12. Sei $f : (0, \infty] \to \mathbb{R}$ zweimal stetig differenzierbar. Für $a \in \mathbb{R}^3$ sei $F : \mathbb{R}^3 \setminus \{a\} \to \mathbb{R}$ durch $F(x) := f(\|x - a\|)$ erklärt. Zeigen Sie

$$\Delta F(x) = f''(\|x - a\|) + \frac{2}{\|x - a\|}.\, f'(\|x - a\|)$$

13. Beweisen Sie den folgenden Satz (Eulerscher Satz für homogene Funktionen):
Ist die Funktion $f : \mathbb{R}^n \to \mathbb{R}$ differenzierbar auf $\mathbb{R}^n$ und homogen vom Grade α (d. h., gilt $f(tx) = t^\alpha f(x)$ für alle $x \in \mathbb{R}^n$ und alle $t > 0$), so besteht die Eulersche Relation

$$\frac{\partial f(x)}{\partial x_1} x_1 + \frac{\partial f(x)}{\partial x_2} x_2 + \cdots + \frac{\partial f(x)}{\partial x_n} x_n = \alpha f(x).$$

14. Stellen Sie $w = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in Polarkoordinaten dar.

15. Berechnen Sie das Oberflächenintegral

$$\oiint_{\mathcal{O}} \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} d\vec{o},$$

wobei $\mathcal{O}$ die Berandung des Bereichs

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 4 \wedge x^2 + y^2 \leq 1\}$$

bezeichnet. Berechnen Sie das Integral auch mit Hilfe des Integralsatzes von Gauß.

12)

$$\frac{\partial^2 F}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial u}\frac{\partial u}{\partial x_i}\right) = \frac{\partial F}{\partial u}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) + \frac{\partial u}{\partial x_i}\frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial x_i}$$

$$u = \|x-a\| = \sqrt{\sum_{i=1}^{3}(x_i-a_i)^2} \implies \frac{\partial u}{\partial x_i} = \frac{1}{2}\frac{1}{u}2(x_i-a_i) = \frac{x_i-a_i}{u}$$

$$\frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_i} = \frac{u-(x_i-a_i)^2\frac{1}{u}}{u^2}$$

$$\implies \frac{\partial^2 F}{\partial x_i^2} = \left(\frac{x_i-a_i}{u}\right)^2\frac{\partial^2 F}{\partial u^2} + \frac{1}{u}\frac{\partial F}{\partial u}\left(1-\frac{(x_i-a_i)^2}{u^2}\right)$$

$$\implies \Delta F = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 F}{\partial x_3^2} = \frac{\partial^2 F}{\partial u^2} + \frac{1}{u}\frac{\partial F}{\partial u}\left(3-\underbrace{\frac{1}{u^2}\sum_{i=1}^{3}(x_i-a_i)^2}_{1}\right) = f'' + \frac{2}{\|x-a\|}f'$$

13)

1) $$\frac{\partial}{\partial t}f(tx) = \frac{\partial f(tx)}{\partial(tx_1)}\frac{\partial(tx_1)}{\partial t} + \frac{\partial f(tx)}{\partial(tx_2)}\frac{\partial(tx_2)}{\partial t} + \ldots = \sum_{i=1}^{n}\frac{\partial f(tx)}{\partial(tx_i)}x_i =$$

2) $$= \frac{\partial}{\partial t}t^\alpha f(x) = \alpha t^{\alpha-1}f(x) \longrightarrow \text{setze } t=1$$

$$\implies \alpha f(x) = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}x_i$$

14)

$$w = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad x = r\cos\varphi, \quad y = r\sin\varphi$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) = \frac{\partial}{\partial r}\left(u_x\cos\varphi + u_y\sin\varphi\right) = u_{xx}\cos^2\varphi + u_{yy}\sin^2\varphi + 2u_{xy}\sin\varphi\cos\varphi$$

$$\frac{\partial^2 u}{\partial \varphi^2} = \frac{\partial}{\partial \varphi}\left(u_x(-r\sin\varphi) + u_y r\cos\varphi\right) = \underbrace{u_x(-r\cos\varphi) - u_y r\sin\varphi}_{-ru_r} + u_{xx}r^2\sin^2\varphi + u_{yy}r^2\cos^2\varphi - 2r^2u_{xy}\sin\varphi\cos\varphi$$

$$= -ru_r + r^2\left(u_{xx}\sin^2\varphi + u_{yy}\cos^2\varphi\right) - 2r^2u_{xy}\sin\varphi\cos\varphi$$

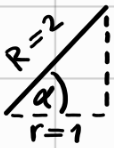
$$\implies u_{xx}\sin^2\varphi + u_{yy}\cos^2\varphi + u_{xx}\cos^2\varphi + u_{yy}\sin^2\varphi = u_{xx} + u_{yy} = w$$

$$= w = u_{\varphi\varphi}\frac{1}{r^2} + u_r\frac{1}{r} + u_{rr}$$

15)

•)

$$x^2+y^2+z^2 \leq 4 \quad \wedge \quad x^2+y^2 \leq 1 \quad \Longrightarrow$$



$z_g = \sqrt{4-1} = \sqrt{3}$

$\sin\alpha = \frac{\sqrt{3}}{2}$

$\Rightarrow \alpha = 60° = \pi/3$

1) Halbkugel: $\vec{n} = \frac{1}{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1/2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$I_1 = \int_0^{\pi/3}\int_0^{2\pi} \vec{F}\cdot\vec{n}\, r^2 \sin\vartheta \, d\varphi \, d\vartheta + \int_{2/3\pi}^{\pi}\int_0^{2\pi} \vec{F}\cdot\vec{n}\, r^2 \sin\vartheta \, d\varphi \, d\vartheta =$$

$$= \int_0^{\pi/3}\int_0^{2\pi} 1/2 \cdot 3xyz\, r^2 \sin\vartheta \, d\varphi \, d\vartheta + \int_{2/3\pi}^{\pi}\int_0^{2\pi} 1/2 \cdot 3xyz\, r^2 \sin\vartheta \, d\varphi \, d\vartheta$$

$$= \int_0^{\pi/3}\int_0^{2\pi} 3/2\, r^5 \sin^3\vartheta \cos\vartheta \underline{\sin\varphi\cos\varphi} \, d\varphi \, d\vartheta + \int_{2/3\pi}^{\pi}\int_0^{2\pi} 3/2\, r^5 \sin^3\vartheta \cos\vartheta \underline{\sin\varphi\cos\varphi} \, d\varphi \, d\vartheta = 0$$

* über 2π integriert

2) Cylinder:

$$\vec{n} = \frac{1}{r}\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$$I_2 = \int_{2/3\pi}^{2\pi/3}\int_0^{2\pi} \vec{F}\cdot\vec{n}\, r^2 \sin\vartheta \, d\varphi \, d\vartheta = \int_{2/3\pi}^{2\pi/3}\int_0^{2\pi} 2xyz\, r^2 \sin\vartheta \, d\varphi \, d\vartheta = 0$$ ... gleich wie oben

$$I = I_1 + I_2 = \underline{\underline{0}}$$

•) $I = \int_B \operatorname{div}(\vec{F})\, dV$, $\operatorname{div}(\vec{F}) = 0+0+0 = 0$

$$\Rightarrow I = \underline{\underline{0}}$$