Pricing and hedging of lookback options in hyper-exponential jump diffusion models

Markus Hofer* Philipp Mayer †

Abstract

In this article we consider the problem of pricing lookback options in certain exponential Lévy market models. While in the classic Black-Scholes models the price of such options can be calculated in closed form, for more general asset price model one typically has to rely on (rather time-intense) Monte-Carlo or P(I)DE methods. However, for Lévy processes with double exponentially distributed jumps the lookback option price can be expressed as one-dimensional Laplace transform (cf. Kou [Kou et al., 2005]). The key ingredient to derive this representation is the explicit availability of the first passage time distribution for this particular Lévy process, which is well-known also for the more general class of hyper-exponential jump diffusions (HEJD). In fact, Jeannin and Pistorius [Jeannin and Pistorius, 2010] were able to derive formulae for the Laplace transformed price of certain barrier options in market models described by HEJD processes. Here, we similarly derive the Laplace transforms of floating and fixed strike lookback option prices and propose a numerical inversion scheme, which allows, like Fourier inversion methods for European vanilla options, the calculation of lookback options with different strikes in one shot. Additionally, we give semi-analytical formulae for several Greeks of the option price and discuss a method of extending the proposed method to generalised hyper-exponential (as e.g. NIG or CGMY) models by fitting a suitable HEJD process. Finally, we illustrate the theoretical findings by some numerical experiments.

1 Introduction

It has been known for many years that the classic Black-Scholes model suffers from many shortcomings and is not capable of explaining many important stylised facts of financial markets, like skewed and heavy tailed return distributions, or the thereby introduced volatility smile/skew. Thus, despite the superior analytical tractability of the geometric Brownian motion model, many authors proposed the more general class of Lévy processes as underlying model for prices of financial quantities. Most definitely we cannot do justice to the vast literature in this field and we limit ourselves to cite just three classics related to our work, namely [Barndorff-Nielsen, 1998], [Carr et al., 2002] or [Kou, 2002], and refer the reader to those and references therein for more details on the use of Lévy processes in finance. However, the extra flexibility of Lévy driven financial models often comes at the cost of more complicated pricing algorithms for exotic path-dependent options. The purpose of this article is thus to contribute to the development of more efficient pricing algorithms for certain popular exotic derivatives. More precisely, we will calculate the

*Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: markus.hofer@tugraz.at. Research supported by the Austrian Research Foundation (FWF) and by the DOC [Doctoral Fellowship Programme of the Austrian Academy of Sciences], Project S9603-N23.

†Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: mayer@finanz.math.tugraz.at

Mathematics Subject Classification: 91G20 (60J75, 91B70)

Keywords: exotic options, lookback options, option pricing, delta hedging, Greeks, hyper-exponential jump diffusion, Lévy processes, Laplace tranform

(time-)Laplace transformed price of different kinds of lookback options and propose and test an efficient inversion algorithm for this transform.

Loosely speaking, there are three approaches for pricing derivatives related to the maximum or minimum of the asset price: Monte Carlo methods, Partial (integro)-differential equations (PIDE) schemes, and Laplace transform based methods, where the latter ones, if applicable, are in general preferable in terms of performance. Focusing on the Laplace transform approach we would like to mention the very nice theoretical discussion regarding this kind of methods for general Lévy processes by Eberlein et al. [Eberlein et al., 2011], where very general formulae for the (multi)-Laplace transformed prices of many different option types were derived. For general Lévy processes these formulae have the drawback that the inversion of the Laplace transform is typically quite involved and for a numerical evaluation several numerical integrations need to be performed. However, for some particular Lévy processes these formulae simplify significantly and option prices can be calculated by applying just a standard one-dimensional inversion. For example, Kou [Kou, 2002] proposed a financial market model (typically called Kou model), in which the logarithmic asset price process is described by a jump diffusion with two-sided exponential jumps and showed that in this setting, the Laplace transform of several exotic options, including lookback options, can be given in an analytic way (see [Kou et al., 2005]). Notably, for the same class of processes, Sepp [Sepp, 2005] presents a PIDE approach for the pricing of lookback options.

The Kou model also sets the basis for the more general hyper-exponential jump diffusion model (HEJD), where the up- and downward jumps are not modeled by a single exponential random variable, but by a mixture of several exponential random variables with different parameters. Apart from the obvious advantage of more flexibility the main motivation for considering this kind of models was established by Jeannin and Pistorius [Jeannin and Pistorius, 2010], who showed that many frequently used Lévy based financial models can be approximated arbitrarily well by HEJD processes. More precisely, any process in the class of the so-called general hyper-exponential Lévy processes, that includes e.g. the normal inverse Gaussian (NIG) [Barndorff-Nielsen, 1998], or the CGMY process [Carr et al., 2002] to name only two, can be represented as a limit of a sequence of HEJD processes. Moreover, Jeannin and Pistorius also derived the time-Laplace transforms prices of barrier and digital options, and some sensitivities, within the framework of HEJD models. Pricing of double barrier options in HEJD models was discussed by N. Cai et al. [Cai et al., 2009] where also formulae for the first passage time and related identities of HEJD processes are given. Those two papers also form the basis of this work, where we slightly extend the existing results to apply them to the problem of pricing lookback options.

The rest of the paper is organised as follows: in Section 2 we give a brief introduction to HEJD processes and the Wiener-Hopf factorisation for HEJD processes. In the third section, we derive prices and sensitivities for different types of lookback options and in Section 4 we justify the approximation of lookback option price under a NIG process by corresponding prices under a HEJD process. A numerical analysis of our methods concludes the paper in Section 5.

2 Introduction to HEJD models and preliminary results

We will consider lookback options and similar derivatives on an underlying asset, the price process of which, $(S_t)_{t\geq 0}$, is given as $S_t = S_0 e^{X_t}$, where $S_0 > 0$. We assume $\mathbb{E}[e^{X_t}] = e^{rt}$ for all $t \geq 0$, where r denotes the risk free interest rate and $(X_t)_{t\geq 0}$ to be a Lévy process with $X_0 = 0$ a.s.

To valuate a lookback option we have to analyse the supremum and infimum process of the asset price process. Let us hence define

$$\overline{X}_t = \sup_{0\leq s\leq t} X_s, \qquad \underline{X}_t = \inf_{0\leq s\leq t} X_s$$

and recall the well-known Wiener-Hopf factorisation.

Theorem 2.1 (Wiener-Hopf factorisation) *Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}$ and $(\overline{X}_t)_{t\geq 0}$ and $(\underline{X}_t)_{t\geq 0}$*

its supremum and infimum process, respectively. Furthermore, let θ be an exponentially distributed random variable with parameter q. Then the characteristic function of $(X_t)_{t\geq 0}$ at the random time θ can be factorised in the following way:

$$\mathbb{E}[e^{izX_\theta}] = \mathbb{E}[e^{iz\overline{X}_\theta}]\,\mathbb{E}[e^{iz\underline{X}_\theta}], \qquad \forall z \in \mathbb{R},$$

or equivalently

$$\frac{q}{q - \log(\phi_X(z))} = \phi_q^+(z)\phi_q^-(z), \qquad \forall z \in \mathbb{R},$$

where $\phi(z)$ is the characteristic function of X_1, $\phi_q^+(z) = \mathbb{E}[e^{iz\overline{X}_\theta}]$ and $\phi_q^-(z) = \mathbb{E}[e^{iz\underline{X}_\theta}]$.

Additionally, formulae for the Wiener-Hopf factors ϕ_q^- and ϕ_q^+ can be given (see e.g. Sato [Sato, 1999]). For general Lévy processes, however, the actual computation of the factors affords numerical evaluations of high-dimensional numerical integrals. Of course, for some particular types of Lévy processes it is possible to give explicit formulae for ϕ_q^- and ϕ_q^+ (see e.g. Kyprianou [Kyprianou, 2006, Chapter 6]).
A class of Lévy processes, which is well suited for asset price models and allows for considerably simplified formulae for the Wiener-Hopf factors and other identities are jump diffusions with phase-type distributed jumps (cf. Asmussen et al. [Asmussen et al., 2004]). Here, at least in the first sections, we concentrate on a special kind of the this last category of Lévy processes, more precisely on so called hyper-exponential jump-diffusions.

Definition 2.1 (Hyper-exponential jump-diffusion) *Let X_t be a Lévy process with $X_0 = 0$ a.s. , then X_t is called hyper-exponential jump-diffusion (HEJD), if it has the following representation*

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_+} Y_i^+ + \sum_{j=1}^{N_-} Y_j^-, \qquad t \geq 0,$$

where W is a Wiener process, N_+, N_- are Poisson processes with parameters $\lambda_+ > 0$ and $\lambda_- > 0$, respectively and $(Y_j^+), (Y_j^-)$ are i.i.d. families of mixed exponential random variables, i.e.

$$Y_j^+ = \sum_{i=1}^{n^+} p_i^+ Z_i^+, \qquad Y_j^- = \sum_{i=1}^{n^-} p_i^- Z_i^-,$$

where $\sum_{i=1}^{n^+} p_i^+ = \sum_{j=1}^{n^-} p_j^- = 1$, $p_i^+ > 0, i = 1, \ldots, n^+$, $p_j^- > 0, j = 1, \ldots, n^-$ and Z_i^+, Z_i^- are exponentially distributed with means $\alpha_i^+ > 0$ and $\alpha_i^- > 0$, respectively. Moreover, all random variables and processes are assumed to be independent.

By the Lévy-Khinchin formula (see e.g. Sato [Sato, 1999]), the characteristic exponent of a HEJD can be written as

$$\begin{aligned}\phi(u) = \log(E[e^{iuX_1}]) = ui\mu - \frac{\sigma^2}{2}u^2 + \lambda^+ \sum_{k=1}^{n^+} p_k^+ \Big(\frac{\alpha_k^+}{\alpha_k^+ - ui} - 1\Big) \\ + \lambda^- \sum_{j=1}^{n^-} p_j^- \Big(\frac{\alpha_j^-}{\alpha_j^- + ui} - 1\Big).\end{aligned} \tag{1}$$

The function $\phi(u)$ can be extended analytically (cf. e.g. Sato [Sato, 1999, Chapter 9]) to the whole complex plane except for the finite sets $\{-\mathrm{i}\alpha_i^+,\ \text{for } i = 1, \ldots, n^+\}, \{-\mathrm{i}\alpha_i^-,\ \text{for } i = 1, \ldots, n^-\}$ and we will

denote the roots of the Cramér-Lundberg equation $-q+\phi(-is)=0$ with positive and negative real part, by $\rho_i^+(q), i=1,\ldots,n^++1$ and $\rho_i^-(q), i=1,\ldots,n^-+1$, respectively.
Applying the formulae for general two-sided phase-type distributed jumps on HEJD processes, we find

$$\phi_q^+(u)=\frac{\prod_{k=1}^{n^+}\left(1-\frac{ui}{\alpha_k^+}\right)}{\prod_{k=1}^{m^+}\left(1-\frac{ui}{\rho_k^+(q)}\right)} \qquad \text{and} \qquad \phi_q^-(u)=\frac{\prod_{k=1}^{n^-}\left(1+\frac{ui}{\alpha_k^-}\right)}{\prod_{k=1}^{m^-}\left(1-\frac{ui}{\rho_k^-(q)}\right)}.$$

Moreover, the time-Laplace transforms of the distributions of $\overline{X}_t$ and $\underline{X}_t$ can be calculated explicitly (cf. Mordecki [Mordecki, 2002])

$$\int_0^\infty e^{-qt}P(\overline{X}_t\le z)dt=\frac{1}{q}\Big(1-\sum_{k=1}^{m^+}A_k^+(q)e^{-\rho_k^+(q)z}\Big), \qquad z\ge 0 \tag{2}$$

$$\int_0^\infty e^{-qt}P(-\underline{X}_t\le z)dt=\frac{1}{q}\Big(1-\sum_{k=1}^{m^-}A_k^-(q)e^{\rho_k^-(q)z}\Big), \qquad z\ge 0 \tag{3}$$

where the coefficients $A_k^+(q)$ and $A_k^-(q)$ are given by

$$A_k^+(q)=\frac{\prod_{v=1}^{n^+}\left(1-\frac{\rho_k^+(q)}{\alpha_v^+}\right)}{\prod_{v=1,v\neq k}^{m^+}\left(1-\frac{\rho_k^+(q)}{\rho_v^+(q)}\right)}, \tag{4}$$

$$A_k^-(q)=\frac{\prod_{v=1}^{n^-}\left(1+\frac{\rho_k^-(q)}{\alpha_v^-}\right)}{\prod_{v=1,v\neq k}^{m^-}\left(1-\frac{\rho_k^-(q)}{\rho_v^-(q)}\right)}. \tag{5}$$

Let us shortly note here, that another way to understand the above formula is that for any exponentially distributed random variable θ, $\overline{X}_\theta$ and $\underline{X}_\theta$ are hyper-exponential distributed random variables.
With the notable exception of the Kou model (for which $n^+=n^-=1$) the roots of the Cramér Lundberg equation cannot be calculated analytically. However, due to favorable structural properties of the Cramér Lundberg equation the numerical computation of the roots is not difficult and can be efficiently implemented. The following Lemma 2.1, which is a slight extension of [Cai, 2009, Lemma 1], states the precise result.

Lemma 2.1 (Characterisation of the moment generating function of X_t) *The function $\phi(-is)$ is a convex function for $s\in(-\alpha_1^-,\alpha_1^+)$. Furthermore:*

- *If $\sigma>0$, the equation $-q+\phi(-is)=0$ for $q\in\mathbb{R}^+$ has roots $\rho_k^+, k=1,\ldots,n^++1=m^+$ and $\rho_j^-, j=1,\ldots,n^-+1=m^-$, which satisfy the condition*

$$-\infty<-\rho_{n^-+1}^-(q)<-\alpha_{n^-}^-<-\rho_{n^-}^-(q)<\ldots<-\rho_2^-(q)<-\alpha_1^-<-\rho_1^-(q)<0,$$
$$0<\rho_1^+(q)<\alpha_1^+<\rho_2^+(q)<\ldots<\rho_{n^+}^+(q)<\alpha_{n^+}^+<\rho_{n^++1}^+(q)<\infty.$$

- *If $\sigma=0$ and $\mu>0$, the equation $-q+\phi(-is)=0$ for $q\in\mathbb{R}^+$ has roots $\rho_k^+, k=1,\ldots,n^++1=m^+$ and $\rho_j^-, j=1,\ldots,n^-=m^-$, which satisfy the condition*

$$-\infty<-\alpha_{n^-}^-<-\rho_{n^-}^-(q)<\ldots<-\rho_2^-(q)<-\alpha_1^-<-\rho_1^-(q)<0,$$
$$0<\rho_1^+(q)<\alpha_1^+<\rho_2^+(q)<\ldots<\rho_{n^+}^+(q)<\alpha_{n^+}^+<\rho_{n^++1}^+(q)<\infty$$

- *if $\sigma = 0$ and $\mu < 0$, the equation $-q+\phi(-is) = 0$ for $q \in \mathbb{R}^+$ has roots $\rho_k^+, k = 1, \ldots, n^+ = m^+$ and $\rho_j^-, j = 1, \ldots, n^- + 1 = m^-$, which satisfy the condition*

$$-\infty < -\rho^-_{n^-+1}(q) < -\alpha^-_{n^-} < -\rho^-_{n^-}(q) < \ldots < -\rho_2^-(q) < -\alpha_1^- < -\rho_1^-(q) < 0,$$
$$0 < \rho_1^+(q) < \alpha_1^+ < \rho_2^+(q) < \ldots < \rho^+_{n^+}(q) < \alpha^+_{n^+} < \infty.$$

- *if $\sigma = 0$ and $\mu = 0$, the equation $-q+\phi(-is) = 0$ for $q \in \mathbb{R}^+$ has roots $\rho_k^+, k = 1, \ldots, n^+ = m^+$ and $\rho_j^-, j = 1, \ldots, n^- = m^-$, which satisfy the condition*

$$-\infty < -\alpha^-_{n^-} < -\rho^-_{n^-}(q) < \ldots < -\rho_2^-(q) < -\alpha_1^- < -\rho_1^-(q) < 0,$$
$$0 < \rho_1^+(q) < \alpha_1^+ < \rho_2^+(q) < \ldots < \rho^+_{n^+}(q) < \alpha^+_{n^+} < \infty.$$

Proof:
For simplicity of notation, we set $\psi(s) = \phi(-is)$. Note that in every case, $\psi(s)$ is a convex function on $(-\alpha_1^-, \alpha_1^+)$, because it is a sum of convex functions on this interval.
Furthermore, $\psi(s)$ has poles on on the sets $\{\alpha_i^+, \text{ for } i = 1, \ldots, n^+\}, \{\alpha_i^-, \text{ for } i = 1, \ldots, n^-\}$. For a positive pole α_i^+ it follows $\psi(\alpha_i^+-) = +\infty$ and $\psi(\alpha_i^++) = -\infty$ and for a negative pole α_i^- it follows $\psi(\alpha_i^--) = -\infty$ and $\psi(\alpha_i^-+) = +\infty$. Furthermore $\psi(s)$ is continuous between two poles, so that there is always at least one root of the equation $-q + \phi(-is) = 0$ between two such poles. From the fact that $\psi(0) = 0$ and the convexity of ψ in $(-\alpha_1^-, \alpha_1^+)$, we conclude that there is exactly one root on each of the intervals $(-\alpha_1^-, 0)$ and $(0, \alpha_1^+)$. While all of the observations so far hold in every of the four cases, we will now consider different combinations of σ and μ separately.
If $\sigma > 0$, it follows by simple transformations that the equation $-q + \psi(s) = 0$ has two more roots than $\psi(s)$ has poles and that $\lim_{s\to+\infty} \psi(s) = \lim_{s\to-\infty} \psi(s) = +\infty$. Because of these facts, there is exactly one root in $(-\infty, \alpha^-_{n^-})$ and $(\alpha^+_{n^+}, +\infty)$. Hence there is exactly one root in each of the intervals $(\alpha_i^+, \alpha_{i+1}^+)$ for $i = 1, \ldots, m^+ - 1$, $(\alpha_{i+1}^-, \alpha_i^-)$ for $i = 1, \ldots, m^- - 1$, $(\alpha^+_{n^+}, +\infty)$ and $(-\infty, \alpha^-_{n^-})$.
The argumentation is similar in the three remaining cases, where $\sigma = 0$. If $\mu \neq 0$, then $-q + \psi(s) = 0$ has one more root than $\psi(s)$ has poles. Because of $\lim_{s\to+\infty} \psi(s) = +\infty$ if $\mu > 0$ and $\lim_{s\to-\infty} \psi(s) = +\infty$ if $\mu < 0$, there must be a root on $(\alpha^+_{n^+}, +\infty)$ and $(-\infty, \alpha^-_{n^-})$, respectively. The case $\mu = 0$ and $\sigma = 0$ follows directly from the above considerations. □

3 Prices and Greeks of lookback options

In this section we will give pricing formulae for different lookback options on an underlying asset, that is modeled by the exponential of a HEJD. More precisely, we will assume the asset price S to be given as:

$$S_t = S_0 e^{X_t},$$

where X_t is a HEJD process with $\alpha_1 > 1$. This last assumption guarantees that the expectation of the stock price is finite.
We consider two classes of lookback options, namely floating and fixed strike lookback options. Denoting the maturity by T and the strike price by K, the payoff of fixed strike calls and puts are defined by $(\max_{0\le t\le T} S_t - K)^+$ with $K \geq S_0$ and $(K - \min_{0\le t\le T} S_t)^+$ with $0 < K \leq S_0$, respectively. The prices of these options are given by

$$\mathrm{LC}_{\text{fixed}}(T, S_0, K) = \mathbb{E}[e^{-rT}(\max_{0\le t\le T} S_t - K)^+], \quad K \geq S_0, \tag{6}$$

and

$$\mathrm{LP}_{\text{fixed}}(T,S_0,K)=\mathbb{E}[e^{-rT}(K-\min_{0\le t\le T}S_t)^+],\quad 0<K\le S_0,$$

respectively. In the same manner the prices of puts and calls of floating strike lookback options are defined as expectations of their payoffs $(\max\{M,\max_{0\le t\le T}S_t\}-S_T)$ and $(S_T-\min\{N,\min_{0\le t\le T}S_t\})$, respectively, where $M\ge S_0\ge N$. Thus

$$\begin{aligned}\mathrm{LP}_{\text{float}}(T,S_0,M)&=\mathbb{E}[e^{-rT}(\max\{M,\max_{0\le t\le T}S_t\}-S_T)]\\&=\mathbb{E}[e^{-rT}(\max\{M,\max_{0\le t\le T}S_t\})]-S_0\\&=\mathbb{E}[e^{-rT}(\max_{0\le t\le T}S_t-M)^+]+e^{-rT}M-S_0\\&=\mathrm{LC}_{\text{fixed}}(T,S_0,M)+e^{-rT}M-S_0\end{aligned}\tag{7}$$

and

$$\begin{aligned}\mathrm{LC}_{\text{float}}(T,S_0,N)&=\mathbb{E}[e^{-rT}(S_T-\min\{N,\min_{0\le t\le T}S_t\})]\\&=S_0-\mathbb{E}[e^{-rT}(\min\{N,\min_{0\le t\le T}S_t\})],\\&=S_0-e^{-rT}N+\mathbb{E}[e^{-rT}(N-\min_{0\le t\le T}S_t)^+]\\&=S_0-e^{-rT}N+\mathrm{LP}_{\text{fixed}}(T,S_0,N)\end{aligned}\tag{8}$$

It follows by (7) and (8) that the price of a floating strike lookback put option is just the sum of the price of a fixed strike lookback call option and a constant with respect to X_t. An analogous statement applies to floating strike lookback call options. We will use these facts frequently in the proofs of the following theorems and corollaries.

3.1 Prices of lookback options

As mentioned before the aim is to calculate the Laplace transform of prices of lookback options and the following lemma will prove useful for this.

Lemma 3.1 *Let X_t be a HEJD. Then*

$$\lim_{y\to\infty}e^y\,\mathbb{P}[\overline{X}_T\ge y]=0,\qquad \textit{and}\qquad \lim_{y\to-\infty}e^y\,\mathbb{P}[\underline{X}_T\le y]=0,\qquad \forall T\ge 0.$$

Proof:
Observe that $(e^{\theta X_t-\phi(-i\theta)t})_{t\ge0}$ is a martingale for any $\theta\in(-\alpha_1^-,\alpha_1^+)$. Since $\alpha_1^+>1$, ϕ is continuous and $\phi(-i)=r>0$, there exists some $\beta\in(1,\alpha_1^+)$ such that $\phi(-i\theta)>0$. Hence

$$e^y\,\mathbb{P}[\overline{X}_T\ge y]=e^{(1-\beta)y}e^{\beta y}\,\mathbb{P}[\tau_y\le T],$$

where τ_y denotes the first passage time of the process X over a level y. By the optimal sampling theorem the second term can be dominated by

$$e^{\beta y}\,\mathbb{P}[\tau_y\le T]\le\mathbb{E}[e^{\beta X_{(\tau_y\wedge T)}}]\le e^{\phi(-i\beta)T}\,\mathbb{E}[e^{\beta X_{(\tau_y\wedge T)}-\phi(-i\beta)(\tau_y\wedge T)}]=e^{\phi(-i\beta)T},\tag{9}$$

where the second inequality follows from the fact that $\mathbb{E}[e^{\phi(-i\beta)(T-(\tau_y\wedge T))}]>1$ and the required result follows since $\beta>1$. The second limit result follows by applying the same arguments on the dual reflecting process $-\overline{X}_t$. □

Theorem 3.1 *Let $A_k^+(q)$ and $A_k^-(q)$ be given as in* (4) *and* (5) *and let the negative and positive roots of the equation $\phi(-is)-q=0$ be given by $\rho_k^-(q), k=1,\ldots,m^-$ and $\rho_k^+(q), k=1,\ldots,m^+$, respectively. Then the Laplace transform of the price of a fixed strike lookback call is given by*

$$\int_0^\infty e^{-\alpha T}\,\mathrm{LC}_{\textit{fixed}}(T,S_0,K)dT = S_0\frac{1}{\alpha+r}\sum_{k=1}^{m^+} A_k^+(\alpha+r)\frac{e^{-\log(K/S_0)(\rho_k^+(\alpha+r)-1)}}{\rho_k^+(\alpha+r)-1} \qquad K\geq S_0,$$

while for a fixed strike lookback put option we have

$$\int_0^\infty e^{-\alpha T}\,\mathrm{LP}_{\textit{fixed}}(T,S_0,K)dT = S_0\frac{1}{\alpha+r}\sum_{k=1}^{m^-} A_k^-(\alpha+r)\frac{e^{-\log(K/S_0)(\rho_k^-(\alpha+r)-1)}}{1-\rho_k^-(\alpha+r)} \qquad K\leq S_0.$$

Proof:
We need to calculate the Laplace transform of $\mathbb{E}[e^{-rT}(S_0e^{\overline{X}_T}-K)^+], K\geq S_0$. Defining

$$z=\log(K/S_0)\geq 0,$$

we have

$$\mathbb{E}[e^{-rT}(S_0e^{\overline{X}_T}-K)^+] = S_0\,\mathbb{E}[e^{-rT}(e^{\overline{X}_T}-e^z)\mathbf{1}_{\{\overline{X}_T\geq z\}}]. \tag{10}$$

Applying integration-by-parts and Lemma 3.1 yields

$$\begin{aligned}\mathbb{E}[e^{-rT}e^{\overline{X}_T}\mathbf{1}_{\{\overline{X}_T\geq z\}}] &= -e^{-rT}\int_z^\infty e^y d\,\mathbb{P}[\overline{X}_T\geq y]\\ &= -e^{-rT}\Big(-e^z\,\mathbb{P}[\overline{X}_T\geq z]-\int_z^\infty e^y\,\mathbb{P}[\overline{X}_T\geq y]dy\Big)\\ &= \mathbb{E}[e^{-rT}e^z\mathbf{1}_{\{\overline{X}_T\geq z\}}]+e^{-rT}\int_z^\infty e^y\,\mathbb{P}[\overline{X}_T\geq y]dy.\end{aligned}$$

Hence

$$S_0\,\mathbb{E}[e^{-rT}(e^{\overline{X}_T}-e^z)\mathbf{1}_{\{\overline{X}_T\geq z\}}] = S_0e^{-rT}\int_z^\infty e^y\,\mathbb{P}[\overline{X}_T\geq y]dy$$

and for all $\alpha>0$

$$\begin{aligned}\int_0^\infty e^{-\alpha T}S_0\,\mathbb{E}[e^{-rT}(e^{\overline{X}_T}-e^z)\mathbf{1}_{\{\overline{X}_T\geq z\}}]dT &= S_0\int_0^\infty e^{-\alpha T}e^{-rT}\int_z^\infty e^y\,\mathbb{P}[\overline{X}_T\geq y]dydT\\ &= S_0\int_z^\infty e^y\int_0^\infty e^{-(\alpha+r)T}\,\mathbb{P}[\overline{X}_T\geq y]dTdy,\end{aligned}$$

where changing the order of integration in the last step is justified by Tonelli's theorem.
Note that the inner integral in the above is exactly the Laplace transform of the distribution of the supremum process $\overline{X}$ and is hence given by equation (2), i.e. we have

$$\int_0^\infty e^{-(\alpha+r)T}\,\mathbb{P}[\overline{X}_T\geq y]dT = \frac{1}{\alpha+r}\sum_{k=1}^{m^+}A_k^+(\alpha+r)e^{-\rho_k^+(\alpha+r)y}.$$

By Lemma 2.1 and $\phi(-i)=r$ we have $\min_k\rho_k^+(\alpha+r)>\min_k\rho_k^+(r)=1$ for $\alpha>0$ and therefore,

$$\int_0^\infty e^{-\alpha T}S_0\,\mathbb{E}[e^{-rT}(e^{\overline{X}_T}-e^z)\mathbf{1}_{\{\overline{X}_T\geq z\}}]dT = S_0\int_z^\infty\frac{1}{\alpha+r}\sum_{k=1}^{m^+}A_k^+(\alpha+r)e^{y(1-\rho_k^+(\alpha+r))}dy$$

$$= S_0 \frac{1}{\alpha + r} \sum_{k=1}^{m^+} A_k^+(\alpha + r) \frac{e^{-\log(K/S_0)(\rho_k^+(\alpha+r)-1)}}{\rho_k^+(\alpha + r) - 1},$$

which proves the first statement.
The second result follows from similar reasoning. $\square$

Corollary 3.1 *Let $0 < N \leq S_0 \leq M$ and let A_k^+, A_k^-, ρ_k^+ and ρ_k^- be as in Theorem 3.1, and denote the maturity by T. Then we have*

$$\int_0^\infty e^{-\alpha T} \,\mathrm{LP}_{\mathit{float}}(T, S_0, M) dT = S_0 \frac{1}{\alpha + r} \sum_{k=1}^{m^+} A_k^+(\alpha + r) \frac{e^{-\log(M/S_0)(\rho_k^+(\alpha+r)-1)}}{\rho_k^+(\alpha + r) - 1} + \frac{M}{\alpha + r} - \frac{S_0}{\alpha}, \tag{11}$$

$$\int_0^\infty e^{-\alpha T} \,\mathrm{LC}_{\mathit{float}}(T, S_0, N) dT = \frac{S_0}{\alpha} + S_0 \frac{1}{\alpha + r} \sum_{k=1}^{m^-} A_k^-(\alpha + r) \frac{e^{-\log(N/S_0)(\rho_k^-(\alpha+r)-1)}}{1 - \rho_k^-(\alpha + r)} - \frac{N}{\alpha + r}. \tag{12}$$

Proof:
The proof follows immediately from Theorem 3.1, (7) and (8). $\square$

3.2 Greeks of lookback options

In this subsection, we use the results of the previous subsections to give formulae for the Laplace transforms of sensitivities of lookback options. We derive expressions for Θ_V, Δ_V and Γ_V, which are defined by

$$\Delta_V = \frac{\partial V}{\partial S_0}, \qquad \Gamma_V = \frac{\partial^2 V}{\partial S_0^2}, \qquad \Theta_V = \frac{\partial V}{\partial T},$$

where S_0 denotes the initial price of the underlying asset, T is the maturity of the option and V is the price of an option on the underlying asset.

Theorem 3.2 *Suppose X_t is a HEJD process with $\sigma > 0$ and let $\alpha > 0$. Then the Laplace transforms of $\Delta_{\mathrm{LC}_{\mathit{fixed}}}$ and $\Gamma_{\mathrm{LC}_{\mathit{fixed}}}$ are given by*

$$\widehat{\Delta}_{\mathrm{LC}_{\mathit{fixed}}}(\alpha) = \frac{1}{\alpha + r} \sum_{k=1}^{m^+} A_k^+(\alpha + r)\rho_k^+(\alpha + r) \frac{e^{-\log(K/S_0)(\rho_k^+(\alpha+r)-1)}}{\rho_k^+(\alpha + r) - 1}, \qquad 0 < S_0 \leq K, \tag{13}$$

$$\widehat{\Gamma}_{\mathrm{LC}_{\mathit{fixed}}}(\alpha) = \frac{1}{\alpha + r} \frac{1}{S_0} \sum_{k=1}^{m^+} A_k^+(\alpha + r)\rho_k^+(\alpha + r) e^{-\log(K/S_0)(\rho_k^+(\alpha+r)-1)}. \qquad 0 < S_0 \leq K. \tag{14}$$

The Greeks of fixed strike lookback put option are given by

$$\widehat{\Delta}_{\mathrm{LP}_{\mathit{fixed}}}(\alpha) = \frac{1}{\alpha + r} \sum_{k=1}^{m^-} A_k^-(\alpha + r)\rho_k^-(\alpha + r) \frac{e^{-\log(K/S_0)(\rho_k^-(\alpha+r)-1)}}{1 - \rho_k^-(\alpha + r)}, \qquad 0 \leq K \leq S_0, \tag{15}$$

$$\widehat{\Gamma}_{\mathrm{LP}_{\mathit{fixed}}}(\alpha) = -\frac{1}{\alpha + r} \frac{1}{S_0} \sum_{k=1}^{m^-} A_k^-(\alpha + r)\rho_k^-(\alpha + r) e^{-\log(K/S_0)(\rho_k^-(\alpha+r)-1)}, \qquad 0 \leq K \leq S_0. \tag{16}$$

Proof:
First note that Δ_V and Γ_V exist and are continuous, since $\mathrm{LP}_{\text{fixed}}$ and $\mathrm{LP}_{\text{fixed}}$ can be understood (viewed as function of S_0) as convolution of the continuous density of $\overline{X}_T$ and the function $f(x) = (x-c)^+$, the second derivative of which in the sense of distributions is given by the Dirac-Delta measure.
Formulae (13) – (16) all directly follow by interchanging differentiation and the Laplace transform. So we only have to show that changing the order is in fact justified. To this end note that

$$\frac{\partial}{\partial S_0}\int_0^\infty e^{-\alpha T}\,\mathrm{LC}_{\text{fixed}}(T,S_0,K)dT = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^\infty e^{-\alpha T}\Big(\mathrm{LC}_{\text{fixed}}(T,S_0+\epsilon,K) - \mathrm{LC}_{\text{fixed}}(T,S_0,K)\Big)dT.$$

Now the aim is to apply the dominated convergence theorem on the difference quotient. Observe that

$$\begin{aligned}
&\frac{1}{\epsilon}e^{-\alpha T}\Big|\mathrm{LC}_{\text{fixed}}(T,S_0+\epsilon,K) - \mathrm{LC}_{\text{fixed}}(T,S_0,K)\Big| \\
&\quad = \frac{1}{\epsilon}e^{-(\alpha+r)T}\,\mathbb{E}\Big[\big((S_0+\epsilon)e^{\overline{X}_T} - K\big)\mathbf{1}_{\{e^{\overline{X}_T}\geq K/(S_0+\epsilon)\}} - \big(S_0e^{\overline{X}_T}-K\big)\mathbf{1}_{\{e^{\overline{X}_T}\geq K/S_0\}}\Big] \\
&\quad = \frac{1}{\epsilon}e^{-(\alpha+r)T}\,\mathbb{E}\Big[\epsilon e^{\overline{X}_T}\mathbf{1}_{\{e^{\overline{X}_T}\geq K/S_0\}} + \big((S_0+\epsilon)e^{\overline{X}_T}-K\big)\mathbf{1}_{\{K/(S_0+\epsilon)\leq e^{\overline{X}_T}\leq K/S_0\}}\Big] \\
&\quad \leq e^{-(\alpha+r)T}\Big(\mathbb{E}[e^{\overline{X}_T}\mathbf{1}_{\{e^{\overline{X}_T}\geq K/S_0\}}] + \frac{K}{S_0}\Big) \\
&\quad \leq e^{-(\alpha+r)T}\Big(\mathbb{E}\big[\max(e^{\overline{X}_T},K/S_0)\big] + \frac{K}{S_0}\Big).
\end{aligned}$$

Furthermore we have that

$$\begin{aligned}
&\int_0^\infty e^{-(\alpha+r)T}\Big(\mathbb{E}\big[\max(e^{\overline{X}_T},K/S_0)\big] + \frac{K}{S_0}\Big)dT \\
&= \int_0^\infty e^{-(\alpha+r)T}\,\mathbb{E}\big[\max(e^{\overline{X}_T},K/S_0)\big]dT + \frac{K}{(\alpha+r)S_0},
\end{aligned}$$

where the first term on the right hand-side was already calculated and shown to be finite in the proof of Theorem 3.1. Thus the dominated convergence theorem can be applied to justify the interchange of integration and differentiation and we have

$$\begin{aligned}
\frac{\partial}{\partial S_0}\int_0^\infty e^{-\alpha T}\,\mathrm{LC}_{\text{fixed}}(T,S_0,K)dT &= \int_0^\infty e^{-\alpha T}\frac{\partial}{\partial S_0}\,\mathrm{LC}_{\text{fixed}}(T,S_0,K)dT \\
&= \int_0^\infty e^{-\alpha T}\Delta_{\mathrm{LC}_{\text{fixed}}}dT = \widehat{\Delta}_{\mathrm{LC}_{\text{fixed}}}(\alpha).
\end{aligned}$$

The argumentation in the case of $\Gamma_{\mathrm{LC}_{\text{fixed}}}$ is similar. Again we consider the differentiation quotient and again we want to apply the dominated convergence theorem. First note that

$$\begin{aligned}
&\frac{1}{\epsilon^2}e^{-\alpha T}\big|\big(\,\mathrm{LC}_{\text{fixed}}(T,S_0+\epsilon,K) - 2\,\mathrm{LC}_{\text{fixed}}(T,S_0,K) + \mathrm{LC}_{\text{fixed}}(T,S_0-\epsilon,K)\big| \\
&\qquad = \frac{1}{\epsilon^2}e^{-(\alpha+r)T}\Big|\mathbb{E}\Big[\epsilon e^{\overline{X}_T}\mathbf{1}_{\{e^{\overline{X}_T}\geq K/S_0\}} + ((S_0+\epsilon)e^{\overline{X}_T}-K)\mathbf{1}_{\{K/(S_0+\epsilon)\leq e^{\overline{X}_T}\leq K/S_0\}}\Big] \\
&\qquad\qquad - \mathbb{E}\Big[\epsilon e^{\overline{X}_T}\mathbf{1}_{\{e^{\overline{X}_T}\geq K/(S_0-\epsilon)\}} + (S_0e^{\overline{X}_T}-K)\mathbf{1}_{\{K/S_0\leq e^{\overline{X}_T}\leq K/(S_0-\epsilon)\}}\Big]\Big| \\
&\qquad \leq \frac{1}{\epsilon^2}e^{-(\alpha+r)T}\,\mathbb{E}\Big[\epsilon e^{\overline{X}_T}|\mathbf{1}_{\{e^{\overline{X}_T}\geq K/S_0\}} - \mathbf{1}_{\{e^{\overline{X}_T}\geq K/(S_0-\epsilon)\}}|\Big] \\
&\qquad\quad + \frac{1}{\epsilon^2}e^{-(\alpha+r)T}\,\mathbb{E}\Big[\epsilon e^{\overline{X}_T}\mathbf{1}_{\{K/(S_0+\epsilon)\leq e^{\overline{X}_T}\leq K/S_0\}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\epsilon^2} e^{-(\alpha+r)T}\, \mathbb{E}\Big[\big|(S_0 e^{\overline{X}_T} - K)(\mathbf{1}_{\{e^{\overline{X}_T} \geq K/S_0\}} - \mathbf{1}_{\{e^{\overline{X}_T} \geq K/(S_0-\epsilon)\}})\big|\Big] \\
\leq\ & \frac{1}{\epsilon} e^{-(\alpha+r)T}\, \mathbb{E}\Big[e^{\overline{X}_T}\mathbf{1}_{\{K/(S_0+\epsilon)\leq e^{\overline{X}_T} \leq K/(S_0-\epsilon)\}}\Big] \\
&+ \frac{1}{\epsilon^2} e^{-(\alpha+r)T}\, \mathbb{E}\Big[\big|S_0 e^{\overline{X}_T} - K\big|\mathbf{1}_{\{K/(S_0+\epsilon)\leq e^{\overline{X}_T} \leq K/(S_0-\epsilon)\}}\Big] \\
\leq\ & e^{-(\alpha+r)T}\frac{K+1}{S_0-\epsilon}\frac{\mathbb{P}\big[K/(S_0+\epsilon) \leq e^{\overline{X}_T} \leq K/(S_0-\epsilon)\big]}{\epsilon}.
\end{aligned}
$$

Hence the dominated convergence theorem can be applied, if

$$
\int_0^\infty e^{-(\alpha+r)T}\frac{\mathbb{P}\big[K/(S_0+\epsilon) \leq e^{\overline{X}_T} \leq K/(S_0-\epsilon)\big]}{\epsilon} dT < \infty,
$$

for any sufficiently small $\epsilon > 0$. In fact, this is easily seen to be the case, if the distribution of $\overline{X}_T$ admits a density.

Hence, finally, we have to argue why the density of the distribution of $\overline{X}_T$ exists. For this purpose, we use a result of Chaumont [Chaumont, 2010, Theorem 2], who states that $\overline{X}_T$ is absolutely continuous for $T > 0$ with respect to the Lebesgue measure on $\mathbb{R}^+$ if and only if the potential measure of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$ and 0 is a regular point for $(-\infty, 0)$ and $(0, \infty)$. Since $\sigma^2 > 0$, 0 is a regular point for both intervals in our case and following Bertoin [Bertoin, 1996, Theorem II.16] we get that the absolute continuity of the potential measure is equivalent to

$$
\int_{\mathbb{R}} \Re\Big(\frac{1}{q+\phi(x)}\Big) dx < \infty, \tag{17}
$$

where $\phi(x)$ is the characteristic exponent of X_T given in (1). Using Lemma 2.1, we conclude that all singularities and roots of $q + \log(\phi(x))$ have non-zero imaginary part and since $\sigma^2 > 0$, $\phi(x)$ is a polynomial of degree $n^+ + n^- + 2$ divided by a polynomial of degree $n^+ + n^-$, thus it follows that integral in (17) is finite.

The equations (15) and (16) follow by similar arguments. □

Corollary 3.2 *Under the assumptions of Theorem 3.2, the first and second order derivatives of prices of floating strike lookback options with respect to S_0 are given by*

$$
\begin{aligned}
\widehat{\Delta}_{\mathrm{LP}_{\mathit{float}}}(\alpha) &= \frac{1}{\alpha+r}\sum_{k=1}^{m^+} A_k^+(\alpha+r)\rho_k^+(\alpha+r)\frac{e^{-\log(M/S_0)(\rho_k^+(\alpha+r)-1)}}{\rho_k^+(\alpha+r)-1} - \frac{1}{\alpha}, && 0 < S_0 \leq M, \\
\widehat{\Gamma}_{\mathrm{LP}_{\mathit{float}}}(\alpha) &= \frac{1}{\alpha+r}\frac{1}{S_0}\sum_{k=1}^{m^+} A_k^+(\alpha+r)\rho_k^+(\alpha+r)e^{-\log(M/S_0)(\rho_k^+(\alpha+r)-1)}. && 0 < S_0 \leq M, \\
\widehat{\Delta}_{\mathrm{LC}_{\mathit{float}}}(\alpha) &= \frac{1}{\alpha} + \frac{1}{\alpha+r}\sum_{k=1}^{m^-} A_k^-(\alpha+r)\rho_k^-(\alpha+r)\frac{e^{-\log(N/S_0)(\rho_k^-(\alpha+r)-1)}}{1-\rho_k^-(\alpha+r)}, && 0 \leq N \leq S_0, \\
\widehat{\Gamma}_{\mathrm{LC}_{\mathit{float}}}(\alpha) &= -\frac{1}{\alpha+r}\frac{1}{S_0}\sum_{k=1}^{m^-} A_k^-(\alpha+r)\rho_k^-(\alpha+r)e^{-\log(N/S_0)(\rho_k^-(\alpha+r)-1)}, && 0 \leq N \leq S_0.
\end{aligned}
$$

Proof:

The corollary follows directly by Corollary 3.1, Theorem 3.2, (7) and (8). □

Theorem 3.3 *Suppose X_t is a HEJD process with σ and let $\alpha > 0$. Then the Laplace transforms of the sensitivities of fixed strike lookback options with respect to the maturity T are given by*

$$\widehat{\Theta}_{\mathrm{LC}_{\mathit{fixed}}}(\alpha) = \alpha\widehat{\mathrm{LC}}_{\mathit{fixed}}(\alpha) = S_0\frac{\alpha}{\alpha+r}\sum_{k=1}^{m^+} A_k^+(\alpha+r)\frac{e^{-\log(K/S_0)(\rho_k^+(\alpha+r)-1)}}{\rho_k^+(\alpha+r)-1}, \qquad K \geq S_0, \tag{18}$$

$$\widehat{\Theta}_{\mathrm{LP}_{\mathit{fixed}}}(\alpha) = \alpha\widehat{\mathrm{LP}}_{\mathit{fixed}}(\alpha) = S_0\frac{\alpha}{\alpha+r}\sum_{k=1}^{m^-} A_k^-(\alpha+r)\frac{e^{-\log(K/S_0)(\rho_k^-(\alpha+r)-1)}}{1-\rho_k^-(\alpha+r)}, \qquad K \leq S_0. \tag{19}$$

Proof:
Note that the first equations in (18) and (19) are classic for Laplace transforms, given that $\mathrm{LC}_{\text{fixed}}$ and $\mathrm{LP}_{\text{fixed}}$ are differentiable (with respect to T). Here we will show that both are Lipschitz continuous and thus almost everywhere differentiable, which is sufficient for the before-mentioned results to apply.
Thus let us turn to the proof of the Lipschitz continuity and note that

$$\begin{aligned}
&|\mathrm{LC}_{\text{fixed}}(T+\epsilon, S_0, K) - \mathrm{LC}_{\text{fixed}}(T, S_0, K)| \\
=&\left|\mathbb{E}\big[e^{-r(T+\epsilon)}(S_0e^{\overline{X}_{T+\epsilon}} - K)^+\big] - \mathbb{E}\big[e^{-rT}(S_0e^{\overline{X}_T} - K)^+\big]\right| \\
\leq&\left|\mathbb{E}\big[e^{-r(T+\epsilon)}S_0e^{\overline{X}_{T+\epsilon}}\big] - \mathbb{E}\big[e^{-rT}S_0e^{\overline{X}_T}\big]\right| \\
=&\left|\mathbb{E}\big[e^{-rT}S_0(e^{\overline{X}_{T+\epsilon}} - e^{\overline{X}_T})\big] + \mathbb{E}\big[S_0e^{\overline{X}_{T+\epsilon}}(e^{-r(T+\epsilon)} - e^{-rT})\big]\right| \\
=&\left|e^{-rT}S_0\Big(\mathbb{E}\big[e^{\overline{X}_T}(e^{\overline{X}_{T+\epsilon}-\overline{X}_T} - 1)\big] + (e^{-r\epsilon} - 1)\,\mathbb{E}\big[e^{\overline{X}_{T+\epsilon}}\big]\Big)\right| \\
\leq& c_1\left|\mathbb{E}\big[e^{\overline{X}_T}(e^{\overline{X}_{T+\epsilon}-\overline{X}_T} - 1)\big]\right| + c_2(e^{-r\epsilon} - 1) \\
\leq& c_1\left|\mathbb{E}\big[e^{\overline{X}_T}\big]\,\mathbb{E}\big[(e^{\overline{X}_\epsilon} - 1)\big]\right| + c_3\epsilon \\
\leq& c_4\left|\mathbb{E}\big[(e^{\overline{X}_\epsilon} - 1)\big]\right| + c_3\epsilon,
\end{aligned}$$

where the c_i's denote some constants and we used the independence of the increments of the Lèvy process X, the fact that $\mathbb{E}[e^{\overline{X}_T}] < \infty$, and the local Lipschitz continuity of the exponential function.
Furthermore for any $1 < \beta < \alpha_1$ we have

$$\begin{aligned}
\left|\mathbb{E}\big[(e^{\overline{X}_\epsilon} - 1)\big]\right| &\leq \Big(\mathbb{E}\big[(e^{\overline{X}_\epsilon} - 1)^\beta\big]\Big)^{1/\beta} \\
&\leq \frac{\beta}{\beta-1}\Big(\mathbb{E}\big[(e^{X_\epsilon} - 1)^\beta\big]\Big)^{1/\beta} \\
&= \frac{\beta}{\beta-1}\Big(e^{\epsilon\phi(-i\beta)} - 1\Big) < c_5\epsilon
\end{aligned}$$

where we applied Jensen's inequality, Doob's martingale inequality, and again the local Lipschitz continuity of the exponential function.
This completes the proof of the Lipschitz continuity of $\mathrm{LC}_{\text{fixed}}$. The second result in (18) follows by similar arguments. □

Corollary 3.3 *Let $0 < N \leq S_0 \leq M$, $\alpha > 0$ and let the assumptions of Theorem 3.3 be satisfied. Then* *the Laplace transforms of* $\Theta_{\mathrm{LP}_{float}}$ *and* $\Theta_{\mathrm{LC}_{float}}$ *are given by*

$$\widehat{\Theta}_{\mathrm{LP}_{float}}(\alpha) = \alpha\widehat{\mathrm{LP}}_{float}(\alpha) = S_0\left(\frac{\alpha}{\alpha+r}\sum_{k=1}^{m^+}A_k^+(\alpha+r)\frac{e^{-\log(M/S_0)(\rho_k^+(\alpha+r)-1)}}{\rho_k^+(\alpha+r)-1}-1\right)+\frac{\alpha M}{\alpha+r}, \tag{20}$$

$$\widehat{\Theta}_{\mathrm{LC}_{float}}(\alpha) = \alpha\widehat{\mathrm{LC}}_{float}(\alpha) = S_0\left(1+\frac{\alpha}{\alpha+r}\sum_{k=1}^{m^-}A_k^-(\alpha+r)\frac{e^{-\log(N/S_0)(\rho_k^-(\alpha+r)-1)}}{1-\rho_k^-(\alpha+r)}\right)-\frac{\alpha N}{\alpha+r}, \tag{21}$$

respectively.

Proof:
By (7) it follows that the price of a floating strike lookback put option is the sum of the price of a fixed strike lookback call option and an exponential function with respect to T. Thus by the proof of the previous theorem the price of a floating strike lookback put option is the sum of two Lipschitz continuous functions and therefore Lipschitz continuous, which proves (20). The second statement (21) follows analogously. $\square$

4 Estimation of infinite activity processes via HEJD processes

Having seen that lookback options can be priced efficiently in HEJD-model markets the goal is now to apply these results to more general Lévy processes. While a direct generalisation is due to the lack of explicit formulae for the Laplace transforms of the supremum and infimum processes typically not possible, for so-called generalised hyper-exponential processes, there is another possibility, which we will discuss now.

Definition 4.1 (Generalised hyper-exponential Lévy process) *A Lévy process is called generalised hyper - exponential Lèvy process (GHE), if its Lévy measure admits a density* k *of the form* $k(x) = k_+(x)\mathbf{1}_{\{x>0\}} + k_-(-x)\mathbf{1}_{\{x<0\}}$*, where* k_+, k_- *are completely monotone functions on* $(0,\infty)$*.*

Obviously the class of hyper-exponential jump diffusions is a subclass of the GHE processes, since their Lévy density can be written as

$$k_{HEJD}(x) = \lambda^+\sum_{i=1}^{n^+}p_i^+\alpha_i^+e^{-\alpha_i^+x}\mathbf{1}_{\{x>0\}} + \lambda^-\sum_{j=1}^{n^-}p_j^-\alpha_j^-e^{-\alpha_j^-x}\mathbf{1}_{\{x<0\}}. \tag{22}$$

Another well-known member of the GHE class is the NIG process which has the following representations of its Lévy densities:

$$k_{NIG}(x) = \frac{\delta\alpha}{\pi}e^{\beta x}\frac{K_1(\alpha x)}{x}\mathbf{1}_{\{x>0\}} + \frac{\delta\alpha}{\pi}e^{\beta x}\frac{K_1(-\alpha x)}{-x}\mathbf{1}_{\{x<0\}}, \tag{23}$$

where $\alpha > |\beta| > 0, \delta > 0$ and K_1 is the McDonald function

$$K_1(x) = x\int_1^\infty e^{-vx}(v^2-1)^{1/2}dv.$$

Jeannin and Pistorius [Jeannin and Pistorius, 2010] show, that for every process X in GHE, a sequence of HEJD processes $(X^n)_{n\geq 0}$ can be constructed which converges weakly to X in the Skorokhod topology on the space of real-valued cadlag functions on $\mathbb{R}_+$. They also show that the sequence of maximum processes $(\overline{X}^n)_{n\geq 0}$ converges in distribution to the maximum process $\overline{X}$. The next theorem states that also the sequence of lookback option prices converges in distribution to the lookback option price under X.

Theorem 4.1 *Let X be a GHE process, which is not a compound Poisson process, let the price process be given as $S_t = S_0 e^{X_t}$ and let $\mathrm{LC}_{float}(S_0, K, T)$ be the pricing function of a floating strike lookback call option. Let $(X^n)_{n\geq 0}$ be a sequence of HEJD processes, with $X^n \to X$ for $n \to \infty$. Then the sequence of floating strike lookback put option prices LC^n_{float} under the approximated processes X^n converges to LC_{float}.*

Proof:
Following the proof of Theorem 3.1, by using equation (10) it is sufficient to show

$$\lim_{n\to\infty} \mathbb{E}[e^{-rT}(e^{\overline{X}^n_T} - e^z)\mathbf{1}_{\{\overline{X}^n_T \geq z\}}] = \mathbb{E}[e^{-rT}(e^{\overline{X}_T} - e^z)\mathbf{1}_{\{\overline{X}_T \geq z\}}],$$

which is, by the arguments used in the proof of Theorem 3.1, equivalent to

$$\lim_{n\to\infty} \int_z^\infty e^y\, \mathbb{P}[\overline{X}^n_T \geq y]dy = \int_z^\infty e^y\, \mathbb{P}[\overline{X}_T \geq y]dy.$$

By using inequality (9) from Lemma 3.1, we can dominate $e^y\, \mathbb{P}[\overline{X}^n_T \geq y]$ and apply the dominated convergence theorem, thus the proof is complete. $\square$

Remark 4.1 *The convergence of prices of fixed strike lookback options and floating strike lookback put options follows by similar arguments.*

5 Numerical results

In this last section we give numerical values of prices and Greeks of lookback options, which result by applying the Gaver-Stehfest algorithm for numerical Laplace transform inversion (see e.g. [Gaver, 1966]) to the formulae given in the Theorems 3.1 - 3.3 and Corollaries 3.1 - 3.3. These results are compared to corresponding values derived via Monte Carlo integration. The main advantage of our method is that computing the numerical Laplace inversion of prices and Greeks takes only 1 second per option while the Monte Carlo simulation values takes several minutes.
The numerical analysis is divided into three subsections: in the first subsection we analyse the numerical error of the Gaver-Stehfest algorithm, by comparing results from the presented Laplace inversion method with a Monte Carlo (MC) simulation of a HEJD process. In the second subsection we give prices of lookback options under a HEJD process which is fitted to a NIG process and compare them to a Monte Carlo simulation of the original NIG process. In the last section, we compare sensitivities resulting from our technique with the corresponding simulated values of a NIG process. All computations were done in Mathematica 7, but for the Laplace inversion the authors a specially implemented Gaver-Stehfest algorithm instead of the build-in function. Furthermore note that a high numerical precision is required in such computations.
The problem of fitting a HEJD process to a NIG process is considered in the articles of Crosby, LeSaux

and Mijatovic [Crosby et al., 2009] and Jeannin and Pistorius [Jeannin and Pistorius, 2010]. In both papers, a HEJD process is fitted to a NIG process with parameters $\alpha = 8.858, \beta = -5.808, \delta = 0.176$. All methods use a mixture of seven exponentially distributed upward jump variables Z_i^+ and a mixture of seven exponentially distributed downward jump variables Z_i^- to model the jumps of the logarithmic price process. In [Jeannin and Pistorius, 2010] the parameters $\alpha_i^\pm, i = 1, \dots, 7$ are fixed in the beginning and the remaining parameters $\lambda^\pm$, σ and $p_i^\pm, i = 1, \dots, 7$ are derived by a least squares approximation. The parameter μ follows from no-arbitrage considerations. Crosby, LeSaux and Mijatovic present several fitting methods which use more complicated optimisation techniques. In the following numerical examples we use the parameter set corresponding to method c) in [Crosby et al., 2009] because it showed the best performance. These parameters are given in Table 1.

Parameter set CLM

Parameter	Value
$\sigma; \lambda_+; \lambda_-$	0.04062; 3.09468; 4.55662
p^+	{0.07858, 0.15033, 0.20017, 0.22039, 0.20704, 0.14327, 0.00022}
p^-	{0.05004, 0.12865, 0.22579, 0.21569, 0.18166, 0.13097, 0.06717}
α^+	{70.53135, 64.58179, 54.96035, 43.32801, 31.69567, 22.07423, 16.12466}
α^-	{4.58662, 10.85414, 20.98976, 33.24374, 45.49773, 55.63335, 61.90087}

Table 1: Parameters of the calibrated HEJD process fitted to a NIG process with parameters $\alpha = 8.858, \beta = -5.808, \delta = 0.176$ (method by Crosby, LeSaux and Mijatovic [Crosby et al., 2009]).

5.1 Error of the Gaver-Stehfest algorithm

As a benchmark for our analysis, we use an unbiased Monte Carlo simulation method for the HEJD, similar to the technique presented in [Baldeaux, 2008], which applies to general jump diffusion processes. This approach uses the fact that the underlying process is a diffusion between two jump times, thus the maximum can be simulated exactly as the maximum of a Brownian bridge. The authors first simulated the jump times and sizes and then, conditionally on these values, the maxima of the Brownian bridges. This technique significantly speeds up the procedure. Addtional variance reduction technique were not applied.

The number of simulated paths is 100.000 and the computation time for one price is about 5 minutes. The results for prices of fixed strike lookback call and put options are given in the Tables 2 and 3, respectively. The numerical errors in the approximation of sensitivities of fixed strike lookback put options are given in Table 4.

Prices of fixed strike lookback call options in the HEJD model

S_0	MC Price	95%-conf.int.	CLM
70	0.00116	(0.00048; 0.00185)	0.00088
75	0.00387	(0.00227; 0.00546)	0.00322
80	0.00953	(0.00722; 0.01185)	0.01071
85	0.03632	(0.03091; 0.04174)	0.03253
90	0.09638	(0.08842; 0.10435)	0.09083
95	0.24363	(0.23116; 0.25609)	0.23522
100	0.56749	(0.54821; 0.58677)	0.56626
105	1.25536	(1.22719; 1.28353)	1.26059
110	2.53137	(2.49160; 2.57114)	2.56846
115	4.78483	(4.73138; 4.83828)	4.74117
120	7.93318	(7.86722; 7.99914)	7.90993
122.5	9.83583	(9.76352; 9.90814)	9.86508
125	12.03170	(11.95470; 12.10870)	12.0539
127.5	14.44220	(14.36186; 14.52254)	14.4661
128	14.98910	(14.90728; 15.07092)	14.975
128.5	15.57960	(15.49714; 15.66206)	15.4928
129	16.00890	(15.92693; 16.09087)	16.0197
129.5	16.55780	(16.47508; 16.64052)	16.556
130	17.01520	(16.93203; 17.09837)	17.102

Table 2: Prices of fixed strike lookback call options with varying initial asset price S_0, strike price $K = 130$ and maturity $T = 1$.

Prices of fixed strike lookback put options in the HEJD model

S_0	MC Price	95%-conf.int.	CLM
70	6.52566	(6.47190; 6.57942)	6.53762
70.5	6.13835	(6.08455; 6.19215)	6.17411
71	5.85091	(5.79742; 5.90440)	5.88262
71.5	5.63913	(5.58548; 5.69278)	5.62039
72	5.39465	(5.34158; 5.44772)	5.37709
72.5	5.12923	(5.07689; 5.18157)	5.14933
75	4.16099	(4.11118; 4.21080)	4.19454
77.5	3.43517	(3.38830; 3.48204)	3.46899
80	2.92254	(2.87822; 2.96686)	2.90436
85	2.03532	(1.99668; 2.07396)	2.09797
90	1.57349	(1.53832; 1.60866)	1.56554
95	1.19229	(1.16122; 1.22336)	1.19899
100	0.95105	(0.92272; 0.97937)	0.93796
105	0.72813	(0.70269; 0.75356)	0.74684
110	0.56268	(0.53989; 0.58548)	0.60358
115	0.47751	(0.45603; 0.49899)	0.49407
120	0.39429	(0.37399; 0.41458)	0.40891
125	0.33014	(0.31157; 0.34870)	0.34171
130	0.26002	(0.24218; 0.27786)	0.28801

Table 3: Prices of fixed strike lookback put options with varying initial asset price S_0, strike price $K = 70$ and maturity $T = 1$.

The prices derived by our method are located in almost all cases in the 95%-confidence interval of the Monte Carlo estimator. Therefore, we conclude that numerical error resulting from the Gaver-Stehfest algorithm is very small, especially when the difference between the initial asset price and the strike price is not too large.

The Monte Carlo sensitivities in Table 4 are estimated by unbiased central finite difference estimators as described in Glasserman [Glasserman, 2004, Chapter 7]. To derive unbiased MC estimators for the sensitivities, we use the same set of random paths for each price computation, therefore a comparison of the MC prices with prices resulting from our method is omitted.

Greeks of fixed strike lookback put options in the HEJD model

S_0	MC Δ	MC Γ	CLM Δ	CLM Γ	Δ diff. in %	Γ diff. in %
70	-0.86616	0.43778	-0.87705	0.90464	1.26%	106.64%
72.5	-0.44270	0.05355	-0.44126	0.05496	-0.33%	2.64%
75	-0.32967	0.04057	-0.33031	0.03604	0.20%	-11.16%
77.5	-0.25327	0.02434	-0.25452	0.02544	0.49%	4.52%
80	-0.19952	0.02035	-0.20002	0.01862	0.25%	-8.50%
82.5	-0.15964	0.01346	-0.15965	0.01396	0.00%	3.68%
85	-0.12936	0.00842	-0.12911	0.01065	-0.19%	26.58%
87.5	-0.10584	0.00736	-0.10563	0.00826	-0.20%	12.15%
90	-0.08711	0.00741	-0.08731	0.00649	0.23%	-12.43%
92.5	-0.07259	0.00568	-0.07283	0.00516	0.33%	-9.23%
95	-0.06109	0.00363	-0.06126	0.00414	0.28%	14.25%
97.5	-0.05159	0.00346	-0.05192	0.00336	0.64%	-2.87%
100	-0.04420	0.00268	-0.04430	0.00275	0.23%	2.80%

Table 4: Prices of fixed strike lookback put options with varying initial asset price S_0, strike price $K = 70$ and maturity $T = 1$.

The relative differences in the last two columns are calculated using the following formulae:

$$\Delta\text{diff.} = \frac{CLM\ \Delta - MC\ \Delta}{MC\ \Delta}, \quad \Gamma\text{diff.} = \frac{CLM\ \Gamma - MC\ \Gamma}{MC\ \Gamma}. \tag{24}$$

The numerical error in the computation of the sensitivities is relatively small, although the values of the error of the second derivative vary quite a lot. Especially, when S_0 is close to the strike price the Monte Carlo estimator and the Laplace inversion values differ.

5.2 Error of the parameter fit

The next step is to compare prices derived by our numerical Laplace inversion method with a Monte Carlo simulation of the corresponding NIG process. The paths of the NIG process were simulated on an equidistant grid, which of course introduces a bias, but numerical experiments show that a simulation of 100.000 simulated paths with 1.000 grid points provides a reasonable accuracy.

The computation times are about one hour for the Monte Carlo method and 1 second for the computation of one price together with the corresponding sensitivities using the presented Laplace inversion method. Our first example (see Table 5) is a fixed strike lookback call option. In Table 6, prices of floating strike lookback put options are compared.

Prices of fixed strike lookback call options in the NIG model

S_0	MC Price	95%-conf.Int.	CLM
70	0.00116	(0.00051; 0.00182)	0.00088
75	0.00438	(0.00273; 0.00603)	0.00322
80	0.01202	(0.00931; 0.01474)	0.01071
85	0.03254	(0.02776; 0.03733)	0.03253
90	0.08844	(0.08092; 0.09595)	0.09083
95	0.21810	(0.20634; 0.22987)	0.23522
100	0.54534	(0.52699; 0.56369)	0.56626
105	1.23040	(1.20301; 1.25779)	1.26059
110	2.52913	(2.49046; 2.56780)	2.56846
115	4.60783	(4.55722; 4.65844)	4.74117
120	7.78941	(7.72576; 7.85306)	7.90993
122.5	9.64449	(9.57600; 9.71298)	9.86508
125	11.9014	(11.8273; 11.9754)	12.0539
127.5	14.3110	(14.2333; 14.3886)	14.4661
128	14.83000	(14.7516; 14.9083)	14.975
128.5	15.3338	(15.2547; 15.4128)	15.4928
129	15.8319	(15.7524; 15.9113)	16.01972
129.5	16.3457	(16.2658; 16.4255)	16.556
130	16.9719	(16.8910; 17.0527)	17.102

Table 5: Prices of fixed strike lookback call options with varying initial asset price S_0, strike price $K = 130$ and maturity $T = 1$.

Prices of floating strike lookback put options in the NIG model

S_0	MC Price	95%-conf.Int.	CLM
70	56.1569	(56.0765; 56.2372)	56.1588
75	51.1523	(51.0661; 51.2384)	51.1611
80	46.1821	(46.0902; 46.2739)	46.1686
85	41.1983	(41.1016; 41.2949)	41.1906
90	36.1890	(36.0873; 36.2906)	36.2488
95	31.3931	(31.2864; 31.4997)	31.3932
100	26.7064	(26.5971; 26.8156)	26.7242
105	22.3340	(22.2236; 22.4443)	22.4185
110	18.6218	(18.5126; 18.7309)	18.7265
115	15.8606	(15.7538; 15.9673)	15.8990
120	13.9640	(13.8601; 14.0678)	14.0679
122.5	13.3252	(13.2226; 13.4277)	13.5231
125	13.0204	(12.9178; 13.1229)	13.2119
127.5	12.9782	(12.8746; 13.0817)	13.1239
128	12.9709	(12.8670; 13.0747)	13.1330
128.5	12.8558	(12.7526; 12.9589)	13.1509
129	13.0367	(12.9317; 13.1416)	13.1778
129.5	12.9958	(12.8911; 13.1004)	13.2141
130	13.0514	(12.9466; 13.1561)	13.2599

Table 6: Prices of floating strike lookback put options with varying initial asset price S_0, initial maximum $M = 130$ and maturity $T = 1$.

The fitting procedure for HEJD processes is accurate and robust in the case of vanilla options, see [Crosby et al., 2009]. Nevertheless, especially for values of S_0 near K and M, respectively, there is a remarkable difference between the corresponding prices. A possible improvement could be to consider a fitting method which concentrates more on the tail behavior of the distribution of the increments of the underlying process. See [Asmussen et al., 2007], for a fitting method which takes that into account in the case of fitting a HEJD to a CGMY process.

5.3 Overall error of the sensitivity estimators

The purpose of this subsection is to compare sensitivities of prices of fixed strike lookback options computed with a Monte Carlo method with our method, using the parameter set CLM. The last two columns in every of the following tables are calculated via (24).

Greeks of fixed strike lookback call options in the NIG model

S_0	MC Δ	MC Γ	CLM Δ	CLM Γ	Δ Diff. in %	Γ Diff. in %
70	0.02012	0.00471	0.0215133	0.005021	6.92%	6.61%
72.5	0.03534	0.00781	0.0379496	0.008369	7.38%	7.16%
75	0.06035	0.01230	0.0647412	0.013370	7.28%	8.71%
77.5	0.10287	0.01968	0.10642	0.020297	3.45%	3.14%
80	0.16421	0.02750	0.1677	0.028963	2.13%	5.32%
82.5	0.24754	0.03792	0.251891	0.038380	1.76%	1.21%
85	0.35310	0.04624	0.358668	0.046664	1.58%	0.92%
87.5	0.47680	0.05184	0.482484	0.051717	1.19%	-0.24%
90	0.60902	0.05424	0.613796	0.0527	0.78%	-2.84%
92.5	0.73850	0.05136	0.743544	0.050765	0.68%	-1.16%
95	0.86106	0.04752	0.866602	0.047605	0.64%	0.18%
97.5	0.97660	0.04320	0.982458	0.045664	0.60%	5.70%
100	1.09440	0.04800	1.10199	0.052875	0.69%	10.16%

Table 7: Prices of fixed strike lookback call options with varying initial asset price S_0, strike price $K = 100$ and maturity $T = 1$.

Greeks of fixed strike lookback put options in the NIG model

S_0	MC Δ	MC Γ	CLM Δ	CLM Γ	Δ Diff. in %	Γ Diff. in %
70	-0.87740	0.31746	-0.87704	0.90464	-0.04%	184.96%
72.5	-0.44139	0.05647	-0.44126	0.05496	-0.03%	-2.67%
75	-0.32881	0.03777	-0.33031	0.03604	0.46%	-4.58%
77.5	-0.25198	0.02701	-0.25452	0.02544	1.01%	-5.81%
80	-0.19713	0.01994	-0.20001	0.01861	1.46%	-6.63%
82.5	-0.15679	0.01551	-0.15964	0.01395	1.82%	-10.02%
85	-0.12624	0.01051	-0.12911	0.01065	2.28%	1.36%
87.5	-0.10313	0.00858	-0.10562	0.00825	2.42%	-3.75%
90	-0.08494	0.00425	-0.08730	0.00648	2.78%	52.64%
92.5	-0.07081	0.00436	-0.07282	0.00515	2.85%	18.28%
95	-0.05947	0.00340	-0.06125	0.00414	3.01%	21.87%
97.5	-0.05034	0.00291	-0.05191	0.00336	3.13%	15.54%
100	-0.04280	0.00293	-0.04430	0.00275	3.51%	-6.06%

Table 8: Prices of fixed strike lookback call options with varying initial asset price S_0, strike price $K = 70$ and maturity $T = 1$.

Note that the computation of the Greeks (Δ, Γ) means almost no additional computational effort as one can see for example by comparing the formulae in Theorem 3.1 and 3.2. As in Subsection 5.1, the error of the Γ values is relatively high, especially near $S_0 = K$.

6 Conclusion

In this paper, we present explicit formulae for the Laplace transforms of prices and sensitivities of lookback options in a hyper-exponential jump diffusion model. Since a wide class of exponential Lévy processes can be approximated arbitrarily close by HEJD processes, these results give the possibility to efficiently approximate prices of lookback options for a vast class of processes used in financial modelling. The effectiveness of the inversion of the Laplace transformed values was illustrated in several numerical examples.

Bibliography

S. Asmussen, F. Avram, and M. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and Applications*, 109:79–111, 2004.

S. Asmussen, D. Madan, and M. Pistorius. Pricing Equity Default Swaps under an approximation to the CGMY Lévy Model. *Journal of Computational Finance*, 11:79–93, 2007.

J.F. Baldeaux. Unbiased Monte Carlo Methods for Lookback options in Jump-diffusion models. *School of Mathematics and Statistics, University of New South Wales: working paper*, 2008.

O.E. Barndorff-Nielsen. Processes of Normal Inverse Gaussian type. *Finance and Stochastics*, 2:41–68, 1998.

J. Bertoin. *Lévy processes*. Cambridge University Press, Cambridge, 1996.

N. Cai. On the first passage time of a hyper exponential jump diffusion process. *Operation Research Letters*, 37(2): 127–134, 2009.

N. Cai, N. Chen, and X. Wan. Pricing double-barrier options under a flexible jump diffusion model. *Operation Research Letters*, 37(3):163–167, 2009.

P. Carr, P. Geman, D.B. Madan, and M. Yor. The fine structure of asset returns: an empirical investigation. *Journal of Business*, 75:305–332, 2002.

L. Chaumont. On the law of the supremum of Lévy processes. *arXiv:1011.4151v1*, 2010.

J. Crosby, N. LeSaux, and A. Mijatovic. Approximating Lévy processes with a view to option pricing. *Interantional Journal of Theoretical and Applied Finance*, 13(1):63–91, 2009.

E. Eberlein, K. Glau, and A. Papapantoleon. Analyticity of the Wiener-Hopf factors and valuation of exotic options in Lévy models. *In: Advanced Mathematical Methods for Finance, G. Di Nunno, B. Øksendal (eds.), Springer*, 2011.

D. P. Gaver. Observing stochastic processes and approximate transform inversion. *Operations Research*, 14:444–489, 1966.

P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer Verlag, New York, 2004.

M. Jeannin and M. Pistorius. A transform approach to calculate prices and Greeks of barrier options driven by a class of Lévy processes. *Quantitative Finance*, 10(6), 2010.

S.G. Kou. A Jump-Diffusion Model for Option Pricing. *Management Science*, 48:1086–1101, 2002.

S.G. Kou, G. Petrella, and H. Wang. Pricing Path-Dependent Options with Jump Risk via Laplace Transforms. *The Kyoto Economic Review*, 74(9):1–23, 2005.

A.E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin, 2006.

E. Mordecki. The distribution of the maximum of a Lévy process with positive jumps of phase-type. *Theory of Stochastic Processes*, 8:309–316, 2002.

K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.

A. Sepp. Analytical Valuation of Lookback and Russian Options in a Double-Exponential Jump-Diffusion Model. *Working paper*, 2005.