28. Finden Sie für $n = 0, 1, 2$ eine Orthonormalbasis der Kugelflächenfunctionen und stellen Sie diese Funktionen dann als Funktionen von θ und ϕ (Kugelkoordinaten) dar.

29. Bestimmen Sie die Legendre-Polynome $P_n(x)$ für $n = 1, \ldots, 5$.

30. Sei P_n das n-te Legendre-Polynom. Bestimmen Sie das Integral

$$\oiint_{\|\vec{y}\|=1} P_m(\langle \vec{x}, \vec{y} \rangle) P_n(\langle \vec{y}, \vec{z} \rangle) \, do(\vec{y})$$

für $\|\vec{x}\| = \|\vec{z}\| = 1$ und $m, n \in \mathbb{N}_0$.

31. Für $\|\vec{x}\| = 1$ ($\vec{x} = (x, y, z)$) sei die Funktion

$$f(\vec{x}) = P_2(x) - 3P_3(y) + P_5\left(\frac{2x - y + 2z}{3}\right)$$

gegeben. Bestimmen Sie die Lösung von $\triangle u = 0$ auf dem Inneren der Einheitskugel, die der Randbedingung $u(\vec{x}) = f(\vec{x})$ für $\|\vec{x}\| = 1$ genügt.

28)

Normierung: $\int_{2\Omega} Y_{ni} Y_{mj} \, d\Omega = 4\pi \delta_{nm} \delta_{ij}$

0) aus Bsp. 19) $H_0 = \{1\}$

$$\int_0^{\pi}\int_0^{2\pi} (N \cdot 1)^2 \sin\vartheta \, d\varphi \, d\vartheta = 4\pi N^2 \overset{!}{=} 4\pi \quad \Rightarrow \quad N = 1$$

$\Rightarrow$ $Y_{00} = 1$

1) aus Bsp. 19) $H_1 = \{x, y, z\}$

$$\int_0^{\pi}\int_0^{2\pi} xy \sin\vartheta \, d\varphi \, d\vartheta = \int_0^{\pi}\int_0^{2\pi} \sin^3\vartheta \sin\varphi \cos\varphi \, d\varphi \, d\vartheta = 0$$

$$\int_0^{\pi}\int_0^{2\pi} N^2 x^2 \sin\vartheta \, d\varphi \, d\vartheta = \int_0^{\pi}\int_0^{2\pi} N^2 \sin^3\vartheta \cos^2\varphi \, d\varphi \, d\vartheta = \int_0^{\pi} N^2 \pi \sin^3\vartheta \, d\vartheta = \frac{4}{3}\pi N^2 \overset{!}{=} 4\pi$$

$\Rightarrow N = \sqrt{3}$

$\Rightarrow$
$Y_{10} = \sqrt{3}x = \sqrt{3}\sin\vartheta\cos\varphi$
$Y_{11} = \sqrt{3}y = \sqrt{3}\sin\vartheta\sin\varphi$
$Y_{12} = \sqrt{3}z = \sqrt{3}\cos\vartheta$

2) aus Bsp. 19) $H_2 = \{xy, xz, yz, x^2-y^2, x^2-z^2\}$

$$\int_0^{\pi}\int_0^{2\pi} N_1^2 x^2 y^2 \sin\vartheta \, d\varphi \, d\vartheta = \int_0^{\pi}\int_0^{2\pi} N_1^2 \sin^5\vartheta \cos^2\varphi \sin^2\varphi \, d\varphi \, d\vartheta = \int_0^{\pi} N_1^2 \sin^5\vartheta \frac{\pi}{4} \, d\vartheta = \frac{4}{15}\pi N_1^2 \overset{!}{=} 4\pi$$

$\Rightarrow N_1 = \sqrt{15}$

$$\int_0^{\pi}\int_0^{2\pi} xy(x^2-y^2)\sin\vartheta\, d\varphi\, d\vartheta = \int_0^{\pi}\int_0^{2\pi} \sin^5\vartheta \underbrace{\sin\varphi\cos\varphi\cos 2\varphi}_{1/4 \sin 4\varphi}\, d\varphi\, d\vartheta = 0$$

$$\int_0^{\pi}\int_0^{2\pi} N_2^2 (x^2-y^2)^2 \sin\vartheta\, d\varphi\, d\vartheta = \int_0^{\pi}\int_0^{2\pi} N_2^2 \sin^5\vartheta \cos^2 2\varphi\, d\varphi\, d\vartheta = \int_0^{\pi} N_2^2 \sin^5\vartheta\, \pi\, d\varphi\, d\vartheta = \frac{16}{15}\pi N_2^2 \overset{!}{=} 4\pi$$

$$\Rightarrow N_2 = \frac{\sqrt{15}}{2}$$

$$\int_0^{\pi}\int_0^{2\pi} N_2 (x^2-y^2)(x^2-z^2)\sin\vartheta\, d\varphi\, d\vartheta = \int_0^{\pi}\int_0^{2\pi} \frac{\sqrt{15}}{2}\sin^3\vartheta \cos 2\varphi \left(\sin^2\vartheta\cos^2\varphi - \cos^2\vartheta\right) d\varphi\, d\vartheta =$$

$$= \int_0^{\pi} \frac{\sqrt{15}}{2}\,\frac{\pi}{2}\sin^5\vartheta\, d\vartheta = \frac{\pi\sqrt{15}}{4}\,\frac{16}{15} = \frac{4\pi}{\sqrt{15}} = \left\langle \tfrac{\sqrt{15}}{2}(x^2-y^2) \middle| (x^2-z^2)\right\rangle$$

$\rightarrow$ Gram-Schmidt: $|V_5'\rangle = (x^2-z^2) - \left\langle \frac{\sqrt{15}}{2}(x^2-y^2) \middle| (x^2-y^2)\right\rangle \frac{1}{4\pi}\left(\frac{\sqrt{15}}{2}(x^2-y^2)\right)$

$$= x^2 - z^2 - \frac{1}{\sqrt{15}}\,\frac{\sqrt{15}}{2}(x^2-y^2) = \frac{1}{2}(y^2-x^2) + (x^2-z^2) = 1/2\,(x^2+y^2) - z^2$$

$$\int_0^{\pi}\int_0^{2\pi} N_3^2 \left(1/2\,(x^2+y^2) - z^2\right)^2 \sin\vartheta\, d\varphi\, d\vartheta = \int_0^{\pi}\int_0^{2\pi} N_3^2 \left(1/2 \sin^2\vartheta - \cos^2\vartheta\right)^2 \sin\vartheta\, d\varphi\, d\vartheta =$$

$$= \int_0^{\pi} 2\pi N_3^2 \left(1/4 \sin^5\vartheta - \sin^3\vartheta\cos^2\vartheta + \sin\vartheta\cos^4\vartheta\right) d\vartheta = 2\pi N_3^2 \left(1/4 \cdot 16/15 - 4/15 + 2/5\right) =$$

$$= \frac{4}{5}\pi N_3^2 \overset{!}{=} 4\pi \qquad \Rightarrow N_3 = \sqrt{5}$$

$$\Rightarrow \quad Y_{20} = \sqrt{15}\, xy = \sqrt{15}/2 \,\sin^2\vartheta \sin 2\varphi$$

$$Y_{21} = \sqrt{15}\, xz = \sqrt{15}/2 \,\sin 2\vartheta \cos\varphi$$

$$Y_{22} = \sqrt{15}\, yz = \sqrt{15}/2 \,\sin 2\vartheta \sin\varphi$$

$$Y_{23} = \frac{\sqrt{15}}{2}(x^2-y^2) = \sqrt{15}/2 \,\sin^2\vartheta \cos 2\varphi$$

$$Y_{24} = \sqrt{5}\left(1/2(x^2+y^2) - z^2\right) = \sqrt{5}/2\,(1 - 3\cos^2\vartheta)$$

29)

$$P_n = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n = \left(\frac{d}{dx}\right)^n \frac{1}{2^n n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^{2k}$$

$$= \frac{1}{2^n n!}\sum_{k\geq n/2}^{n}\binom{n}{k}(-1)^{n-k}\frac{(2k)!}{(2k-n)!}x^{2k-n} =$$

$$= \frac{1}{2^n}\sum_{k\geq n/2}^{n}\binom{n}{k}\binom{2k}{n}(-1)^{n-k}x^{2k-n}$$

für $n=1 \quad k\in\{1\}$
$n=2 \quad k\in\{1,2\}$
$n=3 \quad k\in\{2,3\}$
$n=4 \quad k\in\{2,3,4\}$
$n=5 \quad k\in\{3,4,5\}$

$$P_1 = \tfrac{1}{2}\left(2x^{2-1}\right) = \underline{x}$$

$$P_2 = \tfrac{1}{4}\left(2\cdot 1\cdot(-1)x^0 + 1\cdot 6\cdot 1\cdot x^2\right) = \underline{\tfrac{3}{2}x^2 - \tfrac{1}{2}}$$

$$P_3 = \tfrac{1}{8}\left(3\cdot 4\cdot(-1)x + 1\cdot 20\cdot 1\cdot x^3\right) = \underline{\tfrac{5}{2}x^3 - \tfrac{3}{2}x}$$

$$P_4 = \tfrac{1}{16}\left(6\cdot 1\cdot 1\cdot x^0 + 4\cdot 15\cdot(-1)x^2 + 1\cdot 70\cdot 1\cdot x^4\right) = \underline{\tfrac{1}{8}(35x^4 - 30x^2 + 3)}$$

$$P_5 = \tfrac{1}{32}\left(10\cdot 6\cdot 1\cdot x + 5\cdot 56\cdot(-1)x^3 + 1\cdot 252\cdot 1\cdot x^5\right) = \underline{\tfrac{1}{8}(63x^5 - 70x^3 + 15x)}$$

30)

$$I = \oiint_{\|\vec{y}\|=1} P_m\langle x|y\rangle \, P_n\langle y|z\rangle \, do(\vec{y})$$

$$P_m\langle x|y\rangle = \frac{1}{2m+1} \sum_{l=-m}^{m} Y_m^l(\vec{x}) Y_m^l(\vec{y})$$

$$I = \iint \frac{1}{2m+1} \sum_{l=-m}^{m} Y_m^l(\vec{x}) Y_m^l(\vec{y}) \frac{1}{2n+1} \sum_{k=-n}^{n} Y_n^k(\vec{y}) Y_n^k(\vec{z}) \, do(\vec{y})$$

$$= \frac{1}{2m+1} \frac{1}{2n+1} \sum_{l=-m}^{m} \sum_{k=-n}^{n} Y_m^l(\vec{x}) \, Y_n^k(\vec{z}) \iint Y_m^l(\vec{y}) \, Y_n^k(\vec{y}) \, do(\vec{y})$$

Kugelflächenfunktionen sind orthonormal:

$$\langle f_i | f_j \rangle = \int_{\partial\Omega} f_i \, f_j \, d\Omega = \delta_{ij} \, 4\pi$$

$$I = \frac{1}{2m+1} \frac{1}{2n+1} \sum_{l=-m}^{m} \sum_{k=-n}^{n} Y_m^l(\vec{x}) \, Y_n^k(\vec{z}) \, \delta_{mn} \, \delta_{lk} \, 4\pi =$$

$$= \left(\frac{1}{2n+1}\right)^2 \sum_{l=-n}^{n} Y_n^l(\vec{x}) \, Y_n^l(\vec{z}) \, \delta_{mn} \, 4\pi =$$

$$= \left(\frac{1}{2n+1}\right)^2 \frac{2n+1}{1} P_n(\langle x|z\rangle) \delta_{mn} \, 4\pi =$$

$$= \frac{4\pi}{2n+1} \delta_{mn} P_n\langle x|z\rangle$$

31)

$$u(r,\vartheta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(A_{nm}r^n+\cancel{\frac{B_{nm}}{r^{n+1}}}\right)Y_n^m(\vartheta,\varphi)$$! da $r\leq 1$

$$f(\vec{x})=\sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}Y_n^m(\vartheta,\varphi), \qquad A_{nm}=\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\vec{x})Y_n^m \sin\vartheta\, d\varphi\, d\vartheta$$

$$f(x)=P_2(x)-3P_3(y)+P_5\left(\frac{2x-y+2z}{3}\right)$$

$$=P_2(\langle x|V_1\rangle)-3P_3(\langle x|V_2\rangle)+P_5(\langle x|V_3\rangle) \qquad V_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad V_2=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad V_3={}^1/_3\begin{pmatrix}2\\-1\\2\end{pmatrix}$$

$$P_m\langle x|y\rangle=\frac{1}{2m+1}\sum_{l=-m}^{m}Y_m^l(\vec{x})Y_m^l(\vec{y})$$

$$4\pi A_{nm}=\int_{\partial\Omega}Y_n^m\left(\frac{1}{2\cdot2+1}\sum_{l=-2}^{2}Y_2^l(\vec{x})Y_2^l(\vec{V}_1)-\frac{3}{2\cdot3+1}\sum_{k=-3}^{3}Y_3^k(\vec{x})Y_3^k(\vec{V}_2)+\frac{1}{2\cdot5+1}\sum_{j=-5}^{5}Y_5^j(\vec{x})Y_5^j(\vec{V}_3)\right)d\Omega$$

$$=\left(\frac{1}{2\cdot2+1}\sum_{l=-2}^{2}Y_2^l(\vec{V}_1)\right)\int_{\partial\Omega}Y_n^m(x)Y_2^l(\vec{x})\,d\Omega-3\left(\frac{1}{2\cdot3+1}\sum_{k=-3}^{3}Y_3^k(\vec{V}_2)\right)\int_{\partial\Omega}Y_n^m(x)Y_3^k(\vec{x})\,d\Omega+\left(\frac{1}{2\cdot5+1}\sum_{j=-5}^{5}Y_5^j(\vec{V}_3)\right)\int_{\partial\Omega}Y_n^m(x)Y_5^j(\vec{x})\,d\Omega$$

$$=\left(\frac{4\pi}{2\cdot2+1}\sum_{l=-2}^{2}Y_2^l(\vec{V}_1)\right)\delta_{2n}\delta_{ml}-3\left(\frac{4\pi}{2\cdot3+1}\sum_{k=-3}^{3}Y_3^k(\vec{V}_2)\right)\delta_{3n}\delta_{mk}+\left(\frac{4\pi}{2\cdot5+1}\sum_{j=-5}^{5}Y_5^j(\vec{V}_3)\right)\delta_{5n}\delta_{mj}$$

$$=\frac{4\pi}{5}Y_2^m(\vec{V}_1)\delta_{2n}-3\frac{4\pi}{7}Y_3^m(\vec{V}_2)\delta_{3n}+\frac{4\pi}{11}Y_5^m(\vec{V}_3)\delta_{5n}$$

$$u(r,\vartheta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{nm}r^nY_n^m(\vartheta,\varphi)=\sum_{n=0}^{\infty}r^n\sum_{m=-n}^{n}A_{nm}Y_n^m(\vartheta,\varphi)$$

$$=\sum_{n=0}^{\infty}r^n\sum_{m=-n}^{n}Y_n^m(\vartheta,\varphi)\left(\frac{1}{5}Y_2^m(\vec{V}_1)\delta_{2n}-3\frac{1}{7}Y_3^m(\vec{V}_2)\delta_{3n}+\frac{1}{11}Y_5^m(\vec{V}_3)\delta_{5n}\right)$$

$$=\sum_{n=0}^{\infty}r^n\left(\delta_{2n}P_2(x)-3\delta_{3n}P_{3n}(y)+\delta_{5n}P_5\left(\frac{2x-y+2z}{3}\right)\right)$$

$$=r^2P_2(x)-3r^3P_3(y)+r^5P_5\left(\frac{2x-y+2z}{3}\right)$$