Lambda^{-1}}\underbrace{\frac{\mathbf{e}_{\mathbf{k}(n)}}{n}}_{\to 1} \xrightarrow{n\to\infty} \frac{1}{\Lambda} \quad \text{almost surely.} \tag{5.2}$$

□

Corollary 5.5. *The rate of escape w.r.t. the normal form word length exists and satisfies*

$$\lim_{n\to\infty} \frac{\|X_n\|}{n} = \frac{2}{\Lambda}.$$

Proof: This is an immediate consequence of Proposition 5.4 together with the fact that

$$2|g|_t - 1 \leq \|g\| \leq 2|g|_t + 1 \quad \text{for all } g \in G.$$

We remark that existence follows also from Kingman's subadditive ergodic theorem. □

Now we extend Proposition 5.4 to existence of the rate of escape w.r.t. arbitrary length functions ℓ of polynomial growth. For $(w_0t_0h_0, m) \in \mathbb{D}$, define $\widetilde{\ell}(w_0t_0h_0, m) := \ell(w_0t_0)$ and set

$$\Delta := \int \widetilde{\ell}\, d\pi = \sum_{(w_0t_0h_0,m)\in\mathbb{D}} \ell(w_0t_0) \cdot \pi(w_0t_0h_0, n) < \infty,$$

where finiteness follows from Lemma 5.3. We obtain:

Theorem 5.6. *Let $\ell \not\equiv 0$ be a length function on $G_0 \cup \{t, t^{-1}\}$ which is of polynomial growth. Then the rate of escape w.r.t. ℓ exists and is given by the almost sure positive constant number*

$$\lambda_\ell = \lim_{n\to\infty} \frac{\ell(X_n)}{n} = \frac{\Delta}{\Lambda} > 0.$$

Proof: We can write $X_{\mathbf{e}_{\mathbf{k}(n)}} = g_1 t_1 \dots g_{\mathbf{k}(n)} t_{\mathbf{k}(n)} g'_{\mathbf{k}(n)+1}$ in normal form as in (2.1). Observe that $g'_{\mathbf{k}(n)+1} \in A \cup B$. Then the ergodic theorem for positive recurrent Markov chain yields

$$\lim_{n\to\infty} \frac{\ell(X_{\mathbf{e}_{\mathbf{k}(n)}})}{\mathbf{k}(n)} = \lim_{n\to\infty} \frac{1}{\mathbf{k}(n)} \sum_{i=1}^{\mathbf{k}(n)} \ell(g_i t_i) \xrightarrow{n\to\infty} \Delta \text{ almost surely.}$$

By assumption on ℓ, there are $C > 0$ and $\kappa \in \mathbb{N}$ such that $\ell(g_0) \leq C \cdot |g_0|^\kappa$ for all $g \in G_0$. By Lemma 5.3, we have $\Lambda_\kappa = \lim_{n\to\infty} \frac{1}{k} \sum_{j=1}^{k} \mathbf{i}_j^\kappa < \infty$ almost surely. Setting $M := \max\{\ell(t), \ell(t^{-1})\}$ we get almost surely:

$$\begin{aligned} 0 &\leq \frac{\ell(X_n) - \ell(X_{\mathbf{e}_{\mathbf{k}(n)}})}{n} \leq \frac{C \cdot (n - \mathbf{e}_{\mathbf{k}(n)})^\kappa + M \cdot (n - \mathbf{e}_{\mathbf{k}(n)})}{n} \\ &\leq \frac{C \cdot (\mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)})^\kappa + M \cdot (\mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)})}{n} \\ &= \frac{C \cdot \mathbf{i}_{\mathbf{k}(n)+1}^\kappa + M \cdot \mathbf{i}_{\mathbf{k}(n)+1}}{n} \xrightarrow{n\to\infty} 0. \end{aligned}$$

The rest follows as in (5.2). Observe that $\Delta > 0$ if $\ell \not\equiv 0$. □

We are now able to prove Theorem 2.7, where we derive an alternative formula for the drift λ_ℓ, which will be useful in Section 7.

Proof of Theorem 2.7: Existence of λ_ℓ was already shown in Theorem 5.6. Recall that, for $g = g_1 t_1 \dots g_k t_k g_{k+1}$ in normal form, we write $[g] := g_1 t_1 \dots g_k t_k$. We set $\mathbb{E}_\pi\big[\ell([X_{\mathbf{e}_2}]) - \ell([X_{\mathbf{e}_1}])\big]$ as

$$\begin{aligned} &\sum_{\substack{x=(w_1 t_1 h_1, m_1),\\ y=(w_2 t_2 h_2, m_2)\in\mathbb{D}}} \pi(x) \cdot q\big(x,y\big) \cdot \big(\ell(w_1 t_1 w_2 t_2) - \ell(w_1 t_1)\big) \\ = &\sum_{(w_2 t_2 h_2, m_2)\in\mathbb{D}} \pi(w_2 t_2 h_2, m_2) \cdot \ell(w_2 t_2) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_\pi[\mathbf{e}_2 - \mathbf{e}_1] &:= \sum_{\substack{(w_1 t_1 h_1, m_1),\\ (w_2 t_2 h_2, m_2)\in\mathbb{D}}} \pi(w_1 t_1 h_1, m_1) \cdot q\big((w_1 t_1 h_1, m_1), (w_2 t_2 h_2, m_2)\big) \cdot m_2 \\ &= \sum_{(w_2 t_2 h_2, m_2)\in\mathbb{D}} \pi(w_2 t_2 h_2, m_2) \cdot m_2. \end{aligned}$$

That is, we take the expectations w.r.t. the invariant measure of the positive recurrent Markov chain $\big((\mathbf{W}_k, \mathbf{i}_k), (\mathbf{W}_{k+1}, \mathbf{i}_{k+1})\big)_{k\in\mathbb{N}}$. Finiteness of both expectations follows from Lemma 5.3 together with at most polynomial growth of ℓ.

By the ergodic theorem for positive recurrent Markov chains, we obtain

$$\frac{1}{\mathbf{k}(n)} \sum_{i=1}^{\mathbf{k}(n)} \big(\ell([X_{\mathbf{e}_i}]) - \ell([X_{\mathbf{e}_{i-1}}])\big) \xrightarrow{n\to\infty} \mathbb{E}_\pi\big[\ell([X_{\mathbf{e}_2}]) - \ell([X_{\mathbf{e}_1}])\big] \text{ almost surely.}$$

Furthermore, we observe that

$$\frac{1}{\mathbf{k}(n)} \sum_{j=2}^{\mathbf{k}(n)} \mathbf{e}_j - \mathbf{e}_{j-1} = \frac{1}{\mathbf{k}(n)} \sum_{j=2}^{\mathbf{k}(n)} \mathbf{i}_j \xrightarrow{n\to\infty} \mathbb{E}_\pi[\mathbf{i}_2] = \mathbb{E}_\pi[\mathbf{e}_2 - \mathbf{e}_1] \text{ almost surely..}$$

Hence,

$$\frac{\mathbf{e}_{\mathbf{k}(n)}}{\mathbf{k}(n)} \xrightarrow{n\to\infty} \mathbb{E}_\pi[\mathbf{e}_2 - \mathbf{e}_1] = \Lambda \text{ almost surely.}$$

Since

$$0 \leq \frac{n - \mathbf{e}_{\mathbf{k}(n)}}{\mathbf{k}(n)} \leq \frac{\mathbf{e}_{\mathbf{k}(n)+1} - \mathbf{e}_{\mathbf{k}(n)}}{\mathbf{k}(n)} \xrightarrow{n\to\infty} 0 \text{ almost surely,}$$

we get

$$\frac{n}{\mathbf{k}(n)} = \frac{n - \mathbf{e}_{\mathbf{k}(n)}}{\mathbf{k}(n)} + \frac{\mathbf{e}_{\mathbf{k}(n)}}{\mathbf{k}(n)} \xrightarrow{n\to\infty} \mathbb{E}_\pi[\mathbf{e}_2 - \mathbf{e}_1] \text{ almost surely.}$$

From the proof of Theorem 5.6 follows now the claim:

$$\begin{aligned} \lambda_\ell &= \lim_{n\to\infty} \frac{\ell(X_{\mathbf{e}_{\mathbf{k}(n)}})}{n} = \lim_{n\to\infty} \frac{\mathbf{k}(n)}{n} \frac{1}{\mathbf{k}(n)} \sum_{i=1}^{\mathbf{k}(n)} \bigl(\ell([X_{\mathbf{e}_i}]) - \ell([X_{\mathbf{e}_{i-1}}])\bigr) \\ &= \frac{\mathbb{E}_\pi[\ell([X_{\mathbf{e}_2}]) - \ell([X_{\mathbf{e}_1}])]}{\mathbb{E}_\pi[\mathbf{e}_2 - \mathbf{e}_1]} \quad \text{almost surely.} \end{aligned}$$

□

Remark 5.7. The required condition of a length function ℓ of at most polynomial growth can be relaxed to the condition that

$$\sum_{(w_0 t_0 h_0, n_0)\in\mathbb{D}} \max\bigl\{\ell(w_0 t_0), n\bigr\} \cdot \pi(w_0 t_0 h_0, n_0) < \infty.$$

However, this condition is in general hard to prove, because it needs good knowledge of π. Nonetheless, we may allow word length functions of the following form: let be $\varrho \in \bigl(1, R(\mathcal{K})\bigr)$, where $R(\mathcal{K})$ is the radius of convergence of $\mathcal{K}(z)$; assume that ℓ satisfies $\ell(g_0) \leq C \cdot \varrho^{|g_0|}$ for all $g_0 \in G_0$. Then one can show analogously to Lemma 5.3 that

$$\sum_{(w_0 t_0 h_0, n_0)\in\mathbb{D}} \pi(w_0 t_0 h_0, n_0) \cdot \ell(w_0 t_0) < \infty.$$

Once again, $R(\mathcal{K})$ is hard to determine, so we restricted the proofs to a general class of meaningful length functions.

As an application we derive an upper bound for the random walk's *entropy*, which is given by the non-negative constant h such that

$$h = \lim_{n\to\infty} -\frac{1}{n} \log \pi_n(X_n) \qquad \text{almost surely,}$$

where π_n is the distribution of X_n. Again, existence of the entropy is well-known due to Kingman's subadditive ergodic theorem.

For $g \in G$, define $F(e,g) := \mathbb{P}[\exists n \in \mathbb{N} : X_n = g]$. We choose now the *Greenian distance* as length function, that is,

$$\ell(g) := \ell_G(g) := -\log F(e,g) \qquad \text{for } g \in G_0 \cup \{t, t^{-1}\};$$

compare with Blachère et al. (2008). If the minimal single step transition probability is given by $\varepsilon_0 := \min\{p(e,g) \mid g \in G, p(e,g) > 0\}$, then

$$\ell_G(g) = -\log F(e,g) \leq -\log \varepsilon_0^{|g|} = -|g| \log \varepsilon_0,$$

that is, ℓ is of polynomial growth, and therefore λ_{ℓ_G} exists due to Theorem 2.7. Moreover, we get a simple upper bound for the entropy:

Corollary 5.8. *$\lambda_{\ell_G} \geq h$.*

Proof: By Benjamini and Peres (1994), the asymptotic entropy can be rewritten as

$$h = \lim_{n\to\infty} -\frac{1}{n}\log G(e, X_n|1). \tag{5.3}$$

For $m, n \in \mathbb{N}$, $m < n$, $x_1, \ldots, x_m, x \in G_0$, we have

$$\mathbb{P}[X_n = x] \geq \mathbb{P}\bigl[\exists k_1 < k_2 < \ldots < k_m < n : X_{k_1} = x_1, \ldots, X_{k_m} = x_m, X_n = x\bigr].$$

By conditioning on the first visits to $x_1, \ldots, x_m, x$ we obtain due to vertex transitivity:

$$\begin{aligned} G(e,x) &\geq F(e,x_1)\cdot F(x_1,x_2)\cdot \ldots \cdot F(x_m,x) \\ &= F(e,x_1)\cdot F(e,x_1^{-1}x_2)\cdot \ldots \cdot F(e,x_m^{-1}x). \end{aligned} \tag{5.4}$$

If $X_{\mathbf{e}_{i-1}} = g_1t_1\ldots g_{i-1}t_{i-1}h_{i-1}$ and $X_{\mathbf{e}_i} = g_1t_1\ldots g_it_ih_i$ are in normal form, then $X_{\mathbf{e}_{i-1}}^{-1}X_{\mathbf{e}_i} = h_{i-1}^{-1}g_it_ih_i = h_{i-1}^{-1}g_i\varphi^{\delta}(h_i)t_i$, where $\delta = 1$, if $t_i = t^{-1}$, and $\delta = -1$, if $t_i = t$. Therefore, setting $X_{\mathbf{e}_0} := e$, we may apply the inequality (5.4) twice, which yields

$$G(e, X_{\mathbf{e}_n}) \geq \prod_{i=1}^{n} F(e, X_{\mathbf{e}_{i-1}}^{-1}X_{\mathbf{e}_i}) \geq \prod_{i=1}^{n} F(e, X_{\mathbf{e}_{i-1}}^{-1}X_{\mathbf{e}_i}t_i^{-1})F(e,t_i).$$

We obtain the proposed upper bound for h as follows:

$$\begin{aligned} h &= \lim_{n\to\infty} -\frac{1}{\mathbf{e}_n}\log G(e, X_{\mathbf{e}_n}) \leq \lim_{n\to\infty} -\frac{1}{\mathbf{e}_n}\log\prod_{i=1}^{n} F(e, X_{\mathbf{e}_{i-1}}^{-1}X_{\mathbf{e}_i}) \\ &\leq \lim_{n\to\infty} -\frac{1}{\mathbf{e}_n}\sum_{i=1}^{n}\log\bigl[F(e, X_{\mathbf{e}_{i-1}}X_{\mathbf{e}_i}t_i^{-1})\cdot F(e,t_i)\bigr] \\ &= \lim_{n\to\infty} \frac{1}{\mathbf{e}_n}\sum_{i=1}^{n}\bigl(\ell_G(\underbrace{X_{\mathbf{e}_{i-1}}X_{\mathbf{e}_i}t_i^{-1}}_{\in G_0}) + \ell_G(t_i)\bigr) = \lim_{n\to\infty}\frac{1}{\mathbf{e}_n}\ell_G(X_{\mathbf{e}_n}) = \lambda_{\ell_G}. \end{aligned}$$

□

Remark 5.9.
At the end of this section let us discuss why it is considerably more difficult to study the rate of escape w.r.t. the natural graph metric and why the reasoning above can *not* be applied straight-forwardly. It is unclear under which (natural) conditions the graph metric can be expressed by length functions. This is due to the fact that shortest paths in HNN extensions may follow a subtle behaviour, which seems to be quite cryptic how to cut shortest paths into i.i.d. pieces, which stabilize as $n \to \infty$. In order to give an idea of the obstacles consider a group G_0 with finite isomorphic subgroups $A, B \subsetneq G_0$ such that $A \cap B \neq \{e\}$ and $\varphi(A\cap B) = A \cap B$. Take any $a \in A\cap B$, $a \neq e$, and suppose that $\mu_0(a) > 0$. For $n \in \mathbb{N}$, a shortest path (i.e., a sequence of vertices $(v_0, v_1, \ldots, v_m) \in G^{m+1}$ with $\mu(v_{i-1}^{-1}v_i) > 0$ and m minimal) from e to $g := t^n\varphi^n(a)$ is given by

$$\Pi_1 = \bigl(e, a, t\varphi(a), t^2\varphi^2(a), \ldots, t^n\varphi^n(a)\bigr);$$

this path has length $n+1$. Note that $d\bigl(e, \varphi^n(a)\bigr)$ could be large. Moreover, the unique shortest path from e to t^n is given by $\Pi_2 = (e, t, t^2, \ldots, t^n)$, a path of length n. Thus, if the random walk stands at time k at $X_k = t^n\varphi^n(a)$ and at some time $l > k$ at $X_l = t^n$, then the path Π_1, which is a shortest path from e to X_k, has to be changed at all points in order to transform it into the path Π_2, which is now a shortest path from e to X_l. In other words, in this situation no initial part of a shortest path from e to X_n, $n > k$, may have stabilized yet.

Note also that a shortest path from e to $a \in A$ could be $(e, t, t\varphi(a) = at, a)$, that is, shortest paths to elements in G_0 could make abbreviations through the "exterior" of G_0.

It is unclear if and how paths can be chosen such that initial parts stabilize. Further deeper investigation is needed in order to understand the behaviour of shortest paths from e to X_n as $n \to \infty$, requiring a different approach which would go beyond the scope of this article.

6. Central Limit Theorem

In this section we derive a central limit theorem for the word length w.r.t. the length function ℓ. We still assume that ℓ has at most polynomial growth and satisfies $\ell(g_0) \leq C \cdot |g_0|^\kappa$ for some $\kappa \in \mathbb{N}$ and all $g_0 \in G_0$. Before we are able to prove Theorem 2.8 we have to introduce further notation. Observe that $s_0 := (e_0 t e_0, 1) \in \mathbb{D}$ is a state, which can be taken by the Markov chain $(\mathbf{W}_k, \mathbf{i}_k)_{k\in\mathbb{N}}$ with positive probability. Define $\tau_0 := \inf\{m \in \mathbb{N} \mid (\mathbf{W}_m, \mathbf{i}_m) = s_0\}$ and inductively for $k \geq 1$

$$\tau_k := \inf\bigl\{m > \tau_{k-1} \,\big|\, (\mathbf{W}_m, \mathbf{i}_m) = s_0\bigr\}.$$

Positive recurrence of $(\mathbf{W}_k, \mathbf{i}_k)_{k\in\mathbb{N}}$ yields $\tau_k < \infty$ almost surely for all $k \in \mathbb{N}$. Furthermore, we define for $i \in \mathbb{N}_0$:

$$T_i := \mathbf{e}_{\tau_i}. \tag{6.1}$$

The following two lemmas contain the keys for later proofs.

Lemma 6.1. *The random variable $\tau_1 - \tau_0$ has exponential moments, that is, there is a constant $c_\tau > 0$ such that $\mathbb{E}\bigl[\exp\bigl(c_\tau(\tau_1 - \tau_0)\bigr)\bigr] < \infty$.*

Proof: We will just prove the lemma for the case $A, B \subsetneq G_0$; the remaining case of $A = B = G_0$ with $p \neq \frac{1}{2}$ is outsourced to Lemma A.2 in the Appendix.

For every state $(g_0 t_0 h_0, n_0) \in \mathbb{D}$ of $(\mathbf{W}_k, \mathbf{i}_k)_{k\in\mathbb{N}}$, the probability of reaching $(e_0 t e_0, 1)$ in two steps is strictly positive: assume $A, B \subsetneq G_0$ and let be $x \in X \setminus \{e_0\}$ and $n_{h_0} \in \mathbb{N}$ with $\mu_0^{(n_{h_0})}(h_0^{-1}x) > 0$; then

$$\begin{aligned} q\bigl((g_0 t_0 h_0, n_0), (x t e_0, n_{h_0} + 1)\bigr) &\geq \frac{\xi(te_0)}{\xi(t_0 h_0)} \cdot \alpha^{n_{h_0}} \cdot \mu_0^{(n_{h_0})}(h_0^{-1}x) \cdot (1-\alpha) \cdot p > 0, \\ q\bigl((x t e_0, n_{h_0} + 1), (e_0 t e_0, 1)\bigr) &\geq \frac{\xi(te_0)}{\xi(te_0)} \cdot (1-\alpha) \cdot p > 0, \end{aligned}$$

which provides

$$q := \min_{h_0 \in A\cup B} q\bigl((g_0 t_0 h_0, n_0), (x t e_0, n_{h_0} + 1)\bigr) \cdot q\bigl((x t e_0, n_{h_0} + 1), (e_0 t e_0, 1)\bigr) > 0.$$

This leads to the following exponential decaying upper bound:

$$\mathbb{P}[\tau_1 - \tau_0 = n] \leq (1-q)^{\lfloor \frac{n}{2} \rfloor},$$

that is, the random variable $\tau_1 - \tau_0$ has exponential moments. □

Furthermore, we can also show:

Lemma 6.2. *The random variables T_0 and $T_1 - T_0$ have exponential moments, that is, there are constants $c_0 > 0$ and $c_1 > 0$ such that $\mathbb{E}\bigl[\exp\bigl(c_0 T_0\bigr)\bigr] < \infty$ and $\mathbb{E}\bigl[\exp\bigl(c_1(T_1 - T_0)\bigr)\bigr] < \infty$.*

Proof: Once again we only consider the case $A, B \neq G_0$; the remaining case $A = B = G_0$ with $p \neq \frac{1}{2}$ works similarly, see Lemma A.2.

Let be $x \in X \setminus \{e_0\}$. Similarly as in the proof of Lemma 6.1, at any time $n \in [T_0, T_1)$ the random walk $(X_n)_{n\in\mathbb{N}_0}$ can realize the time T_1 within the next $N := \max\{n_h \mid h \in A \cup B\} + 2$ steps, where $n_h := \min\{m \in \mathbb{N} \mid \mu_0^{(m)}(h^{-1}x) > 0\}$: if $X_n = g_1 t_1 \dots g_j t_j g_{j+1}$ (in the form of (2.1)) then one can walk inside the set of words having prefix $[X_n]$ via $g_1 t_1 \dots g_j t_j x$ and $g_1 t_1 \dots g_j t_j x t$ to $g_1 t_1 \dots g_j t_j x t t$, where T_1 can be generated. Hence, there is some $q_T \in (0,1)$ such that

$$\mathbb{P}[T_1 - T_0 = n] \leq (1 - q_T)^{\lfloor \frac{n}{N} \rfloor},$$

which yields existence of exponential moments of $T_1 - T_0$. The same reasoning shows existence of exponential moments of T_0. □

Assume now that $(X_n)_{n\in\mathbb{N}_0}$ tends to some $g_1 t_1 g_2 t_2 \ldots \in \mathcal{B}$ in the sense of Proposition 4.2. For $i \in \mathbb{N}$, we define:

$$\begin{aligned} \widetilde{L}_i &:= \sum_{j=\tau_{i-1}+1}^{\tau_i} \ell(g_j t_j) = \ell([X_{T_i}]) - \ell([X_{T_{i-1}}]) \\ \text{and} \quad L_i &:= \widetilde{L}_i - (T_i - T_{i-1}) \cdot \lambda_\ell. \end{aligned} \tag{6.2}$$

Lemma 6.3.

$$\sigma_L^2 := \mathrm{Var}(L_1) \in (0, \infty).$$

Proof: Since

$$\begin{aligned} \widetilde{L}_1 &= \sum_{j=\tau_0+1}^{\tau_1} \ell(g_j t_j) \leq C \cdot \sum_{j=\tau_0+1}^{\tau_1} \mathbf{i}_j^\kappa + \max\{\ell(t), \ell(t^{-1})\} \cdot (\tau_1 - \tau_0) \\ &\leq C \cdot (T_1 - T_0)^\kappa + \max\{\ell(t), \ell(t^{-1})\} \cdot (\tau_1 - \tau_0), \end{aligned}$$

finiteness of σ_L^2 follows from Lemmas 6.1 and 6.2. Since μ_0 generates G_0 as a semigroup, the random walk can perform arbitrarily many circles in a copy of G_0 (in the underlying Cayley graph) in the time interval $[T_0, T_1]$; therefore, L_1 is *not* constant, and consequently we obtain $\sigma_L^2 > 0$. □

Completely analogously to Theorem 2.7 one can prove that

$$\lambda_\ell = \frac{\mathbb{E}\bigl[\ell([X_{T_1}]) - \ell([X_{T_0}])\bigr]}{\mathbb{E}[T_1 - T_0]} = \frac{\mathbb{E}[\widetilde{L}_1]}{\mathbb{E}[T_1 - T_0]}. \tag{6.3}$$

Observe that we may take expectations w.r.t. the underlying probability measure induced from μ (that is, w.r.t. the initial distribution $\mathbb{P}[\mathbf{W}_1 = \cdot, \mathbf{i}_1 = \cdot]$), and *not* w.r.t. the invariant probability measure π as initial distribution; this is possible since the random times T_i are regeneration times.

Corollary 6.4.

$$\mathbb{E}[L_1] = 0 \quad \textit{and} \quad \sigma_L^2 = \mathbb{E}\Bigl[\bigl(\ell([X_{T_1}]) - \ell([X_{T_0}]) - (T_1 - T_0)\lambda_\ell\bigr)^2\Bigr].$$

Proof: We obtain $\mathbb{E}[L_1] = 0$ immediately from (6.3), and therefore the proposed formula for σ_L^2. □

Now we can prove the proposed central limit theorem, where we use a similar reasoning as in Haïssinsky et al. (2018, Thm. 1.1):

Proof of Theorem 2.8: First, observe that Corollary 6.4 together with Lemma 6.3 ensures that σ^2 as defined in Theorem 2.8 is strictly positive.

For $k \in \mathbb{N}$, set

$$R_k := \sum_{i=1}^{k} \widetilde{L}_i, \quad S_k := \sum_{i=1}^{k} L_i = R_k - (T_k - T_0) \cdot \lambda_\ell,$$

and, for $n \in \mathbb{N}$, set

$$\mathbf{t}(n) := \sup\{m \in \mathbb{N}_0 \mid T_m \leq n\}.$$

We note that $\mathbf{t}(n) \to \infty$ almost surely as $n \to \infty$. Observe that Proposition 5.1 immediately implies that $(\tau_i - \tau_{i-1})_{i\in\mathbb{N}}$ and $(T_i - T_{i-1})_{i\in\mathbb{N}}$ are i.i.d. sequences. The sequence $(L_i)_{i\in\mathbb{N}}$ is also an i.i.d. sequence of random variables; for a proof, we refer to Lemma A.3 in the Appendix. Then, by Billingsley (1999, Theorem 14.4),

$$\frac{S_{\mathbf{t}(n)}}{\sigma_L \sqrt{\mathbf{t}(n)}} \xrightarrow{\mathcal{D}} N(0,1).$$

Analogously as in the proof of Theorem 2.7, one can show that

$$\frac{n}{\mathbf{t}(n)} \xrightarrow{n\to\infty} \mathbb{E}[T_1 - T_0] \quad \text{almost surely.}$$

Applying the Lemma of Slutsky gives for n large enough:

$$\frac{S_{\mathbf{t}(n)}}{\sigma_L \sqrt{n}} = \frac{S_{\mathbf{t}(n)}}{\sigma_L \sqrt{\mathbf{t}(n)}} \frac{\sqrt{\mathbf{t}(n)}}{\sqrt{n}} \xrightarrow{\mathcal{D}} N\left(0, \frac{1}{\mathbb{E}[T_1 - T_0]}\right). \tag{6.4}$$

The next step is to prove the following convergence behaviour:

$$\frac{\ell(X_n) - R_{\mathbf{t}(n)}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0. \tag{6.5}$$

Assume that $\mathbf{t}(n) \geq 1$ and that X_n has the form

$$g_1 t_1 \dots g_{\tau_{\mathbf{t}(n)}} t_{\tau_{\mathbf{t}(n)}} g_{\tau_{\mathbf{t}(n)}+1} t_{\tau_{\mathbf{t}(n)}+1} \dots g_m t_m g_{m+1},$$

where $m = |X_n|_t$. Recall that $R_{\mathbf{t}(n)}$ does not contain the weights of the letters of X_{T_0}. Polynomial growth of ℓ yields the following upper bound:

$$\begin{aligned}
& \ell(X_n) - R_{\mathbf{t}(n)} \\
= \ & \sum_{j=\tau_{\mathbf{t}(n)}+1}^{m} \big(\ell(g_j) + \ell(t_j)\big) + \ell(g_{m+1}) + \ell(X_{T_0}) \\
\leq \ & \sum_{j=\tau_{\mathbf{t}(n)}+1}^{m} \big(C \cdot |g_j|^\kappa + \ell(t_j)\big) + C \cdot |g_{m+1}|^\kappa + \sum_{k=1}^{\tau_0} \big(C \cdot |g_k|^\kappa + \ell(t_k)\big) \\
\leq \ & C \cdot (T_{\mathbf{t}(n)+1} - T_{\mathbf{t}(n)})^\kappa + C \cdot T_0^\kappa + \max_{s\in\{t,t^{-1}\}} \ell(s) \cdot \big((T_{\mathbf{t}(n)+1} - T_{\mathbf{t}(n)}) + T_0\big) \\
\leq \ & C' \cdot (T_{\mathbf{t}(n)+1} - T_{\mathbf{t}(n)})^\kappa + C' \cdot T_0^\kappa
\end{aligned}$$

where $C' := 2 \cdot \big(C + \max_{s\in\{t,t^{-1}\}} \ell(s)\big)$. Since $(T_i - T_{i-1})_{i\in\mathbb{N}}$ is an i.i.d. sequence, we obtain for any $\varepsilon > 0$ and n large enough:

$$\begin{aligned}
&\mathbb{P}\big[\ell(X_n) - R_{\mathbf{t}(n)} > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\big]\\
\leq\ & \mathbb{P}\big[C' \cdot (T_{\mathbf{t}(n)+1} - T_{\mathbf{t}(n)})^{\kappa} + C' \cdot T_0^{\kappa} > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\big]\\
\leq\ & \mathbb{P}\Big[\exists k \in \{1,\dots,n\} : (T_{k+1} - T_k)^{\kappa} \geq \frac{\varepsilon}{2C'}\sqrt{n}\Big] + \mathbb{P}\Big[T_0^{\kappa} \geq \frac{\varepsilon}{2C'}\sqrt{n}\Big]\\
\leq\ & n \cdot \mathbb{P}\Big[(T_1 - T_0)^{\kappa} \geq \frac{\varepsilon}{2C'}\sqrt{n}\Big] + \mathbb{P}\Big[T_0^{\kappa} \geq \frac{\varepsilon}{2C'}\sqrt{n}\Big]\\
\leq\ & n \cdot \mathbb{P}\Big[(T_1 - T_0)^{3\kappa} \geq \Big(\frac{\varepsilon}{2C'}\sqrt{n}\Big)^3\Big] + \mathbb{P}\Big[T_0^{\kappa} \geq \frac{\varepsilon}{2}\sqrt{n}\Big]\\
\leq\ & n \cdot \frac{\mathbb{E}\big[(T_1 - T_0)^{3\kappa}\big]}{\big(\frac{\varepsilon}{2C'}\sqrt{n}\big)^3} + \frac{\mathbb{E}\big[T_0^{\kappa}\big]}{\frac{\varepsilon}{2}\sqrt{n}} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

In the last inequality we applied Markov's Inequality and used Lemma 6.2 afterwards. The proposed convergence behaviour in (6.5) follows now from the fact that $\mathbf{t}(n) \to \infty$ almost surely as $n \to \infty$.

By construction of the random times T_i, $i \in \mathbb{N}_0$, we have

$$\mathbf{S}_{\mathbf{t}(n)} = \big(\ell([X_{\mathbf{t}(n)}]) - \ell([X_{T_0}])\big) - (T_{\mathbf{t}(n)} - T_0) \cdot \lambda_\ell.$$

For $\varepsilon > 0$, we get:

$$\begin{aligned}
&\mathbb{P}\Big[\big|\mathbf{S}_{\mathbf{t}(n)} - \big(\ell(X_n) - n \cdot \lambda_\ell\big)\big| > \varepsilon\sqrt{n}\Big] \qquad (6.6)\\
\leq\ & \mathbb{P}\Big[\ell(X_n) - \ell([X_{\mathbf{t}(n)}]) + \ell([X_{T_0}]) \geq \frac{\varepsilon}{2}\sqrt{n}\Big] + \mathbb{P}\Big[\lambda_\ell\big(n - (T_{\mathbf{t}(n)} - T_0)\big) \geq \frac{\varepsilon}{2}\sqrt{n}\Big].
\end{aligned}$$

From (6.5) follows that

$$\frac{\ell(X_n) - \ell([X_{\mathbf{t}(n)}]) + \ell([X_{T_0}])}{\sqrt{n}} = \frac{\ell(X_n) - R_{\mathbf{t}(n)}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0,$$

hence $\mathbb{P}\Big[\ell(X_n) - \ell([X_{\mathbf{t}(n)}]) + \ell([X_{T_0}]) \geq \frac{\varepsilon}{2}\sqrt{n}\Big] \to 0$ as $n \to \infty$. For the second summand in (6.6), we obtain once again from $T_i - T_{i-1} \sim T_1 - T_0$ for all $i \in \mathbb{N}$:

$$\begin{aligned}
&\mathbb{P}\Big[\lambda_\ell \cdot \big(n - (T_{\mathbf{t}(n)} - T_0)\big) \geq \frac{\varepsilon}{2}\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ & \mathbb{P}\Big[\lambda_\ell \cdot \big(T_{\mathbf{t}(n)+1} - (T_{\mathbf{t}(n)} - T_0)\big) \geq \frac{\varepsilon}{2}\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ & \mathbb{P}\Big[\exists k \in \{1,\dots,n\} : T_{k+1} - T_k \geq \frac{\varepsilon}{4\lambda_\ell}\sqrt{n}\Big] + \mathbb{P}\Big[T_0 \geq \frac{\varepsilon}{4\lambda_\ell}\sqrt{n}\Big]\\
\leq\ & n \cdot \mathbb{P}\Big[T_1 - T_0 \geq \frac{\varepsilon}{4\lambda_\ell}\sqrt{n}\Big] + \mathbb{P}\Big[T_0 \geq \frac{\varepsilon}{4\lambda_\ell}\sqrt{n}\Big]\\
=\ & n \cdot \mathbb{P}\Big[(T_1 - T_0)^4 \geq \frac{\varepsilon^4}{(4\lambda_\ell)^4} n^2\Big] + \mathbb{P}\Big[T_0 \geq \frac{\varepsilon}{4\lambda_\ell}\sqrt{n}\Big]\\
\leq\ & n \cdot (4\lambda_\ell)^4 \cdot \frac{\mathbb{E}\big[(T_1 - T_0)^4\big]}{\varepsilon^4 n^2} + 4 \cdot \lambda_\ell \cdot \frac{\mathbb{E}\big[T_0\big]}{\varepsilon\sqrt{n}} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

We applied Markov's Inequality in the last line together with Lemma 6.2. As $\mathbf{t}(n) \to \infty$ almost surely, we obtain

$$\mathbb{P}\big[\big|S_{\mathbf{t}(n)} - \big(\ell(X_n) - n \cdot \lambda_\ell\big)\big| > \varepsilon\sqrt{n}\big] \xrightarrow{\mathbb{P}} 0.$$

Another application of the Lemma of Slutsky together with (6.4) proves the claim. □

7. Analyticity of λ_ℓ

In this section we show that λ_ℓ varies real-analytically in terms of probability measures of constant support. To this end, we show that both nominator and denominator in the formula for λ_ℓ given in (6.3) vary real-analytically in the parameters describing the random walk on G.

First, we describe the problem more formally. Let $S_0 = \{s_1, \dots, s_d\}$ generate G_0 as a semigroup and denote by

$$\mathcal{P}_0(S_0) = \Big\{(p_1,\dots,p_d) \,\Big|\, \forall i \in \{1,\dots,d\} : p_i > 0, \sum_{j=1}^d p_j = 1\Big\}$$

the set of all strictly positive probability measures μ_0 on S_0 with

$$\big(\mu_0(s_1),\dots,\mu_0(s_d)\big) := (p_1,\dots,p_d) \in \mathcal{P}(S_0).$$

Consider the parameter vector

$$\underline{p} := \big(p_1,\dots,p_d,\alpha,\beta,p,q\big) \in \mathcal{P}_0(S) \times (0,1)^4.$$

The set of valid parameter vectors, whose single entries describe uniquely the random walk probability measure μ on G is given by

$$\mathcal{P} := \mathcal{P}_0(S) \times \big\{(\alpha,\beta) \in (0,1)^2 \mid \beta = 1-\alpha\big\} \times \big\{(p,q) \in (0,1)^2 \mid q = 1-p\big\},$$

if $A, B \neq G_0$. In the case $A = B = G_0$ we have to exclude the case $p \neq \frac{1}{2}$ and set

$$\mathcal{P} := \mathcal{P}_0(S) \times \big\{(\alpha,\beta) \in (0,1)^2 \mid \beta = 1-\alpha\big\} \times \big\{(p,q) \in (0,1)^2 \mid q = 1-p, p \neq 1/2\big\}.$$

Our aim is to show that the mapping

$$(\mu_0,\alpha,p) \mapsto \lambda_\ell = \lambda_\ell(\mu_0,\alpha,p)$$

varies real analytically in $(\mu_0,\alpha,1-\alpha,p,1-p) \in \mathcal{P}$, that is, $\lambda_\ell(\mu_0,\alpha,p)$ can be expanded as a multivariate power series in the variables of $\underline{p}$ (with $\beta = 1-\alpha$ and $q = 1-p$) in a neighbourhood of any $\underline{p}_0 \in \mathcal{P}$.

Remark 7.1. At this point let me remark that analyticity of the rate of escape is not obvious: e.g., consider a nearest neighbour random walk $(Z_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with transition probabilities

$$\mathbb{P}[Z_{n+1} = z+1 | Z_n = z] = p_1,\ \mathbb{P}[Z_{n+1} = z-1 | Z_n = z] = 1-p_1$$

for all $z \in \mathbb{Z}, n \in \mathbb{N}$. Then the mapping $(0,1) \ni p_1 \mapsto \lambda = |2p_1 - 1|$ is not analytic. Another counterexample is given in Mairesse and Mathéus (2007b).

We have to give some preliminary remarks, before we present a proof for our analyticity result. Let A_n, $n \in \mathbb{N}_0$, be a event which can be described by paths of length n of the Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on G; e.g., $A_n = [X_n \in G_0]$. By decomposing each such path belonging to A_n w.r.t. the number of steps which are performed w.r.t. the $d+2$ parameters $\mu(s_i), \mu(t^{\pm 1})$, we can rewrite $\mathbb{P}[A_n]$ as

$$\sum_{\substack{n_1,\dots,n_{d+2}\geq 0:\\ n_1+\dots+n_{d+2}=n}} c(n_1,\dots,n_{d+2}) p_1^{n_1}\cdot\ldots\cdot p_d^{n_d}\cdot\alpha^{n_1+\dots+n_d}\cdot\beta^{n_{d+1}+n_{d+2}}\cdot p^{n_{d+1}}\cdot q^{n_{d+2}}, \quad (7.1)$$

where $c(n_1,\ldots,n_{d+2})\in[0,\infty)$. If the generating function $\mathcal{F}(z):=\sum_{n\geq 0}\mathbb{P}[A_n]\,z^n$, $z\in\mathbb{C}$, has radius of convergence strictly bigger than 1, then, for $\delta>0$ small enough,

$$\infty>\mathcal{F}(1+\delta) = \sum_{n\geq 0}\sum_{\substack{n_1,\ldots,n_{d+2}\geq 0:\\ n_1+\cdots+n_{d+2}=n}} c(n_1,\ldots,n_{d+2})\prod_{i=1}^{d}\bigl(\alpha p_i(1+\delta)\bigr)^{n_i} \cdot\bigl(\beta p(1+\delta)\bigr)^{n_{d+1}}\cdot\bigl(\beta q(1+\delta)\bigr)^{n_{d+2}}; \tag{7.2}$$

that is, the mapping $(\mu_0,\alpha,p)\mapsto\mathcal{F}(1)$ varies real-analytically when considered as a power series in $\underline{p}$. This will be very helpful in the proof of the next two lemmas, which are the essential ingredients for the proof of Theorem 2.9.

Lemma 7.2. *The mapping*

$$(\mu_0,\alpha,p)\mapsto\mathbb{E}[T_1-T_0]$$

varies real-analytically.

Proof: First, observe that we can rewrite the expectation as

$$\mathbb{E}[T_1-T_0]=\sum_{n\geq 1}\mathbb{P}[T_1-T_0=n]\cdot n=\frac{\partial}{\partial z}\left[\sum_{n\geq 1}\mathbb{P}[T_1-T_0=n]\cdot z^n\right]\Bigg|_{z=1}.$$

Since T_1-T_0 has exponential moments, the power series $\sum_{n\geq 1}\mathbb{P}[T_1-T_0=n]\cdot z^n$ has radius of convergence strictly bigger than 1. According to the remarks at the beginning of this section it suffices to show that the probabilities $\mathbb{P}[T_1-T_0=n]$, $n\in\mathbb{N}$ can be written in the form of (7.1). We define

$$\mathbb{D}_{m,n}:=\Bigl\{\bigl((g_1t_1h_1,n_1),\ldots,(g_mt_mh_m,n_m)\bigr)\in(\mathbb{D}\setminus\{s_0\})^m\,\Big|\,n_1+\cdots+n_m=n\Bigr\}.$$

By conditioning on the value of T_0 we obtain together with positive recurrence of $(\mathbf{W}_k,\mathbf{i}_k)_{k\in\mathbb{N}}$:

$$\begin{aligned}
&\mathbb{P}[T_1-T_0=n]\\
=&\sum_{k\geq 1}\sum_{w_1,\ldots,w_{k-1}\in\mathbb{D}\setminus\{s_0\}}\mathbb{P}\left[\substack{(\mathbf{W}_1,\mathbf{i}_1)=w_1,\ldots,(\mathbf{W}_{k-1},\mathbf{i}_{k-1})=w_{k-1},\\ (\mathbf{W}_k,\mathbf{i}_k)=s_0}\right]\\
&\cdot\sum_{m=1}^{n}\sum_{(\bar w_1,\ldots,\bar w_{m-1})\in\mathbb{D}_{m-1,n-1}}\mathbb{P}\left[\substack{(\mathbf{W}_{k+1},\mathbf{i}_{k+1})=\bar w_1,\ldots,\\ (\mathbf{W}_{k+m-1},\mathbf{i}_{k+m-1})=\bar w_{m-1},\\ (\mathbf{W}_{k+m},\mathbf{i}_{k+m})=s_0}\,\middle|\,(\mathbf{W}_k,\mathbf{i}_k)=s_0\right]\\
=&\sum_{m=1}^{n}\sum_{(\bar w_1,\ldots,\bar w_{m-1})\in\mathbb{D}_{m-1,n-1}}\mathbb{P}\left[\substack{(\mathbf{W}_1,\mathbf{i}_1)=\bar w_1,\ldots,(\mathbf{W}_{m-1},\mathbf{i}_{m-1})=\bar w_{m-1},\\ (\mathbf{W}_m,\mathbf{i}_m)=s_0}\,\middle|\,(\mathbf{W}_0,\mathbf{i}_0)=s_0\right].
\end{aligned}$$

Due to the formula in Proposition 5.1 for the transition probabilities of the process $(\mathbf{W}_k,\mathbf{i}_k)_{k\in\mathbb{N}}$ we can find a set A_n, $n\in\mathbb{N}$, of paths of length n of the random walk $(X_n)_{n\in\mathbb{N}_0}$ such that we can rewrite $\mathbb{P}[T_1-T_0=n]$ as

$$\mathbb{P}[T_1-T_0=n]=\frac{\xi(te_0)}{\xi(te_0)}\cdot\sum_{\mathrm{Path}\in A_n}\mathbb{P}[\mathrm{Path}]=\sum_{\mathrm{Path}\in A_n}\mathbb{P}[\mathrm{Path}]. \tag{7.3}$$

Since every probability $\mathbb{P}[\mathrm{Path}]$, $\mathrm{Path}\in A_n$, can be rewritten in the form of (7.1), we finally get analyticity of $\mathbb{E}[T_1-T_0]$ as explained in (7.2). □

Analogously, we have the following property:

Lemma 7.3. *The mapping*

$$(\mu_0,\alpha,p)\mapsto \mathbb{E}\bigl[\ell([X_{T_1}])-\ell([X_{T_0}])\bigr]$$

varies real-analytically.

Proof: We start expanding the expectation $\mathbb{E}\bigl[z^{T_1-T_0}\ell([X_{T_1}])-\ell([X_{T_0}])\bigr]$, $z\in\mathbb{C}$, where we will use the notation $\bar{w}_k=(g_kt_kh_k,n_k)$ for $\bar{w}_k\in\mathbb{D}$:

$$\begin{aligned}
&\mathbb{E}\bigl[z^{T_1-T_0}\bigl(\ell([X_{T_1}])-\ell([X_{T_0}])\bigr)\bigr]\\
=&\underbrace{\sum_{k\geq 1}\sum_{w_1,\dots,w_{k-1}\in\mathbb{D}\setminus\{s_0\}}\mathbb{P}\left[\substack{(\mathbf{W}_1,\mathbf{i}_1)=w_1,\dots,(\mathbf{W}_{k-1},\mathbf{i}_{k-1})=w_{k-1},\\ (\mathbf{W}_k,\mathbf{i}_k)=s_0}\right]}_{=\mathbb{P}[T_0<\infty]=1}\\
&\cdot\sum_{n\geq 1}\sum_{m=1}^{n}\sum_{(\bar{w}_1,\dots,\bar{w}_{m-1})\in\mathbb{D}_{m-1,n-1}}\mathbb{P}\left[\substack{(\mathbf{W}_{k+1},\mathbf{i}_{k+1})=\bar{w}_1,\dots,\\ (\mathbf{W}_{k+m-1},\mathbf{i}_{k+m-1})=\bar{w}_{m-1},\\ (\mathbf{W}_{k+m},\mathbf{i}_{k+m})=s_0}\,\middle|\,(\mathbf{W}_k,\mathbf{i}_k)=s_0\right]\\
&\cdot z^{n_1+\dots+n_{m-1}+1}\cdot\left(\sum_{j=1}^{m-1}\ell(g_jt_j)+\ell(e_0t)\right)\\
=&\sum_{n\geq 1}\underbrace{\sum_{m=1}^{n}\sum_{(\bar{w}_1,\dots,\bar{w}_{m-1})\in\mathbb{D}_{m-1,n-1}}\mathbb{P}\left[\substack{(\mathbf{W}_1,\mathbf{i}_1)=\bar{w}_1,\dots,\\ (\mathbf{W}_{m-1},\mathbf{i}_{m-1})=\bar{w}_{m-1},\\ (\mathbf{W}_m,\mathbf{i}_m)=s_0}\,\middle|\,(\mathbf{W}_0,\mathbf{i}_0)=s_0\right]}_{=\mathbb{P}[T_1-T_0=n]}\\
&\cdot z^n\cdot\left(\sum_{j=1}^{m-1}\ell(g_jt_j)+\ell(e_0t)\right).
\end{aligned}$$

For real $z>0$, we can bound this sum from above by

$$\mathbb{E}\bigl[z^{T_1-T_0}\bigl(\ell([X_{T_1}])-\ell([X_{T_0}])\bigr)\bigr]\leq\sum_{n\geq 1}\mathbb{P}[T_1-T_0=n]\cdot z^n\cdot\bigl(C\cdot n^{\kappa}+n\cdot\max\{\ell(t),\ell(t^{-1})\}\bigr).$$

Since the power series $\sum_{n\geq 1}\mathbb{P}[T_1-T_0=n]\cdot z^n$ has radius of convergence strictly bigger than 1 due to existence of exponential moments of T_1-T_0 (see Lemma 6.2), the left hand side of the above inequality converges for $z=1+\delta$ with $\delta>0$ sufficiently small. Rewriting the left hand side yields

$$\begin{aligned}
&\mathbb{E}\bigl[z^{T_1-T_0}\bigl(\ell([X_{T_1}])-\ell([X_{T_0}])\bigr)\bigr]\\
=&\sum_{n\in\mathbb{N}}z^n\cdot\underbrace{\sum_{s\in\mathrm{supp}\left(\ell([X_{T_1}])-\ell([X_{T_0}])\right)}s\cdot\mathbb{P}\bigl[T_1-T_0=n,\ell([X_{T_1}])-\ell([X_{T_0}])=s\bigr]}_{=:a_n}.
\end{aligned}$$

For each $n\in\mathbb{N}$ and each $s\in\mathrm{supp}\bigl(\ell([X_{T_1}])-\ell([X_{T_0}])\bigr)$, we can find – analogously to (7.3) – a set of paths $A_{n,s}$ of length n such that

$$a_n=\sum_{s\in\mathrm{supp}\left(\ell([X_{T_1}])-\ell([X_{T_0}])\right)}\mathbb{P}[A_{n,s}]\cdot s,$$

that is, we can write a_n in the form of (7.1). The rest follows as explained in (7.2), which proves analyticity of $\mathbb{E}\bigl[\ell([X_{T_1}])-\ell([X_{T_0}])\bigr]$. □

Proof of Theorem 2.9: The proof follows now directly from Lemmas 7.2 and 7.3 in view of the drift formula given in (6.3). □

Proof of Theorem 2.10: This can be checked analogously to Lemmas 7.2 and 7.3 with a similar reasoning (without needing any further additional techniques/ideas) due to existence of exponential moments of $T_1 - T_0$. Therefore, we omit a further, detailed proof at this point. □

Appendix A. Remaining proofs

Lemma A.1. *Consider the case $A = B = G_0$ and $p \neq \frac{1}{2}$. Then $G(e,e|z)$ has radius of convergence strictly bigger than* 1.

Proof: The idea is to trace back this case to a non-symmetric nearest neighbour random walk on $\mathbb{Z}$, from which we can derive the required result.

Let $(Z_n)_{n\in\mathbb{N}_0}$ be a random walk on $\mathbb{Z}$ governed by the probability measure $\mu_\mathbb{Z}(1) = p, \mu_\mathbb{Z}(-1) = 1-p$, that is, we have $\mathbb{P}[Z_{n+1} = x+1 \mid Z_n = x] = p$ and $\mathbb{P}[Z_{n+1} = x-1 \mid Z_n = x] = 1-p$ for all $n \in \mathbb{N}$, $x \in \mathbb{Z}$. We define the associated first visit generating functions:

$$\begin{aligned} F_\mathbb{Z}(0,1|z) &:= \sum_{n\geq 1} \mathbb{P}_0[Z_n = 1, \forall m \in \{1,\dots,m-1\} : Z_m \neq 1]\, z^n, \\ F_\mathbb{Z}(0,-1|z) &:= \sum_{n\geq 1} \mathbb{P}_0[Z_n = -1, \forall m \in \{1,\dots,m-1\} : Z_m \neq -1]\, z^n. \end{aligned}$$

The first return generating function is given by

$$U_\mathbb{Z}(z) := \sum_{n\geq 1} \mathbb{P}_0[Z_n = 0, \forall m \in \{1,\dots,m-1\} : Z_m \neq 0]\, z^n.$$

Conditioning on the first step gives the following system:

$$\begin{aligned} F_\mathbb{Z}(0,1|z) &= \mu_\mathbb{Z}(1)\cdot z + \mu_\mathbb{Z}(-1)\cdot z \cdot F_\mathbb{Z}(0,1|z)^2, \\ F_\mathbb{Z}(0,-1|z) &= \mu_\mathbb{Z}(-1)\cdot z + \mu_\mathbb{Z}(1)\cdot z \cdot F_\mathbb{Z}(0,-1|z)^2, \\ U_\mathbb{Z}(z) &= \mu_\mathbb{Z}(1)\cdot z\cdot F_\mathbb{Z}(0,-1|z) + \mu_\mathbb{Z}(-1)\cdot z \cdot F_\mathbb{Z}(0,1|z). \end{aligned}$$

Solving this system leads to the formula

$$U_\mathbb{Z}(z) = (1-p)\cdot z \cdot \frac{1-\sqrt{1-4pz^2+4p^2z^2}}{2pz} + p\cdot z\cdot \frac{1+\sqrt{1-4pz^2+4p^2z^2}}{2pz}.$$

Therefore, $U_\mathbb{Z}(z)$ has radius of convergence strictly bigger than 1 and satisfies $U_\mathbb{Z}(1) < 1$ due to transience, and consequently

$$G_\mathbb{Z}(z) := \sum_{n\geq 0} \mu_\mathbb{Z}^{(n)}(0)\cdot z^n = \frac{1}{1-U_\mathbb{Z}(z)}$$

has also radius of convergence strictly bigger than 1.

We now turn back to our random walk on G. Define the stopping times

$$s(0) := 0, \quad \forall k \in \mathbb{N} : s(k) := \min\bigl\{m > s(k-1) \mid X_{m-1}^{-1} X_m \in \{t, t^{-1}\}\bigr\}.$$

That is, $s(k)$ is the k-th time that the random walk on G performs a step w.r.t. $\delta_{t^{\pm 1}}$. Due to transience and finiteness of $A = B = G_0$, $s(k) < \infty$ almost surely for all $k \in \mathbb{N}$. For $k \geq 1$, $n_0 := 0, n_1, \dots, n_k \in \mathbb{Z}$, define

$$w(n_1,\dots,n_k) := \mathbb{E}\Bigl[z^{s(k)} \mathbb{1}_{[X_{s(j)} \in t^{n_j} G_0 \forall j \in \{1,\dots,k\}]} \Bigm| X_0 = e\Bigr].$$

Claim 1:

$$w(n_1,\ldots,n_k) = \left(\frac{z}{1-\alpha z}\right)^k \cdot \prod_{j=1}^k \mu\bigl(t^{n_j-n_{j-1}}\bigr).$$

Proof of Claim 1: For $k=1$, we decompose all paths by the intermediate steps within G_0 until time $s(1)$ and set $x_0 := e$, $n_0 := 0$:

$$\begin{aligned} w(n_1) &= \sum_{m\geq 1} \underbrace{\sum_{g_1,\ldots,g_{m-1}\in G_0} \mathbb{P}[\forall j\in\{1,\ldots,m-1\}: X_j = g_j]\cdot z^{m-1}}_{=\alpha^{m-1}} \cdot \mu(t^{n_1})\cdot z \\ &= \frac{z}{1-\alpha z}\cdot \mu(t^{n_1}) = \frac{z}{1-\alpha z}\cdot \mu(t^{n_1-n_0}). \end{aligned}$$

We remark that, for all $m\in\mathbb{N}$ and $h\in G_0$, we have the following equation due to group invariance of our random walk on G:

$$\begin{aligned} &\sum_{g_1,\ldots,g_{m-1}\in G_0} \mathbb{P}_{t^{k-1}}[\forall j\in\{1,\ldots,m-1\}: X_j = t^{k-1}g_j] \\ = &\sum_{g_1,\ldots,g_{m-1}\in G_0} \mathbb{P}_{t^{k-1}h}[\forall j\in\{1,\ldots,m-1\}: X_j = t^{k-1}hg_j]. \end{aligned}$$

Now we can conclude analogously by induction:

$$\begin{aligned} & w(n_1,\ldots,n_k) \\ = \; & w(n_1,\ldots,n_{k-1}) \\ & \cdot \sum_{\substack{m\geq 1,\\ g_1,\ldots,g_{m-1}\in G_0}} \mathbb{P}_{t^{n_{k-1}}}[\forall j\in\{1,\ldots,m-1\}: X_j = t^{n_{k-1}}g_j]\cdot \mu\bigl(t^{n_k-n_{k-1}}\bigr)\cdot z^m \\ = \; & \left(\frac{z}{1-\alpha z}\right)^{k-1} \cdot \prod_{j=1}^{k-1} \mu\bigl(t^{n_j-n_{j-1}}\bigr)\cdot \frac{z}{1-\alpha z}\cdot \mu\bigl(t^{n_k-n_{k-1}}\bigr) \\ = \; & \left(\frac{z}{1-\alpha z}\right)^{k} \cdot \prod_{j=1}^{k} \mu\bigl(t^{n_j-n_{j-1}}\bigr). \end{aligned}$$

This finishes the proof of Claim 1.

Now we connect the random walk on $\mathbb{Z}$ with the random walk $(X_n)_{n\in\mathbb{N}_0}$ on G, for which we introduce the notation

$$G(e,A|z) := \sum_{n\geq 0} \mathbb{P}[X_n\in A]\, z^n = \sum_{g_0\in G_0} G(e,g_0|z).$$

Claim 2:

$$G(e,A|z) = G_{\mathbb{Z}}\left(\frac{(1-\alpha)z}{1-\alpha z}\right)\cdot \frac{1}{1-\alpha z}.$$

Proof of Claim 2: First, we recall that $A = G_0$ and observe the following equation:

$$\sum_{n\geq 0} \mathbb{P}[X_n\in A, s(1)>n]\, z^n = \frac{1}{1-\alpha z}.$$

Furthermore, we recall that $\mu(t) = (1-\alpha)p$ and $\mu(t^{-1}) = (1-\alpha)(1-p)$. By decomposing each path from e to A by the number k of transitions from the sets

$t^m G_0$ to $t^{m\pm 1}G_0$, we obtain:

$$\begin{aligned}
& G(e,A|z) \\
= & \frac{1}{1-\alpha z} + \sum_{k\geq 1} \sum_{n_1,\dots,n_{k-1}\in\mathbb{Z}} w(n_1,\dots,n_{k-1},0)\cdot \frac{1}{1-\alpha z} \\
= & \frac{1}{1-\alpha z} + \sum_{k\geq 1} \left(\frac{z}{1-\alpha z}\right)^k \cdot \sum_{n_1,\dots,n_{k-1}\in\mathbb{Z}} \prod_{j=1}^{k-1} \mu\bigl(t^{n_j-n_{j-1}}\bigr)\cdot \mu(t^{-n_{k-1}})\cdot \frac{1}{1-\alpha z} \\
= & \frac{1}{1-\alpha z}\cdot \left[1+\sum_{k\geq 1}\left(\frac{(1-\alpha)z}{1-\alpha z}\right)^k \cdot \underbrace{\sum_{n_1,\dots,n_{k-1}\in\mathbb{Z}} \prod_{j=1}^{k-1} \mu_{\mathbb{Z}}(n_j-n_{j-1})\mu_{\mathbb{Z}}(-n_{k-1})}_{=\mu_{\mathbb{Z}}^{(k)}(0)}\right] \\
\leq & G_{\mathbb{Z}}\left(\frac{(1-\alpha)z}{1-\alpha z}\right)\cdot \frac{1}{1-\alpha z}.
\end{aligned}$$

This finishes the proof of Claim 2.

Since $G(e,A|z)\geq G(e,e|z)$ the lemma follows now from Claim 2 and the fact that $G_{\mathbb{Z}}(z)$ has radius of convergence strictly bigger than 1. □

In the following we give the proof of Lemmas 6.1 and 6.2 in the remaining case:

Lemma A.2. *Consider the case $A=B=G_0$ with $p\in(0,1), p\neq \frac{1}{2}$. Then the random variables $\tau_1-\tau_0$, T_0 and T_1-T_0 have exponential moments.*

Proof: If $A=B=G_0$ and $p\neq\frac{1}{2}$, then $\mathbf{W}_k$ has the form $e_0t^kb_k$, $b_k\in B$, for all $k\in\mathbb{N}$, if $p>\frac{1}{2}$, and $e_0t^{-1}a_k$, $a_k\in A$, for all $k\in\mathbb{N}$, if $p<\frac{1}{2}$: this is an easy consequence of transience of the projected random walk $\bigl(\psi(X_n)\bigr)_{n\in\mathbb{N}_0}$ onto $\mathbb{Z}$ from Lemma 2.4. We show again that $(e_0te_0,1)$ can be reached from any other state of $(\mathbf{W}_k,\mathbf{i}_k)_{k\in\mathbb{N}}$ in two steps, where we restrict ourselves to the case $p>\frac{1}{2}$ (the case $p<\frac{1}{2}$ works analogously). For $(e_0tb,n_0)\in\mathbb{D}$, choose $n_b\in\mathbb{N}$ with $\mu_0^{(n_b)}(b^{-1})>0$; then

$$\begin{aligned}
q\bigl((e_0tb,n_0),(e_0te_0,n_b+1)\bigr) &\geq \frac{\xi(te_0)}{\xi(t_0b)}\cdot\alpha^{n_b}\cdot\mu_0^{(n_b)}(b^{-1})\cdot(1-\alpha)\cdot p>0 \quad\text{and}\\
q\bigl((e_0te_0,n_b+1),(e_0te_0,1)\bigr) &\geq (1-\alpha)\cdot p>0,
\end{aligned}$$

which provides

$$q:=\min_{b\in B} q\bigl((e_0tb,n_0),(e_0te_0,n_b+1)\bigr)\cdot q\bigl((e_0te_0,n_b+1),(e_0te_0,1)\bigr)>0.$$

This leads to the desired exponential decay:

$$\mathbb{P}[\tau_1-\tau_0=n]\leq (1-q)^{\lfloor \frac{n}{2}\rfloor},$$

that is, $\tau_1-\tau_0$ has exponential moments.

Existence of exponential moments of T_1-T_0 follows analogously as in Lemma 6.2: after time T_0 the random walk $(X_n)_{n\in\mathbb{N}_0}$ can produce the next regeneration time T_1 in at most

$$N:=\max\{n_h\mid h\in A\cup B\}+2$$

steps, where $n_h:=\min\{m\in\mathbb{N}\mid \mu_0^{(m)}(h^{-1})\}$. Hence, there is some $q_T\in(0,1)$ such that

$$\mathbb{P}[T_1-T_0=n]\leq (1-q_T)^{\lfloor \frac{n}{N}\rfloor},$$

which yields existence of exponential moments of $T_1 - T_0$. The same reasoning shows existence of exponential moments of T_0, which finishes the proof. $\square$

The following lemma is left from Section 6, where we introduced the sequence of random variables $(L_i)_{i\in\mathbb{N}}$ in (6.2).

Lemma A.3. $(L_i)_{i\in\mathbb{N}}$ *forms an i.i.d. sequence of random variables.*

Proof: Let be $i \in \mathbb{N}$, $z \in \mathbb{R}$. For $x_0 \in G$ with $\mathbb{P}[X_{\mathbf{e}_{\tau_i}} = x_0] > 0$ and $m \in \mathbb{N}$, denote by $\mathcal{P}^{(1)}_{i,x_0,m}$ the set of paths $(e, w_1, \ldots, w_m = x_0) \in G^{m+1}$ (with $\mu(w_{i-1}^{-1}w_i) > 0$) of length m such that $[X_1 = w_1, \ldots, X_m = w_m] \cap [X_m = x_0, \mathbf{e}_{\tau_i} = m] \neq \emptyset$. Furthermore, denote by $\mathcal{P}^{(2)}_{i,x_0,m,n,z}$ the set of paths $(x_0, y_1, \ldots, y_n) \in G^{n+1}$ of length $n \in \mathbb{N}$ such that

$$[X_m = x_0, X_{m+1} = y_1, \ldots, X_{m+n} = y_n] \cap \left[\begin{array}{c} X_m = x_0, \mathbf{e}_{\tau_{i-1}} = m, \\ \mathbf{e}_{\tau_i} = m+n, L_i = z \end{array}\right] \neq \emptyset.$$

By decomposing all paths until time $\mathbf{e}_{\tau_i}$ into the part until time $\mathbf{e}_{\tau_{i-1}}$ and into the part between times $\mathbf{e}_{\tau_{i-1}}$ and $\mathbf{e}_{\tau_i}$ we obtain:

$$\begin{aligned}
&\mathbb{P}[L_i = z] \\
=\ & \sum_{\substack{x_0\in G:\\ \mathbb{P}[X_{\mathbf{e}_{\tau_{i-1}}}=x_0]>0}} \mathbb{P}\bigl[X_{\mathbf{e}_{\tau_{i-1}}} = x_0, L_i = z\bigr] \\
=\ & \sum_{\substack{x_0\in G:\\ \mathbb{P}[X_{\mathbf{e}_{\tau_{i-1}}}=x_0]>0}} \sum_{m\geq 1} \sum_{(e,w_1,\ldots,w_m)\in\mathcal{P}^{(1)}_{i-1,x_0,m}} \mathbb{P}\bigl[X_1 = w_1, \ldots, X_m = w_m\bigr] \\
&\cdot \sum_{n\geq 1} \sum_{(x_0,y_1,\ldots,y_n)\in\mathcal{P}^{(2)}_{i,x_0,m,n,z}} \mathbb{P}_{x_0}\bigl[X_1 = y_1, \ldots, X_n = y_n\bigr] \\
&\cdot \mathbb{P}_{y_n}\bigl[\forall l \geq 1 : X_l \text{ has prefix } [y_n]\bigr] \\
=\ & \underbrace{\sum_{\substack{x_0\in G:\\ \mathbb{P}[X_{\mathbf{e}_{\tau_{i-1}}}=x_0]>0}} \sum_{m\geq 1} \sum_{(e,w_1,\ldots,w_m)\in\mathcal{P}^{(1)}_{i-1,x_0,m}} \mathbb{P}\bigl[X_1 = w_1, \ldots, X_m = w_m\bigr]}_{=\left(1-\xi(te_0)\right)^{-1}} \\
&\cdot \sum_{\substack{n\geq 1,\\ (x_0,y_1,\ldots,y_n)\in\mathcal{P}^{(2)}_{i,x_0,m,n,z}}} \mathbb{P}_t\bigl[X_1 = tx_0^{-1}y_1, \ldots, X_n = tx_0^{-1}y_n\bigr] \cdot \bigl(1-\xi(te_0)\bigr).
\end{aligned}$$

In the last equation we used group invariance of our underlying random walk. Observe that paths $(x_0, y_1, \ldots, y_n) \in \mathcal{P}^{(2)}_{i,x_0,m,n,z}$ lie completely in the set of words having prefix $[x_0]$. Therefore, there is a 1-to-1 correspondence between paths in $\mathcal{P}^{(2)}_{i,x_0,m,n,z}$ and $\mathcal{P}^{(2)}_{1,t,1,n,z}$, which lies completely in the set of words having prefix t, established by the shift $g \mapsto tx_0^{-1}g$. Therefore,

$$\mathbb{P}[L_i = z] = \sum_{n\geq 1} \sum_{(t,y_1,\ldots,y_n)\in\mathcal{P}^{(2)}_{1,t,1,n,z}} \mathbb{P}_t\bigl[X_1 = y_1, \ldots, X_n = y_n\bigr].$$

This proves that the L_i's have the same distribution. An analogous decomposition of all possible paths proves independence, which we leave as an exercise to the interested reader. $\square$

Acknowledgements

The author is grateful to the anonymous referee for several hints regarding content and exposition.

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