Uniform Distribution of generalized Kakutani's sequences of partitions

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Abstract

We consider a generalized version of Kakutani's splitting procedure where an arbitrary starting partition π is given and in each step all intervals of maximal length are split into m parts, according to a splitting rule ρ. We give conditions on π and ρ under which the resulting sequence of partitions is uniformly distributed.

1 Introduction

In this paper we study a generalization of the Kakutani splitting procedure, which was originally introduced in [5].

Definition 1.1 (Kakutani splitting procedure) *If $\alpha \in (0,1)$ and $\pi = \{[t_{i-1}, t_i] : 1 \leq i \leq k\}$ is any partition of $[0,1]$, then $\alpha\pi$ denotes its α -refinement which is obtained by subdividing all intervals of π having maximal length in two parts, proportional to α and $1-\alpha$, respectively.*
The so-called Kakutani's sequence of partitions $(\alpha^n \omega)_{n\in\mathbb{N}}$ is obtained as the successive α -refinement of the trivial partition $\omega = \{[0,1]\}$.

Definition 1.2 (Uniform distribution of sequences of partitions) *Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of partitions of $[0,1]$, with*

$$\pi_n = \{[t_{i-1}^n, t_i^n] : 1 \leq i \leq k(n)\}.$$

Then π_n is uniformly distributed (u.d.), if for any continuous function f on $[0,1]$

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(t_i^n) = \int_0^1 f(t)dt. \tag{1}$$

Remark 1.1 *For a sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ we define the associated sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ by*

$$\mu_n = \frac{1}{k(n)} \sum_{i=1}^{k(n)} \delta_{t_i^n},$$

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where δ_t denotes the Dirac measure concentrated at t. Weak convergence of $(\mu_n)_{n\in\mathbb{N}}$ to the Lebesgue measure on $[0,1]$ is equivalent to condition (1). In other words, a sequence of partitions is u.d. if and only if for every interval $[a,b]\subset[0,1]$

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{k(n)}\mathbf{1}_{[a,b]}(t_i^n)}{k(n)}=b-a.$$

Kakutani [5] proved that for any $\alpha\in(0,1)$ the sequence of partitions $(\alpha^n\omega)_{n\in\mathbb{N}}$ is uniformly distributed. The properties of the sequence $(\alpha^n\omega)_{n\in\mathbb{N}}$ and related problems have been investigated by many authors. For example, see [1] and [7] for a modification of $(\alpha^n\omega)_{n\in\mathbb{N}}$ where the intervals of maximal length are split at a random position. Carbone and Volčič [3] generalized the splitting procedure for sequences of partitions of $[0,1]^d$, $d\geq 2$, and derived a generalization of Kakutani's result in higher dimensions. Recently, the following further modification of Kakutani's splitting procedure was presented by Volčič [8].

Definition 1.3 (ρ - refinement) *Let ρ denote a non-trivial finite partition of $[0,1]$. Then the ρ -refinement of a partition π of $[0,1]$, denoted by $\rho\pi$, is given by subdividing all intervals of maximal length positively homothetically to ρ.*

Volčič [8] proved, by using arguments from ergodic theory, that the sequence $(\rho^n\omega)_{n\in\mathbb{N}}$ is u.d. for every finite partition ρ. Furthermore, he investigated the behavior of associated uniformly distributed sequences of points. The discrepancy of sequences of partitions constructed as ρ -refinements of ω is discussed in Carbone [2] and Drmota and Infusino [4]. The results of Drmota and Infusino are based on the analysis of a special tree evolution process, namely the Khodak algorithm [6], where the generation of nodes has a similar behavior as the splitting of intervals in the Kakutani splitting sequence.
So far results on the uniform distribution of sequences of partitions were only available in the case when the starting partition π is the trivial partition ω. A simple example shows that there exist starting partitions π for which the sequence $(\rho^n\pi)_{n\in\mathbb{N}}$ is not uniformly distributed. Consider $\pi=\left\{\left[0,\frac{2}{5}\right],\left[\frac{2}{5},1\right]\right\}$ and $\rho=\left\{\left[0,\frac{1}{2}\right],\left[\frac{1}{2},1\right]\right\}$. In this case the splitting procedure operates alternating on $\left[0,\frac{2}{5}\right]$ and $\left[\frac{2}{5},1\right]$ and hence the sequences of associated measures corresponding to the subsequences $(\rho^{2n}\pi)_{n\in\mathbb{N}}$ and $(\rho^{2n+1}\pi)_{n\in\mathbb{N}}$ converge to different measures. Volčič [8] formulated the problem in the following form:

> It is worth noticing that it is necessary to put some restriction on the partition π (even in the simplest case of the Kakutani splitting procedure) if we hope for uniform distribution of $(\rho^n\pi)_{n\in\mathbb{N}}$. It would be interesting to find significant sufficient conditions on π in order to obtain the uniform distribution of $(\rho^n\pi)_{n\in\mathbb{N}}$ even for the case of Kakutani's splitting procedure.

The purpose of the present paper is to present a full solution of this problem.

2 The uniform distribution of generalized Kakutani's sequences of partitions

In the sequel we consider a partition ρ of $[0,1]$ consisting of $m\geq 2$ intervals of lengths $p_1,\ldots,p_m$, and a starting partition π of $[0,1]$ consisting of $l\geq 2$ intervals of lengths $\alpha_1,\ldots,\alpha_l$. In the sequel let H denote the entropy of the probability distribution $p_1,\ldots,p_m$, which is defined as

$$\mathrm{H}=p_1\log\left(\frac{1}{p_1}\right)+\ldots+p_m\log\left(\frac{1}{p_m}\right).$$

Definition 2.1 (Rationally related) *The numbers* $\log\left(\frac{1}{p_1}\right),\dots,\log\left(\frac{1}{p_m}\right)$ *are called rationally related if there exists a positive real number* Λ *such that*

$$\log\left(\frac{1}{p_j}\right) = \nu_j\Lambda, \quad \nu_j \in \mathbb{Z}, j = 1,\dots,m.$$

Without loss of generality we choose Λ *as large as possible, which is equivalent to assuming* $\gcd(\nu_1,\dots,\nu_m) = 1$*. If the numbers* $\log\left(\frac{1}{p_1}\right),\dots,\log\left(\frac{1}{p_m}\right)$ *are not rationally related, they are called irrationally related.*

Remark 2.1 *Note that the numbers* $\log\left(\frac{1}{p_1}\right),\dots,\log\left(\frac{1}{p_m}\right)$ *are rationally related if and only if all fractions*

$$\frac{\log p_i}{\log p_j}, \quad i,j = 1,\dots,m,$$

are rational.

For a fixed real number $\epsilon \in (0, p_{\min})$, where $p_{\min} = \min\{p_1,\dots,p_m\}$, let $\mathcal{I}_\epsilon$ denote the set of all intervals that appear in the sequence $(\rho^n\omega)_{n\in\mathbb{N}}$ and have length greater than or equal to ϵ. Let $\mathcal{E}_\epsilon$ be the set of intervals which are generated by splitting an interval in $\mathcal{I}_\epsilon$ and which have length l satisfying $p_{\min}\epsilon \le l < \epsilon$. Denote by $M_\epsilon = |\mathcal{E}_\epsilon|$ the cardinality of $\mathcal{E}_\epsilon$. Note that the set $\mathcal{E}_\epsilon$ changes only for certain values of ϵ, more precisely when ϵ equals the length of at least one interval appearing in $(\rho^n\omega)_{n\in\mathbb{N}}$.

We will use the following result from [4].

Lemma 2.1 *Let* M_ϵ *be defined as above. Then*

1. *if* $\log\left(\frac{1}{p_1}\right),\dots,\log\left(\frac{1}{p_m}\right)$ *are rationally related, let* Λ *be the largest real number for which* $\log\left(\frac{1}{p_j}\right)$ *is an integer multiple of* Λ*, for* $j = 1,\dots,m$*. Then there exist a real number* $\eta > 0$ *and an integer* $d \ge 0$ *such that*

$$M_\epsilon = \frac{m-1}{\epsilon H} Q_1\left(\log\left(\frac{1}{\epsilon}\right)\right) + \mathcal{O}\left((\log(\epsilon))^d \epsilon^{-(1-\eta)}\right), \tag{2}$$

   where

$$Q_1(x) = \frac{\Lambda}{1-e^{-\Lambda}} e^{-\Lambda\{\frac{x}{\Lambda}\}}$$

   and $\{y\}$ *denotes the fractional part of* y*.*

2. *If* $\log\left(\frac{1}{p_1}\right),\dots,\log\left(\frac{1}{p_m}\right)$ *are irrationally related, then*

$$M_\epsilon = \frac{m-1}{\epsilon H} + o\left(\frac{1}{\epsilon}\right). \tag{3}$$

The following theorem gives sufficient and necessary conditions on π and ρ under which $(\rho^n\pi)_{n\in\mathbb{N}}$ is uniformly distributed.

Theorem 2.1 *Let* $\alpha_j, j = 1,\dots,l$ *denote the lengths of the intervals of the starting partition* π*. Then the sequence* $(\rho^n\pi)_{n\in\mathbb{N}}$ *is uniformly distributed if and only if one of the following conditions is satisfied:*

(I) the real numbers $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ *are irrationally related or*

(II) the real numbers $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ *are rationally related with parameter* Λ *and the lengths of the intervals of* π *can be written in the form*

$$\alpha_i = c e^{v_i \Lambda}, \quad c \in \mathbb{R}^+, v_i \in \mathbb{Z}, \tag{4}$$

for $i = 1, \ldots, l$.

Remark 2.2 *Condition (II) includes the special case that the starting partition* π *is a partition consisting of intervals having the same length, and in particular the case when the starting partition is the trivial partition* ω.

For illustration, the next corollary characterizes the starting partitions π for which the original Kakutani's sequence of partitions is u.d. .

Corollary 2.1 *Let the sequence of partitions* $(\rho^n \pi)_{n\in\mathbb{N}}$ *be defined as a* ρ *-refinement with* $\rho = [[0,p],[p,1]]$ *and* $\pi = [[0,\alpha],[\alpha,1]]$. *Then* $(\rho^n \pi)_{n\in\mathbb{N}}$ *is u.d. if and only if one of the following conditions is satisfied:*

(i) $\log(p)/\log(1-p)$ *is irrational, or*

(ii) $\log\left(\frac{1}{p}\right)$ *and* $\log\left(\frac{1}{1-p}\right)$ *are rationally related with parameter* Λ *and* $\alpha = \frac{1}{e^{k\Lambda}+1}$ *for* $k \in \mathbb{Z}$.

The next theorem describes the asymptotic behavior of the distribution of $(\rho^n \pi)_{n\in\mathbb{N}}$ for those cases which are not covered by Theorem 2.1.

Theorem 2.2 *Assume that neither condition (I) nor condition (II) of Theorem 2.1 is satisfied. Then for any interval* $A = [a,b] \subset [0,1]$ *which is completely contained in the* i*-th interval of the starting partition* π *for some* $i, 1 \le i \le l$, *we have*

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{k(n)} \mathbf{1}_{[a,b]}(t_j^n)}{k(n)} = c_1(b-a),$$
$$\liminf_{n\to\infty} \frac{\sum_{j=1}^{k(n)} \mathbf{1}_{[a,b]}(t_j^n)}{k(n)} = c_2(b-a),$$

where

$$c_1 = \left(\sum_{j=1}^{l} \alpha_j \exp\left(-\Lambda\left\{\frac{\log(\alpha_j)-\log(\alpha_i)}{\Lambda}\right\}\right)\right)^{-1} > 1,$$
$$c_2 = \left(\sum_{j=1}^{l} \alpha_j \exp\left(\Lambda\left\{\frac{\log(\alpha_i)-\log(\alpha_j)}{\Lambda}\right\}\right)\right)^{-1} < 1$$

are constants depending on i.

Remark 2.3 *Observe that only if the conditions (I) and (II) fail to hold,* c_1 *is strictly larger and* c_2 *is strictly smaller than* 1 *and the sequence is not u.d. (cf. Remark 1.1).*

At the end of the introduction we mentioned the example $\pi = \left\{\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, 1\right]\right\}$ and $\rho = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]\right\}$. In this case the theorem indicates that the maximal and minimal asymptotic measure of $[0, \frac{2}{5}]$ is $\frac{1}{2}$ and $\frac{1}{3}$, respectively, and accordingly the maximal and minimal measure of $[\frac{2}{5}, 1]$ is $\frac{2}{3}$ and $\frac{1}{2}$, respectively.

3 Proofs

Proof of Theorem 2.1:
Denote the l intervals of π by $I_i, i = 1, \dots, l$. Then I_i has length $\alpha_i, i = 1, \dots, l$. To show that $(\rho^n \pi)_{n \in \mathbb{N}}$ is uniformly distributed it is sufficient to prove that the relative number of intervals of $(\rho^n \pi)_{n\in\mathbb{N}}$ in I_i converges to α_i, for $i = 1, \dots, l$, since by [8, Theorem 2.7] the sequences of partitions within the intervals I_i are u.d.
Assume that (I) holds and let $0 < \epsilon \le (\min_{1\le j\le l} \alpha_j)(\min_{1\le i\le m} p_i)$. Let $h_\epsilon \in \mathbb{N}$ be the smallest number for which $\rho^{h_\epsilon}\pi$ contains only intervals of length $< \epsilon$. Then the set $\{h_\epsilon : 0 < \epsilon \le (\min_{1\le j\le l} \alpha_j)$ $(\min_{1\le i\le m} p_i)\}$ is of the form $\{n \in \mathbb{N}, n \ge n_0\}$ for some n_0. Using the notation of Lemma 2.1, the number of intervals of $\rho^{h_\epsilon}\pi$ which are contained in I_i equals M_{ϵ/α_i} for $i = 1, \dots, l$, where

$$M_{\epsilon/\alpha_i} = \frac{(m-1)\alpha_i}{\epsilon H} + o\left(\frac{1}{\epsilon}\right).$$

For $i = 1, \dots, l$,

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{M_{\epsilon/\alpha_i}}{\sum_{j=1}^{l} M_{\epsilon/\alpha_j}} &= \lim_{\epsilon\to 0} \frac{\frac{(m-1)\alpha_i}{\epsilon H} + o\left(\frac{1}{\epsilon}\right)}{\sum_{j=1}^{l} \frac{(m-1)\alpha_j}{\epsilon H} + o\left(\frac{1}{\epsilon}\right)} \\ &= \frac{\alpha_i}{\sum_{j=1}^{l} \alpha_j} = \alpha_i,\end{aligned}$$

and thus the sequence $(\rho^n\pi)_{n\in\mathbb{N}}$ is u.d. .
Now assume that condition (I) does not hold. Then the numbers $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ are rationally related with some parameter Λ, and the number of intervals of $\rho^{h_\epsilon}\pi$ which are contained in I_i is M_{ϵ/α_i}, where by Lemma 2.1

$$M_{\epsilon/\alpha_i} = \frac{(m-1)\alpha_i Q_1\left(\log\left(\frac{\alpha_i}{\epsilon}\right)\right)}{\epsilon H} + \mathcal{O}\left((\log(\epsilon))^d \epsilon^{-(1-\eta)}\right). \tag{5}$$

Consider

$$\begin{aligned}\frac{M_{\epsilon/\alpha_i}}{\sum_{j=1}^{l} M_{\epsilon/\alpha_j}} &= \frac{\frac{(m-1)\alpha_i Q_1\left(\log\left(\frac{\alpha_i}{\epsilon}\right)\right)}{\epsilon H} + \mathcal{O}\left((\log(\epsilon))^d\epsilon^{-(1-\eta)}\right)}{\sum_{j=1}^{l} \frac{(m-1)\alpha_j Q_1\left(\log\left(\frac{\alpha_j}{\epsilon}\right)\right)}{\epsilon H} + \mathcal{O}\left((\log(\epsilon))^d\epsilon^{-(1-\eta)}\right)} \\ &= \frac{\alpha_i Q_1\left(\log\left(\frac{\alpha_i}{\epsilon}\right)\right) + \mathcal{O}\left((\log(\epsilon))^d\epsilon^{-(1-\eta)}\right)}{\sum_{j=1}^{l} \alpha_j Q_1\left(\log\left(\frac{\alpha_j}{\epsilon}\right)\right) + \mathcal{O}\left((\log(\epsilon))^d\epsilon^{-(1-\eta)}\right)}.\end{aligned} \tag{6}$$

If (II) holds, then

$$\begin{aligned}\left\{\frac{\log\left(\frac{\alpha_j}{\epsilon}\right)}{\Lambda}\right\} &= \left\{\frac{\log\left(\frac{ce^{v_j\Lambda}}{\epsilon}\right)}{\Lambda}\right\} \\ &= \left\{\frac{\log(c) + v_j\Lambda - \log(\epsilon)}{\Lambda}\right\} \\ &= \left\{\frac{\log(c) - \log(\epsilon)}{\Lambda}\right\}\end{aligned}$$

and

$$Q_1\left(\log\left(\frac{\alpha_j}{\epsilon}\right)\right) = \frac{\Lambda e^{-\Lambda\left\{\frac{\log(n)-\log(\epsilon)}{\Lambda}\right\}}}{1-e^{-\Lambda}},$$

for all $j = 1, \dots, l$. Thus for $i = 1, \dots, l$,

$$\lim_{\epsilon \to 0} \frac{M_{\epsilon/\alpha_i}}{\sum_{j=1}^{l} M_{\epsilon/\alpha_j}} = \frac{\alpha_i}{\sum_{j=1}^{l} \alpha_j} = \alpha_i,$$

and $(\rho^n \pi)_{n\in\mathbb{N}}$ is u.d. .

Now assume that neither (I) nor (II) holds. Then the numbers $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ are rationally related with some parameter Λ, and the starting partition π has to consist of at least two elements. Furthermore, note that condition (II) is equivalent to assuming

$$\log(\alpha_j) - \log(\alpha_i) = n_{ij}\Lambda, \quad n_{ij} \in \mathbb{Z}, \tag{7}$$

for $i, j = 1, \dots, l$, so if (II) does not hold there necessarily exist indices i, j for which (7) is not satisfied. Fix such i, j. Then

$$\left\{ \frac{\log(\alpha_j) - \log(\alpha_i)}{\Lambda} \right\} > 0. \tag{8}$$

Let the sequence $(\epsilon_k)_{k\in\mathbb{N}}$ be defined by

$$\epsilon_k = \alpha_i e^{-k\Lambda}, \quad k \geq 1.$$

Then for $k \geq 1$ and $n \in \{1, \dots, l\}$,

$$\begin{aligned}
\left\{ \frac{\log\left(\frac{\alpha_n}{\epsilon_k}\right)}{\Lambda} \right\} &= \left\{ \frac{\log\left(\frac{\alpha_n e^{k\Lambda}}{\alpha_i}\right)}{\Lambda} \right\} \\
&= \left\{ \frac{\log(\alpha_n) + k\Lambda - \log(\alpha_i)}{\Lambda} \right\} \\
&= \left\{ \frac{\log(\alpha_n) - \log(\alpha_i)}{\Lambda} \right\}.
\end{aligned} \tag{9}$$

Hence,

$$Q_1\left(\log\left(\frac{\alpha_i}{\epsilon_k}\right)\right) = \frac{\Lambda}{1 - e^{-\Lambda}}, \tag{10}$$

and

$$Q_1\left(\log\left(\frac{\alpha_j}{\epsilon_k}\right)\right) = \frac{\Lambda e^{-\Lambda\left\{\frac{\log(\alpha_j)-\log(\alpha_i)}{\Lambda}\right\}}}{1 - e^{-\Lambda}}. \tag{11}$$

By using (6), we obtain

$$\lim_{k\to\infty} \frac{\alpha_i Q_1\left(\log\left(\frac{\alpha_i}{\epsilon_k}\right)\right) + \mathcal{O}\left((\log(\epsilon_k))^d \epsilon_k^{-(1-\eta)}\right)}{\sum_{n=1}^{l} \alpha_n Q_1\left(\log\left(\frac{\alpha_n}{\epsilon_k}\right)\right) + \mathcal{O}\left((\log(\epsilon_k))^d \epsilon_k^{-(1-\eta)}\right)} = \frac{\alpha_i}{\sum_{n=1}^{l} \alpha_n e^{-\Lambda\left\{\frac{\log(\alpha_n)-\log(\alpha_i)}{\Lambda}\right\}}}.$$

By (8) and $\Lambda > 0$ it follows that

$$e^{-\Lambda\left\{\frac{\log(\alpha_n)-\log(\alpha_i)}{\Lambda}\right\}} \leq 1, \quad n = 1, \dots, l,$$

and

$$e^{-\Lambda\left\{\frac{\log(\alpha_j)-\log(\alpha_i)}{\Lambda}\right\}} < 1.$$

Thus

$$\sum_{n=1}^{l} \alpha_n e^{-\Lambda\left\{\frac{\log(\alpha_n)-\log(\alpha_i)}{\Lambda}\right\}} < 1$$

and

$$\lim_{k\to\infty} \frac{M_{\epsilon_k/\alpha_i}}{\sum_{j=1}^{l} M_{\epsilon_k/\alpha_j}} \neq \alpha_i.$$

Thus there exists a subsequence along which the relative number of intervals in I_i does not converge to α_i, and hence the sequence $(\rho^n\pi)_{n\in\mathbb{N}}$ cannot be u.d. This proves the theorem. $\square$

Proof of Corollary 2.1:
The corollary is a special case of Theorem 2.1. By Remark 2.1, condition (i) is equivalent to condition (I).
Furthermore, condition (ii) is equivalent to (II). Assume that (II) holds, then $\alpha = ce^{r\Lambda}, 1-\alpha = ce^{q\Lambda}$, for $q, r \in \mathbb{Z}, c \in \mathbb{R}^+$, and thus

$$\begin{aligned} 1 &= ce^{r\Lambda} + ce^{q\Lambda} \\ \Leftrightarrow c &= \frac{1}{e^{r\Lambda} + e^{q\Lambda}} \end{aligned}$$

and

$$\alpha = \frac{e^{r\Lambda}}{e^{r\Lambda} + e^{q\Lambda}} = \frac{1}{e^{(q-r)\Lambda} + 1}.$$

$\square$

Proof of Theorem 2.2:
Let the i-th interval of π be denoted by I_i and let $h_\epsilon \in \mathbb{N}$ be the smallest number for which $\rho^{h_\epsilon}\pi$ contains only intervals of length $< \epsilon$. Then, following the proof of Theorem 2.1, the number of intervals of $\rho^{h_\epsilon}\pi$ which are contained in I_i is M_{ϵ/α_i}, which is given in (5). We denote by $M_A(\epsilon)$ the number of intervals of $\rho^{h_\epsilon}\pi$ which are contained in $A = [a, b] \subseteq I_i$. By [8, Theorem 2.7], the sequences of partitions within I_i are u.d. Hence

$$M_A(\epsilon) = \frac{(b-a)(m-1)Q_1\left(\log\left(\frac{\alpha_i}{\epsilon}\right)\right)}{\epsilon H} + \mathcal{O}\left((\log(\epsilon))^d \epsilon^{-(1-\eta)}\right).$$

Thus the relative number of intervals in A is given by

$$\begin{aligned} \frac{M_A(\epsilon)}{\sum_{j=1}^{l} M_{\epsilon/\alpha_j}} &= \frac{\frac{(b-a)(m-1)Q_1\left(\log\left(\frac{\alpha_i}{\epsilon}\right)\right)}{\epsilon H} + \mathcal{O}\left((\log(\epsilon))^d \epsilon^{-(1-\eta)}\right)}{\sum_{j=1}^{l} \frac{(m-1)\alpha_j Q_1\left(\log\left(\frac{\alpha_j}{\epsilon}\right)\right)}{\epsilon H} + \mathcal{O}\left((\log(\epsilon))^d \epsilon^{-(1-\eta)}\right)} \\ &= \frac{(b-a)Q_1\left(\log\left(\frac{\alpha_i}{\epsilon}\right)\right) + \mathcal{O}\left((\log(\epsilon))^d \epsilon^{-(1-\eta)}\right)}{\sum_{j=1}^{l} \alpha_j Q_1\left(\log\left(\frac{\alpha_j}{\epsilon}\right)\right) + \mathcal{O}\left((\log(\epsilon))^d \epsilon^{-(1-\eta)}\right)}. \end{aligned}$$

Consider

$$\frac{(b-a)Q_1\left(\log\left(\frac{\alpha_i}{\epsilon}\right)\right)}{\sum_{j=1}^{l} \alpha_j Q_1\left(\log\left(\frac{\alpha_j}{\epsilon}\right)\right)} = \frac{(b-a)\Lambda e^{-\Lambda\{\frac{1}{\Lambda}(\log(\alpha_i)-\log(\epsilon))\}}}{\sum_{j=1}^{l} \alpha_j \Lambda e^{-\Lambda\{\frac{1}{\Lambda}(\log(\alpha_j)-\log(\epsilon))\}}}$$

$$= \frac{b-a}{\sum_{j=1}^{l} \alpha_j \exp\left(-\Lambda\left(\{\frac{1}{\Lambda}(\log(\alpha_j)-\log(\epsilon))\} - \{\frac{1}{\Lambda}(\log(\alpha_i)-\log(\epsilon))\}\right)\right)}. \tag{12}$$

For $j \neq i$, one easily sees that the functions

$$f_{i,j}(\epsilon) := \exp\left(-\Lambda\left(\left\{\frac{\log(\alpha_j)-\log(\epsilon)}{\Lambda}\right\} - \left\{\frac{\log(\alpha_i)-\log(\epsilon)}{\Lambda}\right\}\right)\right)$$

are piecewise constant with discontinuities at

$$\epsilon = \alpha_i e^{-k\Lambda} \text{ and } \epsilon = \alpha_j e^{-k\Lambda},$$

for all $k \in \mathbb{Z}$. By

$$\begin{aligned} f_{i,j}(\alpha_i e^{-k_1\Lambda}) &= f_{i,j}(\alpha_i e^{-k_2\Lambda}), \\ f_{i,j}(\alpha_j e^{-k_1\Lambda}) &= f_{i,j}(\alpha_j e^{-k_2\Lambda}), \end{aligned}$$

for all $k_1, k_2 \in \mathbb{Z}$, it follows that $f_{i,j}(\epsilon), 0 < \epsilon < 1$, only takes two different values, which are

$$\exp\left(-\Lambda\left\{\frac{\log(\alpha_j)-\log(\alpha_i)}{\Lambda}\right\}\right) \quad \text{and} \quad \exp\left(\Lambda\left\{\frac{\log(\alpha_i)-\log(\alpha_j)}{\Lambda}\right\}\right).$$

Furthermore, for all $k \in \mathbb{Z}$

$$f_{i,j}(\alpha_i e^{-k\Lambda}) = \exp\left(-\Lambda\left\{\frac{\log(\alpha_j)-\log(\alpha_i)}{\Lambda}\right\}\right)$$

and

$$f_{i,j}(\alpha_j e^{-k\Lambda}) = \exp\left(\Lambda\left\{\frac{\log(\alpha_i)-\log(\alpha_j)}{\Lambda}\right\}\right).$$

By the above arguments it follows that the function

$$\sum_{j=1}^{l} \alpha_j f_{i,j}(\epsilon),$$

where $f_{i,i}(\epsilon) = 1$, can only take at most l different values. Since all the functions $f_{i,j}(\epsilon), 1 \leq j \leq l$, attain their minimal value at the positions $\alpha_i e^{-k\Lambda}, k \in \mathbb{Z}$, it follows that the quotient in equation (12) is maximal at these positions and

$$\begin{aligned} \limsup_{n\to\infty} \frac{\sum_{j=1}^{k(n)} \mathbf{1}_{[a,b]}(t_j^n)}{k(n)} &= \limsup_{\epsilon\to 0} \frac{M_A(\epsilon)}{\sum_{j=1}^{l} M_{\epsilon/\alpha_j}} \\ &= \lim_{k\to\infty} \frac{M_A(\alpha_i \exp(-k\Lambda))}{\sum_{j=1}^{l} M_{(\alpha_i \exp(-k\Lambda))/\alpha_j}} \\ &= \frac{b-a}{\sum_{j=1}^{l} \alpha_j \exp\left(-\Lambda\left\{\frac{\log(\alpha_j)-\log(\alpha_i)}{\Lambda}\right\}\right)}. \end{aligned}$$

This proves the upper bound in Theorem 2.2.
To prove the lower bound in Theorem 2.2, we choose $0 < \gamma < 1$ such that

$$\gamma\alpha_i > \max_{1\leq j\leq l} \max_{k\in\mathbb{Z}} \left\{\alpha_j e^{-k\Lambda} \mid \alpha_j e^{-k\Lambda} < \alpha_i\right\}.$$

Then for all $1 \leq j \leq l$ and for all $k \in \mathbb{Z}$ the functions $f_{i,j}$ attain their maximal value at the positions $\gamma\alpha_i e^{-k\Lambda}$, and

$$f_{i,j}(\gamma\alpha_i e^{-k\Lambda}) = f_{i,j}(\alpha_j e^{-k\Lambda}) = \exp\left(\Lambda\left\{\frac{\log(\alpha_i) - \log(\alpha_j)}{\Lambda}\right\}\right).$$

Therefore, the quotient in equation (12) attains its minimal possible value at the positions $\gamma\alpha_i e^{-k\Lambda}, k \in \mathbb{Z}$, and

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\sum_{j=1}^{k(n)} \mathbf{1}_{[a,b]}(t_j^n)}{k(n)} &= \liminf_{\epsilon\to 0} \frac{M_A(\epsilon)}{\sum_{j=1}^{l} M_{\epsilon/\alpha_j}} \\
&= \lim_{k\to\infty} \frac{M_A(\gamma\alpha_i \exp(-k\Lambda))}{\sum_{j=1}^{l} M_{(\gamma\alpha_i e^{-k\Lambda})/\alpha_j}} \\
&= \frac{b-a}{\sum_{j=1}^{l} \alpha_j \exp\left(\Lambda\left\{\frac{\log(\alpha_i)-\log(\alpha_j)}{\Lambda}\right\}\right)}.
\end{aligned}$$

This proves the theorem. □

References

[1] M.D. Brennan and R. Durrett. Splitting intervals. *Annals of Probability*, 14(3):1024–1036, 1986.

[2] I. Carbone. Discrepancy of LS-sequences of partitions. *Annali di Matematica Pura ed Applicata*, 2011.

[3] I. Carbone and A. Volčič. Kakutani's splitting procedure in higher dimension. *Rend. Ist. Mathem. Univ. Trieste*, pages 1–8, 2007.

[4] M. Drmota and M. Infusino. On the discrepancy of some generalized Kakutani's sequences of partitions. *Preprint TU Wien, available at www.dmg.tuwien.ac.at/drmota/newkakutani.pdf*, 2011.

[5] S. Kakutani. A problem of equidistribution on the unit interval [0,1]. *Proceedings of Oberwolfach Conference on Measure Theory. Lecture Notes in Mathematics*, 541:369–375, 1976.

[6] G.L. Khodak. Connection Between Redundancy and Average Delay for Fixed-Length Coding. *All-Union Conference on Problems of Theoretical Cybernetics (in Russian)*, 12, 1969.

[7] W.R. van Zwet. A proof of Kakutani's conjecture on random partitions. *Annals of Probability*, 6(1):133–137, 1978.

[8] A. Volčič. A generalization of Kakutani's splitting procedure. *Annali di Matematica Pura ed Applicata*, 190:45–54, 2011.