Pierre Liardet (1943–2014) in memoriam

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1 Introduction

Words and numbers cannot describe a man. The three authors of this paper had a special relationship with Pierre Liardet: as a former PhD student, as former collaborators, and also as friends. As such, we will try to give the reader an idea of the numerous contributions of Pierre to number theory and related fields and further, of the fine man behind these results.

Pierre Liardet was born in Gardanne, close to Aix-en-Provence, on March 20, 1943. After receiving his baccalauréat en mathématiques et technologie at the Lycée Saint-Éloi in Aix-en-Provence in 1961, he studied mathematics at the University in Marseille finishing with his doctorate in 1970. Afterwards, he obtained a position as a "Maître Assistant" at the Université de Provence and submitted his Thèse d'État in 1975. Soon afterwards he became a professor and later "Professeur de première classe" at this university. From 1990–2010 he directed the team "Dynamique, Stochastique et Algorithmique" (DSA).

He was an editor of the "Journal de Théorie des Nombres de Bordeaux" and of "Uniform Distribution Theory". In 2005 he organised the biannual international conference on number theory "Journées Arithmétiques". Pierre liked to travel and visited many countries all over the world. This is reflected by the fact that he had coauthors from many countries: Austria, Canada, China, France, Hungary, Japan, the Netherlands, Poland, Romania, USA. He was professor emeritus since 2012 and was still active in research until his last days. He passed away on August 29, 2014.

Even if he turned his interests rapidly to analysis, Pierre thought as an algebraist, as he was more interested in the structures governing mathematical objects and in the operations on them, than in the objects themselves. In this sense he was an inheritor of Bourbaki's tradition.

Figure 1: Pierre hiking in the French Alps

2 Algebraic number theory and algebraic geometry

Pierre Liardet wrote his PhD thesis [L2] under the advice of Gérard Rauzy on a subject from algebraic number theory. The main interest of this research were rational transformations, which leave certain sets of algebraic numbers stable (*cf.* [L3]). One of his results was published in [L1] and concerned a stability property of the set Θ_n of algebraic integers with exactly n conjugates of modulus greater than 1, and all the other conjugates of modulus smaller than 1 (the case $n = 1$ are the Pisot-Vijayaraghavan numbers).

Theorem 1. *Let f be a non-constant rational function defined over an algebraic extension of $\mathbb{Q}$, and E be the complement in Θ_n of a bounded set. Then*

$$\forall \theta \in E : f(\theta) \in \Theta_n \Rightarrow \exists m \in \mathbb{N} : f(x) = \pm x^m.$$

After finishing his doctorate he continued working in algebraic number theory and algebraic geometry until his Thèse d'État [L6] in 1975, devoted to the stability of algebraic properties of sets of polynomials and rational functions (*cf.* [L9, L10]). Two outstanding results from this work were published, namely the disproof of a conjecture of W. Narkiewicz [L4, L8], and the proof of a conjecture of S. Lang. The latter is formulated in terms of intersections of abelian varieties with algebraic curves in [L5]; we give the more elementary formulation from [L7].

Theorem 2. *Let Γ_0 be a finitely generated subgroup of $\mathbb{C}^*$ and let $\Gamma = \{x \in \mathbb{C}^* \mid \exists n \in \mathbb{N}, n \neq 0 : x^n \in \Gamma_0\}$. If $P(X,Y) \in \mathbb{C}[X,Y]$ is such that $P(\alpha,\beta) = 0$ for infinitely many $\alpha, \beta \in \Gamma$, then there exist non-zero integers u, v and $a, b \in \Gamma$, such that $P(ax^u, bx^v)$ is identically zero.*

After this fruitful period, he turned his interest mostly into ergodic theory, dynamical systems, and uniform distribution of sequences.

3 Ergodic theory

Pierre started with reading the most important authors and assimilated the works of Conze, Furstenberg, Kakutani, Keane, Klaus Schmidt, and others. He published papers in "pure" ergodic theory between 1978 [L11] and 2000 [L36]. The first paper where Pierre grappled with ergodic theory is [L11]. It is noteworthy that this paper already deals with the main themes of his further research articles.

Let X be a compact metrisable space, G a compact metrisable group, and $\varphi\colon X \to G$ a continuous map. For T a transformation on X, one can define the skew product $(X \square_\varphi G, T_\varphi)$ by

$$T_\varphi\colon X \times G \to X \times G, \qquad T_\varphi(x,g) = (Tx, g \cdot \varphi(x))$$

and ask about relations between properties of (X,T) and properties of $(X \square_\varphi G, T_\varphi)$. Skew products both furnish an inexhaustible source of examples, allow to describe by isomorphism several flows associated to sequences (there are numerous such examples in Pierre's work, see for instance [L42]) and illustrate the general idea that a good way to understand mathematical objects is to describe how to act on them. For those reasons, Pierre had a continuous and deep interest in this type of product systems. In this first paper, beginning with considerations from uniform distribution theory, Pierre gave characterizations of ergodicity and weak mixing of large families of skew products, extending results of Furstenberg and Conze; he could also mold a theorem of Veech in a more precise form.

The use of the ergodicity of certain types of skew products was one of Pierre's central techniques to prove results for certain sequences like *irregularities of distribution* (see Section 4). Usually, these sequences were shown to be the orbits of points in some other space under an *odometer* (adding machine transformation, see Section 5), which translates to the study of a group rotation. Further investigations on skew products can be found in [L13] and [L36]. K. Schmidt (probably) introduced him to $\mathbb{Z}^d$-actions, on which he published [L28] and [L30]. His deep interest on spectral approaches originates from here.

The paper [L31] deals with the speed of convergence in Birkhoff's ergodic theorem. Let T be a homeomorphism of the compact metric space X and $C_0(X)$ be the space of all real-valued continuous functions f having zero integral with respect to a fixed T-invariant aperiodic measure λ. Then there exists a G_δ set in $C_0(X)$ such that distributions of the random variables $\frac{1}{c_n}\sum_{0\le j<n} f\circ T^j$ are dense in the set of all probability measures on the real line, where $c_n \uparrow \infty$ and and $c_n/n \to 0$. The second part of the paper is devoted to irrational rotations R_α on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Functions defined on $\mathbb{T}$ are often called *cocycles* in this context. Moreover, a cocycle F is a *coboundary* if there exists G such that $F = G - G\circ R_\alpha$. (Hence, coboundaries F make the sums $\sum F\circ T^n$ telescopic). The last theorem in [L31] shows that if α has bounded partial quotients, then there exists a dense G_δ subset of the set of absolutely continuous cocycles with zero integral, which are not coboundaries. Similar results are shown when the sequence of partial quotients is not bounded, and convergence rates of $(S_n)_n$ are given for several subspaces of cocycles in the latter case.

Studies in the same flavour have been done for Anzai's skew product extensions of the two-dimensional torus, that Pierre investigated in [L21] and [L25]. The monograph [L21] contains an elaborate study of cocycles which are either absolutely continuous or step functions, and a general discussion on group extensions, which yield families of interesting examples. Pierre supervised four PhD theses on subjects related to his interests in ergodic theory: J. Mouline (1990), Y. Lacroix (1992), E. Olivier (1997), and C. Guille-Biel (1997).

4 Uniform distribution

Pierre's contributions were mainly focused upon two topics, sequences generated by certain digit expansions of real numbers, and by the application of notions of ergodic theory. Two keywords in this context are *digital sequences* and *irregularities of distribution*.

The dynamical systems approach that Pierre had chosen to study statistical properties of a sequence $\omega = (x_n)_n$ in a

Figure 2: Pierre with Oto Strauch at a conference in Smolence in 2012

compact metrisable space X, with $X = [0,1)^s$, $s \ge 1$, the s-dimensional torus as the most important case, proceeds in two steps: first, one has to find the dynamical system behind the given sequence ω, and, in the second step, one has to employ properties of this system to derive distribution properties of the sequence.

To illustrate this approach, let $\omega = (\mathbf{x}_n)_{n\ge 0}$ be a sequence on the torus $[0,1)^s$. For a (Lebesgue-) measurable subset A of $[0,1)^s$, let $\lambda_s(A)$ denote its Lebesgue measure. The quantity

$$R_N(A,\omega) = \sharp\{n, 0 \le n < N : \mathbf{x}_n \in A\} - N\lambda_s(A)$$

is called *the local discrepancy* of the first N points of ω for the set A or, for short, *the remainder* of A.

For a given sequence ω, is there any hope that the remainders $R_N(A,\omega)$ stay bounded in N? In an impressive series of papers, Wolfgang Schmidt investigated this question of *bounded remainder sets* by techniques from metric number theory. Here, two questions are central:

(i) for sets A with bounded remainder, determine the set of possible volumes $\lambda_s(A)$ (the so-called *admissible volumes*), and

(ii) identify all bounded remainder sets, at least in the case when they are s-dimensional intervals.

Question (i) may be related to topological dynamics and ergodic theory (*cf.* [3, 8]). From this approach one obtains the result that the set of admissible volumes stems from eigenvalues of an isometric operator. The main contributions of Pierre in this context concern general versions of the *coboundary theorems* that are involved here. This allowed Pierre to give at least partial answers to the bounded remainder sets problem in case of certain sequences (see [L16]). This area is still active, which is shown by the recent breakthrough in [4].

Question (ii) is much more difficult to attack and results are known only in very special cases for ω and for A. For example, this is the case for Kronecker sequences $(n\alpha \mod 1)_{n\ge 0}$ in $[0,1)^s$ (see [5] for $s = 1$ and [L16] for $s > 1$). An important result of Pierre deals with polynomial sequences $(p(n) \mod 1)_{n\ge 0}$ (*cf.* [L16]). In the same paper, as an example to his general approach, he has shown the following. Let α be irrational and let $s_g(n)$ be the sum of digits of n to the base $g \ge 2$. Then the only intervals I of the 1-torus $[0,1)$ which are bounded remainder sets for the sequence $(\alpha s_g(n))_{n\ge 0}$ in $[0,1)$ are the trivial ones, that is to say, $|I| = 0$ or 1.

In the early years of Pierre's examination of uniform distribution and discrepancy theory, he gave an ingenious and transparent new proof of W. Schmidt's lower bound for the discrepancy (i.e., the supremum $D_N(\omega)$ of the local discrepancies for all intervals $A \subset [0,1)$) of a sequence $\omega = (x_n)_{n\geq 0}$ in $[0,1)$,

$$D_N(\omega) = \sup_A \left| \frac{R_N(A,\omega)}{N} \right| \geq C \frac{\log N}{N}, \text{ for infinitely many } N,$$

refining Schmidt's constant $C = \frac{1}{66 \log 4}$ to $\frac{3}{40 \log 5}$. Unfortunately, this result was only published in an institute report [L12]; the proof can be found in [2, pp. 41–44]. Pierre mentioned at several occasions that it was his dream to find the best possible constant C, and that its value should be an eigenvalue of an operator related to the problem.

A second series of Pierre's contributions to the theory of uniform distribution of sequences concerned the interplay between sequences related to numeration systems and the qualitative behavior of such sequences. Historically, these investigations started in close cooperation with Coquet [L15] on notions of (statistical) independence of sequences and then proceeded to measure-theoretic aspects of numeration systems in cooperation with Grabner and Tichy [L34], and with Baláž and Strauch [L54].

5 Systems of numeration and digital expansions

A significant part of Pierre's achievement concerns numeration, sequences related to diverse representations of real, complex or natural numbers, and the underlying dynamical systems. Those themes traversed Pierre's mathematical life for 25 years, and he was able to fruitfully exploit their interplay. One can emphasise three seminal papers in this area.

The first one - [L18] - was very important for him, scientifically as well as humanly. During the seventies and eighties, there was an intense interest on statistic properties of sequences, originated in early works of Wiener. Let W be the set of complex sequences $u = (u_n)_n$ such that the correlations

$$\gamma_u(m) = \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} u_{m+n}\overline{u_n}$$

exist for all $m \in \mathbb{N}$. Extended to negative integers by $\gamma_u(-m) = \overline{\gamma_u(m)}$, the correlations $(\gamma_u(m))_m$ form a positive definite sequence, hence are the Fourier coefficients of some Borel measure on the one-dimensional torus $\mathbb{T}$ by the Bochner-Herglotz theorem. This measure is called the *spectral measure* of u. Investigations on this measure yielded to several concepts and results on pseudo-random sequences. Bésineau, Coquet, Kamae, Mendès-France, M. Queffélec, Rauzy, Rhin and others made several contributions to the subject. Pierre was both interested and engaged in that framework and published [L14], where the authors used their abilities in ergodic theory to show several results on spectral disjointness. Pierre published a continuation [L15] of this work, which contains among other things in a more general setting the following result:

Theorem 3. *Let θ be a q-normal real number ($(q^n\theta \pmod 1))_n$ being uniformly distributed) and P a real polynomial. Then the two sequences $(q^n\theta \pmod 1))_n$ and $(P(n) \pmod 1))_n$ are spectrally disjoint.*

Meanwhile, Coquet passed away and it affected Pierre a lot. Coquet's wife communicated Pierre the private drafts of her husband and Pierre wrote a synthesis of them in [L18]. That included the redaction of some proofs, that had only been sketched by Coquet. In this paper, one encounters for the first time in Pierre's work the notion of *systems of numeration*, that is an increasing sequence of integers $(G_n)_n$ with $G_0 = 1$. Any integer $m \in \mathbb{N}$ can be expressed uniquely as

$$m = \sum_{k\geq 0} \varepsilon_k(m) G_k, \text{ provided that } \forall n\colon \sum_{k<n} \varepsilon_k(m) G_k < G_n.$$

Pierre developed the theory of systems of numeration in [L29], which is his most cited paper and the second one we emphasise. Here, the closure of the set of representations is considered, namely the set

$$\mathcal{K}_G = \{(\varepsilon_k)_{k\geq 0};\ \forall n\colon\ \varepsilon_0(m)G_0 + \cdots \varepsilon_{n-1}(m)G_{n-1} < G_n\}.$$

An extension of the addition $\tau\colon m \mapsto m+1$ is constructed on $\mathcal{K}_G$, which yields a dynamical system $(\mathcal{K}_G, \tau)$, called the *odometer*. Surjectivity, continuity and minimality are discussed. Special and important examples are Ostrowski systems, based on continued fraction expansions, where $G_n = q_n$, for $\alpha = [0; a_0, a_1, \ldots] \in [1/2, 1)$, and $\frac{p_n}{q_n} = [0; a_0, a_1, \ldots, a_n]$, and sequences $(G_n)_n$ satisfying linear recursions of particular types. The study of odometers was pursued in several papers, especially [L35] and [L40]. Pierre together with several coauthors also published an overview on numeration in [L45], where they introduced a notion of *fibred numeration system* after Schweiger [7].

Arithmetical functions related to systems of numeration have been extensively studied by the Austrian school and Pierre collaborated in several papers on the subject. We cite [L33], [L39] which he liked very much, and [L42, L47, L49, L50]. Some of those papers were devoted both to arithmetical functions and odometers. Pierre liked to construct dynamical systems and to investigate them, deciphering their structures, comparing them through conjugacy and other accurate notions of isomorphism.

The third paper we want to emphasise is [L20]. Let us consider a trigonometric polynomial

$$P(\mathrm{e}^{i\theta}) = \sum_{n=1}^{N} \epsilon_n \mathrm{e}^{in\theta} \tag{1}$$

with $\epsilon_n \in \{-1, 1\}$. Parseval Theorem shows that $\|P\|_\infty \geq \sqrt{N}$ and it follows from results of Salem and Zygmund that the expected order of magnitude for $\|P\|_\infty$ is $(N \log N)^{1/2}$ for almost all such sequences $\epsilon_n \in \{-1, 1\}$. In 1951, Shapiro, in his thesis, gave an inductive construction of a sequence $(\epsilon_n)_n$ such that $|P(\mathrm{e}^{i\theta})| \ll N^{1/2}$. Rudin rediscovered the sequence in 1959 and it has been since called *Rudin-Shapiro sequence*. It turns out that ϵ_n has a simple digital interpretation: it counts the parity of the number of appearances of the string "11" in the binary expansion of n.

In [L20], Allouche and Liardet extended this result in a way which was typical of Pierre's thinking. First counting occurrences of subwords of the type $a_0 a_1 \cdots a_d$ with $a_0 a_d \neq 00$ (*generalised Rudin-Shapiro sequences*), they focus on the properties of such sequences and introduce a general abstract machinery including those particular cases. A notion of *chained map* on an alphabet $\mathcal{A}$ taking its values in a compact metrisable group G is introduced and investigated, especially through an appropriate matrix formalism. Those maps f satisfy the relation

$$f(\alpha\beta\gamma) = f(\alpha\beta)f(\beta)^{-1}f(\beta\gamma)$$

for non-empty words α, β and γ and are proven to be q-automatic if $\mathcal{A} = \{0, 1, \dots, q-1\}$ and if G is finite. Norm bounds for functions of the form (1) are obtained, where the exponential function is replaced by an irreducible representation of the group G. In a further section, the generalised Rudin-Shapiro sequences are studied for themselves from a dynamic point of view and the authors prove the following:

Theorem 4. *Let $u\colon \mathbb{N} \to \mathbb{U}$ be a generalised multiplicative Rudin-Shapiro sequence. Then u has a correlation function and its spectral measure is the Lebesgue measure.*

The paper ends with the following announcement: "*In a forthcoming paper we shall study the flows associated to chained sequences, proving that they are (except for degenerate cases) strictly ergodic and can be obtained as group extensions of a a-adic rotation; we shall also give the spectral study of these flows.*". Eighteen years later, this program was completed in [L52].

Other number representations, such as continued fractions and Engel series, also caught Pierre's attention. Methods from ergodic theory, especially skew products, are used in [L17] to prove uniform distribution of the numerators p_n and denominators q_n of the convergents in residue classes modulo m. In [L22] the set $E(\alpha)$ consisting of all real numbers β such that $(\beta q_n(\alpha))_n$ tends to 0 modulo 1 is studied depending on diophantine properties of α.

In [L32] (infinite) automata are constructed, which compute the partial quotients of $f(x)$ from the partial quotients of x for certain rational functions f. For instance, the partial quotients of $\sqrt[3]{2}$ can be computed by such an automaton. This work was continued in [L37], where (infinite) automata are given, which compute the Engel series expansion from the continued fraction and *vice versa*.

Pierre's work in that context was the basis for seven doctoral theses: D. Barbolosi (1988), C. Faivre (1990), P. Stambul (1994), G. Barat (1995), N. Loraud (1996), M. Doudékova-Puysdebois (1999), and I. Abou (2008). Several of the above mentioned results were generalised and extended in their work.

6 Applications

It was very typical for Pierre and his point of view of mathematics, and science in general, that he was trying to make his knowledge, especially of ergodic theory, accessible to people working in other areas of mathematics, but also to more applied scientists, especially computer scientists.

At the Conference on Uniform Distribution Theory in Marseille in January 2008, Pierre brought up the idea to make methods from ergodic theory accessible for, and more popular amongst people working on low-discrepancy sequences. This led to many informal discussions, to an educational workshop "Dynamical Aspects of Low-Discrepancy-Sequences" held in Linz in September 2009, and finally, the survey article [L57]. Pierre's part of this survey was a very general but still accessible and pedagogical description of the cutting-stacking construction for interval exchange maps which originated in Rokhlin's work and was developed further by Kakutani.

Over the years, he had intensive cooperations on cryptographic and algorithmic applications, where he contributed his mastery of dynamical systems. He always intended to make the descriptions of these notions and techniques understandable for his readers, but he was also interested in the applications themselves. It is noteworthy that his son Pierre-Yvan is working in computer science, and Pierre was regularly discussing scientific issues with him, even if they did not formally collaborate. Pierre-Yvan likes to narrate how his father helped him for a crucial counting that he needed for [6]. Father and son regularly climbed the Sainte-Victoire - Cézanne's mountain - and it was the occasion of many mathematical discussions. Pierre-Yvan introduced his father to his cryptographic questions and Pierre used his theoretic knowledge to tackle the problem.

Obvious applications of dynamical systems are random number generators, which are used in different branches of computer science. Evolutionary algorithms are special stochastic search algorithms, which allowed Pierre to introduce his view of dynamical systems to optimise the implementation. The results of these efforts are the papers [L41, L44, L51]. The two papers [L55, L56] propose algorithms for the generation of random selections of k elements out of n. These contributions give a considerable improvement in the rate of convergence towards uniformity. Similarly, a randomisation in the construction of the implementation of finite fields by tower fields is proposed in [L61], in order to make the AES cryptosystem less vulnerable to side-channel attacks.

A further branch of Pierre's research with application to computer science is the prediction of binary sequences with automata. This started with Annie Broglio's PhD thesis in 1991, which led to the paper [L23], and continued with [L38].

Pierre supervised three PhD theses on subjects related to applications in computer science, cryptography and automata: A. Broglio (1991), B. Peirani (1994), and S. Rochet (1998).

7 Concluding remarks

Pierre Liardet left a lasting influence in all research areas he was working in. His results on the intersection of algebraic curves with abelian varieties solving a conjecture of Lang were amongst the first results in this direction, see [1].

After he left the area of algebraic geometry and started to work in ergodic theory, he was influenced by the work of Coquet and Mendès France. He contributed several important works on the dynamics of the orbit closure of sequences defined by digital functions. This had been of interest from work of Wiener and Mahler from 1927 and still provides interesting examples of dynamical systems. Besides his studies on the interplay of number theory and ergodic theory he contributed to the development of ergodic theory itself by deep investigations about skew products. The propagation of the study of odometers and other digital representations as dynamical systems was his personal concern. By the dissemination of this idea he influenced – directly and indirectly –

many young mathematicians, especially in France and Austria. The influence of his driving idea of finding dynamical systems behind many mathematical questions even led him to his cooperations with computer scientists. Reading the papers discussed in Section 6 one can immediately pin down his contribution, namely making apparent the dynamical system behind the seemingly unrelated problem originating from computer science.

Pierre Liardet had great influence on the current development of the theory of uniform distribution of sequences modulo one. He was one of the innovators to apply concepts and results of ergodic theory to open problems concerning irregularities of distribution of sequences, which has allowed to advance after about two decades of a stand-still after the ground-breaking results of K. F. Roth and W. Schmidt to this field. Further, his insistence on what he called "the dynamic point of view" of sequences has changed the attitude with which various types of sequences are studied nowadays and has helped to develop a broader view of the phenomena that appear. Pierre was also one of the leading researchers to bring the theory of automata, ergodic theory, and the theory of uniform distribution together to study the fine structure of various types of sequences.

Pierre liked to meet colleagues and organised several meetings and conferences, among others the Journées Arithmétiques in Marseille in 2005, and several meetings at the CIRM on number theory, ergodic theory, uniform distribution and their interplay. He also created more specialised events, like the "Journées de numération", which has become a regular meeting of researchers working in various aspects of digital expansions, ranging from automata theory to topology and diophantine approximation. The last conference in this series took place in Nancy in June 2015. Indeed, during the last thirty years, Numeration has become a dynamical field and Pierre was one of the leading figures of the group.

We have given an overview of Pierre's scientific work, achievements, and also sketched his influence in several fields. Yet besides his constant interest in science, especially mathematics but also medicine and computer science, he was a loving husband to Josy, a father to Pierre-Yvan, Frédéric, who survived his father only by four months, and Christine, and a caring grandfather to his grandchildren. Pierre was a good friend and colleague to many of us. He could be missionary for his mathematical point of view, but he was always open for discussion and dialogue. His mathematical thoughts and his approach to doing mathematics have influenced many of us, especially the three authors. We will miss his constant input and encouragement.

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