On the maximal spectral type of a class of rank one transformations

Christoph Aistleitner*, Markus Hofer†

Abstract

Rank one transformations are transformations which can be obtained by cutting and stacking, using a single column in each step. Such a transformation is defined by a sequence of cutting parameters $(p_k)_{k\geq 1}$ and a sequence of parameters of spacers $((a_m^{(k)})_{m=1}^{p_k})_{k\geq 1}$. Rank one transformations are ergodic and have simple spectrum. By a result of Klemes and Reinhold, a rank one transformation is of singular maximal spectral type if $\sum_{k=1}^{\infty} p_k^{-2} = \infty$. El Abdalaoui showed that for arbitrary $(p_k)_{k\geq 1}$ the transformation has singular maximal spectral type if for each k all the numbers $(a_m^{(k)})_{m=1}^{p_k}$ are of different order of magnitude. In the present paper we prove a counterpart of El Abdalaoui's result: if for infinitely many indices k a small number of coefficients $(a_m^{(k)})_{m=1}^{p_k}$ are all equal, then the transformation is of singular maximal spectral type.

1 Introduction and statement of results

A rank one transformation can be defined inductively by the cutting and stacking method, using a sequence $(p_k)_{k\geq 1}$ of cutting parameters and a sequence $((a_m^{(k)})_{m=1}^{p_k})_{k\geq 1}$ of parameters of spacers. We call a sequence of disjoint intervals of equal length a *tower*. Starting with the initial tower $H_0 = [0,1]$, assume that the k-th tower H_k is already defined. H_k is a partition of the interval $[0, r_k]$, for some $r_k \geq 1$, into disjoint subintervals of equal length, stacked one on top of each other in some order. We write h_k for the height of H_k (that is, the number of subintervals in H_k). To define H_{k+1}, we cut the tower H_k into p_k subcolumns. On the top of m-th subcolumn we add a number $a_m^{(k)}$ of intervals having the same length as the subcolumns, taken from the right of $[0, r_k]$. Then, stacking each subcolumn on the next one (from left to right), we obtain the tower H_{k+1}, which forms a partition of the interval

$$[0, r_{k+1}] := \left[0, r_k + \sum_{m=1}^{p_k} a_m^{(k)} l\right],$$

*Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: aistleitner@math.tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N23.

†Graz University of Technology, Institute of Mathematics A, Steyrergasse 30, 8010 Graz, Austria. e-mail: markus.hofer@tugraz.at. Research supported by the Austrian Research Foundation (FWF), Project S9603-N23.

Mathematics Subject Classification: 37A30, 28D05, 37A05

Keywords: Rank one transformations, maximal spectral type, exponential sums, martingale approximation

where l denotes the length of the subcolumns of H_k. Then H_{k+1} has height

$$h_{k+1} = p_k h_k + \sum_{m=1}^{p_k} a_m^{(k)}.$$

For notational convenience we assume $p_k \geq 2$ for $k \geq 1$. We also assume that the total measure is finite, i.e. that by rescaling by an appropriate constant we can achieve that $\bigcup_{k=1}^{\infty} H_k = [0,1]$. Then the transformation T, which is obtained in this way, is a measure preserving invertible point transformation on the unit interval. This construction is presented in more detail in [11].

Rank one systems we firstly defined in full generality by Ornstein [18], generalizing earlier constructions of Katok and Stepin [14] and Chacon [6]. Each rank one system is ergodic, of entropy zero, and has spectral multiplicity one. Using a probabilistic method Ornstein [18] constructed a family of mixing rank one transformations (which are consequently mixing of all orders, see [13]), in order to answer Banach's question whether there exists a dynamical system with simple Lebesgue spectrum. However, Bourgain [5] proved that Ornstein's transformations almost surely have singular maximal spectral type. Using the same methods, Klemes [15] showed that the staircase transformation has singular maximal spectral type. Klemes and Reinhold [16] proved that a rank one transformation has singular maximal spectral type if $\sum_{k=1}^{\infty} p_k^{-2} = \infty$, and recently El Abdalaoui [8] showed that the maximal spectral type is also singular provided the sequences $(a_m)_{m=1}^{p_k}$ are lacunary for all k (these results will be discussed in more detail in the next section). Klemes and Reinhold conjectured that the maximal spectral type of *any* rank one transformation is singular.

The main result of the present paper is our following Theorem 1.

Theorem 1 *Assume there exists a constant $c > 0$ such that for a subsequence $(k_n)_{n\geq 1}$ of $\mathbb{N}$ the following two conditions are satisfied:*

(i) $\frac{\log p_{k_n}}{h_{k_n}} \to 0$ *as* $n \to \infty$.

(ii) There exist sets $A(k_n)$ containing at least cp_{k_n} consecutive elements of $\{1, \ldots, p_{k_n}\}$ such that $a_m^{(k_n)} = a^{(k_n)}$, $m \in A(k_n)$, for some $a^{(k_n)}$.

Then the transformation has singular maximal spectral type.

As a direct consequence we obtain the following corollary.

Corollary 1 *Assume that there exists a constant c such that for a subsequence $(k_n)_{n\geq 1}$ of $\mathbb{N}$ condition (i) is satisfied, and there exist sets $A(k_n)$ of at least cp_{k_n} consecutive coefficients $a_m^{(k)}$ which are all zero (i.e., for these indices no spacers are added above the corresponding towers). Then the transformation is of singular maximal spectral type.*

The case when in each step all the numbers $(a_m^{(k)})_{1\leq m\leq p_k}$ are equal is a (deterministic) special case of the random construction in [9].
Note that condition (i) only rules out transformations for which the sequence $(p_k)_{k\geq 1}$ grows extremely fast (as a function of k), since by construction $(h_k)_{k\geq 1}$ grows at least exponentially.

2 Preliminaries

For any invertible, measure preserving transformation T on the unit interval and any function $f \in L^2(0,1)$, the corresponding spectral measure σ_f is defined by

$$\hat{\sigma}_f(n) = \langle f, T^n f \rangle = \int_0^1 e^{-2\pi i n \theta} \sigma_f(d\theta), \qquad n \in \mathbb{Z}.$$

The *maximal spectral type* of T is the equivalence class (with respect to absolute continuity) of all Borel measures σ on [0,1] for which $\sigma \ll \sigma_f$ for all $f \in L^2$, and for which for all measures ν which also satisfy $\nu \ll \sigma_f$ for all $f \in L^2$ necessarily $\sigma \ll \nu$ (here the symbol "$\ll$" denotes absolute continuity of measures). There always exists a function $f \in L^2$ such that σ_f is in the equivalence class defining the maximal spectral type (for details, see [17]).

The maximal spectral type σ of a rank one transformation is given (possibly up to some discrete measure) by the weak-*-limit (in the space of bounded Borel measures on the unit circle) of the generalized Riesz products

$$d\sigma = \lim \prod_{k=1}^N |P_k|^2 \ d\lambda,$$

where

$$P_k(\theta) = \frac{1}{\sqrt{p_k}} \left(\sum_{m=1}^{p_k} e^{-\theta\left(m h_k + \sum_{l=1}^m a_l^{(k)}\right)} \right),$$

and where λ denotes the normalized Lebesgue measure on the unit circle (see [7, 16]). If we want to prove $\sigma \perp \lambda$, it is by [5] sufficient to prove that

$$\inf_{N \geq 1,\ k_1 < \cdots < k_N} \int \prod_{m=1}^N |P_{k_m}(\theta)| \ d\lambda = 0. \tag{1}$$

Using this criterion, Klemes [15] proved that the staircase transformation has singular maximal spectral type (see also [7]). Klemes and Reinhold [16] showed that a rank one transformation has singular maximal spectral type, provided

$$\sum_{k=1}^\infty p_k^{-2} = \infty.$$

Recently, El Abdalaoui [8] obtained the following result: suppose that for any $k \geq 1$ and any $m \in \{1, \dots, p_k - 1\}$ we have $a_{m+1}^{(k)} \geq \sum_{l=1}^m a_l^{(k)}$ and $\sum_{m=1}^{p_k} a_m^{(k)} < h_k/2$. Then the transformation has singular maximal spectral type. El Abdalaoui's proof utilizes a new variant of the well-known central limit theorem for lacunary trigonometric series (see [19, 20]). In [5] and [15] the criterion (1) was used together with an asymptotic lower bound for $\int \left| |P_{k_m}|^2 - 1 \right|$, in [8] with an lower bound for $\int ||P_{k_m}| - 1|$. In our proof of Theorem 1 we will apply (1) directly, together with an approximation by martingales.

3 Proof of Theorem 1

For the proof of Theorem 1 we will use a martingale approximation technique, which has some similarities to a method commonly used in the theory of lacunary function systems (cf. for example [1, 2]).
Let $(k_n)_{n\geq 1}$ be a sequence for which conditions (i) and (ii) in the formulation of Theorem 1 hold. By condition (i) we can assume that $(k_n)_{n\geq 1}$ grows sufficiently fast such that

$$\frac{2^n \sqrt{p_{k_n}} \left(p_{k_n} h_{k_n} + \sum_{l=1}^{p_{k_n}} a_l^{(k_n)} \right) \log p_{k_{n+1}}}{h_{k_{n+1}}} \to 0 \qquad \text{as} \qquad n \to \infty. \tag{2}$$

To simplify formulas, we set

$$\begin{aligned} q_n &= p_{k_n}, & n \geq 1, \\ j_n &= h_{k_n}, & n \geq 1, \\ b_m^{(n)} &= a_m^{(k_n)}, & n \geq 1,\ 1 \leq m \leq q_n. \end{aligned}$$

We write $A(n) \subset \{1, \dots, q_n\}$ for the set from condition (ii) of Theorem 1, i.e. for a set of at least cq_n consecutive indices, such that $b_m^{(n)} = b^{(n)}$ for all $m \in A(n)$ for some number $b^{(n)}$. Thus $A(n)$ is of the form

$$\{m: \ s(n) \leq m \leq t(n)\} \qquad \text{for some} \quad 1 \leq s(n) \leq t(n) \leq q_n, \quad t(n) - s(n) \geq cq_n.$$

Furthermore, we set $B(n) = \{1, \dots, q_n\} \backslash A(n)$ and define

$$\begin{aligned} R_n(\theta) &= \frac{1}{\sqrt{q_n}} \left(\sum_{m \in A(n)} e^{-2\pi i \theta \left(m j_n + \sum_{l=1}^m b_l^{(n)} \right)} \right), & \theta \in [0,1], \\ S_n(\theta) &= \frac{1}{\sqrt{q_n}} \left(\sum_{m \in B(n)} e^{-2\pi i \theta \left(m j_n + \sum_{l=1}^m b_l^{(n)} \right)} \right), & \theta \in [0,1]. \end{aligned}$$

Clearly

$$\begin{aligned} R_n(\theta) &= \frac{1}{\sqrt{q_n}} \left(\sum_{m=s(n)}^{t(n)} e^{-2\pi i \theta \left(m j_n + \sum_{l=1}^m b^{(n)} \right)} \right) \\ &= \frac{e^{-2\pi i \theta s(n)(j_n + b^{(n)})}}{\sqrt{q_n}} \cdot \frac{1 - e^{-2\pi i \theta (t(n) - s(n) + 1)(j_n + b^{(n)})}}{1 - e^{-2\pi i \theta (j_n + b^{(n)})}}, \end{aligned}$$

and, by standard trigonometric identities,

$$|R_n(\theta)| = \frac{|\sin \pi\,(t(n) - s(n) + 1)\,(j_n + b^{(n)})\theta|}{\sqrt{q_n} |\sin \pi (j_n + b^{(n)})\theta|}. \tag{3}$$

Let

$$Q_N(\theta) = \prod_{n=1}^{N-1} \left(|R_n(\theta)| + |S_N(\theta)| \right)\ d\theta, \qquad N \geq 1.$$

To prove that the transformation has singular maximal spectral type, by the criterion (1) it is sufficient to show that

$$\lim_{N\to\infty} \int_0^1 Q_N(\theta)\ d\theta = 0. \tag{4}$$

Simply speaking, we will show in the sequel that for large N

$$\begin{aligned} \int_0^1 |Q_N(\theta)R_N(\theta)|\ d\theta &\approx \int_0^1 |Q_N(\theta)|\ d\theta \int_0^1 |R_N(\theta)|\ d\theta, && (5)\\ \int_0^1 |Q_N(\theta)S_N(\theta)|\ d\theta &\approx \int_0^1 |Q_N(\theta)|\ d\theta \int_0^1 |S_N(\theta)|\ d\theta, && (6)\end{aligned}$$

and

$$\int_0^1 |R_N(\theta)|\ d\theta = o(1) \quad \text{as} \quad N\to\infty, \tag{7}$$

$$\int_0^1 |S_N(\theta)|\ d\theta \lessapprox \sqrt{1-c}. \tag{8}$$

Combing (5), (6), (7) and (8) will prove the theorem. The estimate (7) follows from the standard inequality

$$\int_0^1 \frac{|\sin \pi n\theta|}{|\sin \pi\theta|}\ d\theta \le (1+\log n), \qquad n\ge 1, \tag{9}$$

while (8) is a consequence of the Cauchy-Schwartz inequality.

Let

$$d(n) = \left\lceil \log_2\left(2\pi\sqrt{q_n}\left(q_n j_n + \sum_{l=1}^{q_n} b_l^{(n)}\right)\right)\right\rceil + n + 1, \quad n\ge 1,$$

and let $\mathcal{F}_n,\ n\ge 1$ denote the σ-field generated by the sets

$$[l2^{-d(n)}, (l+1)2^{-d(n)}), \quad 0\le l < 2^{d(n)}.$$

By (2),

$$\frac{2^{d(n)}\log q_{n+1}}{j_{n+1}} \to 0 \qquad \text{as} \qquad n\to\infty. \tag{10}$$

We define discrete functions $\overline{R}_n, \overline{S}_n$ as the conditional expectations (with respect to the Lebesgue measure)

$$\overline{R}_n = \mathbb{E}(R_n|\mathcal{F}_n), \quad \overline{S}_n = \mathbb{E}(S_n|\mathcal{F}_n), \qquad n\ge 1.$$

Since the fluctuation of R_n and S_n on any atom of $\mathcal{F}_n$ is at most 2^{-n-1}, we have

$$|R_n(\theta)-\overline{R}_n(\theta)| \le 2^{-n-1}, \quad |S_N(\theta)-\overline{S}_n(\theta)| \le 2^{-n-1}, \qquad \theta\in[0,1]. \tag{11}$$

We define

$$\overline{Q}_N(\theta) = \prod_{n=1}^{N-1}\left(\left|\overline{R}_n(\theta)\right| + \left|\overline{S}_n(\theta)\right| + 2^{-N}\right)\ d\theta.$$

Then

$$\int_0^1 Q_N(\theta)\ d\theta \le \int_0^1 \overline{Q}_N(\theta)\ d\theta,$$

and to show (4) it remains to prove

$$\int_0^1 \overline{Q}_N(\theta)\ d\theta \to 0 \quad \text{as} \quad N \to \infty, \tag{12}$$

which can be deduced immediately from the following lemma.

Lemma 1 *There exists a number $\eta > 0$ such that*

$$\int_0^1 \overline{Q}_{N+1}(\theta)\ d\theta \le (1-\eta) \int_0^1 \overline{Q}_N(\theta)\ d\theta$$

for sufficiently large N.

We will use the following simple observation:

Lemma 2 *For any interval $[\alpha, \beta] \subset [0,1]$, any function f which is periodic with period 1, and any positive integer m,*

$$\left| \int_\alpha^\beta f(mx)\ dx \right| \le \frac{1}{m} \int_0^1 |f(x)|\ dx + (\beta - \alpha) \left| \int_0^1 f(x)\ dx \right|.$$

Proof of Lemma 1: By (11),

$$\int_0^1 \overline{Q}_N(\theta) |\overline{R}_N(\theta)|\ d\theta \le \int_0^1 \overline{Q}_N(\theta) |R_N(\theta)|\ d\theta + 2^{-N-1} \int_0^1 \overline{Q}_N(\theta)\ d\theta. \tag{13}$$

On any atom I of $\mathcal{F}_{N-1}$ the function $\overline{Q}_N$ is constant and, by (3), (9), Lemma 2 and since the function R_N is periodic with period $(j_N + b^{(N)})$,

$$\begin{aligned}
\int_I \overline{Q}_N(\theta) |R_N(\theta)|\ d\theta &\le \overline{Q}_N \int_I \frac{|\sin \pi (t(n) - s(n) + 1)(j_N + b^{(N)})\theta|}{\sqrt{q_N} |\sin \pi (j_N + b^{(N)})\theta|} \\
&\le \frac{\overline{Q}_N}{\sqrt{q_N}} \left(\int_0^1 \frac{|\sin \pi (t(n) - s(n) + 1)\theta|}{\sqrt{q_N} |\sin \pi \theta|}\ d\theta \right) \left(2^{-d(N-1)} + \frac{1}{j_N + b^{(N)}} \right) \\
&\le \frac{\overline{Q}_N (1 + \log q_N)}{\sqrt{q_N}} \left(2^{-d(N-1)} + \frac{1}{j_N} \right).
\end{aligned}$$

Therefore, for arbitrary $\varepsilon > 0$, by (10)

$$\begin{aligned}
\int_0^1 \overline{Q}_N(\theta) |R_N(\theta)|\ d\theta &= \int_0^1 \mathbb{E} \left(\overline{Q}_N(\theta) |R_N(\theta)| \middle| \mathcal{F}_{N-1} \right)\ d\theta \\
&\le \frac{(1 + \log q_N)}{\sqrt{q_N}} \left(1 + \frac{2^{d(N-1)}}{j_N} \right) \int_0^1 \overline{Q}_N(\theta)\ d\theta \\
&\le \varepsilon \int_0^1 \overline{Q}_N(\theta)\ d\theta \qquad \text{for sufficiently large } N.
\end{aligned} \tag{14}$$

By (11),

$$\int_0^1 \overline{Q}_N(\theta)|\overline{S}_N(\theta)|\ d\theta \le \int_0^1 \overline{Q}_N(\theta)|S_N(\theta)|\ d\theta + 2^{-N-1}\int_0^1 \overline{Q}_N(\theta)\ d\theta. \tag{15}$$

On any atom I of $\mathcal{F}_{N-1}$, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\int_I \overline{Q}_N(\theta)|S_N(\theta)|\ d\theta &= \int_I \overline{Q}_N(\theta)|S_N(\theta)|\ d\theta \\ &\le 2^{-d(N-1)/2}\ \overline{Q}_N(\theta)\left(\int_I |S_N(\theta)|^2\ d\theta\right)^{1/2}.\end{aligned} \tag{16}$$

It is easily seen that the function $|S_N(\theta)|^2$ is of the form

$$\frac{1}{q_N}\sum_{m_1,m_2\in B(N)}\left(\cos\left(\left(m_1 j_N + \sum_{l=1}^{m_1} b_l^{(N)}\right) - \left(m_2 j_N + \sum_{l=1}^{m_2} b_l^{(N)}\right)\right)\right).$$

Thus by Lemma 2 and the orthogonality of the trigonometric system,

$$\begin{aligned} q_N\int_I |S_N(\theta)|^2\ d\theta &\le \sum_{\substack{m_1,m_2\in B(N),\\ m_1\ne m_2}} \frac{1}{\left|\left(m_1 j_N + \sum_{l=1}^{m_1} b_l^{(N)}\right) - \left(m_2 j_N + \sum_{l=1}^{m_2} b_l^{(N)}\right)\right|} \\ &\quad + \sum_{\substack{m_1,m_2\in B(N),\\ m_1 = m_2}} \int_I 1\ d\theta \\ &\le \left(\sum_{\substack{m_1,m_2\in B(N),\\ m_1\ne m_2}} \frac{1}{|m_1-m_2| j_N}\right) + (1-c)q_N 2^{-d(N-1)} \\ &\le \frac{2q_N(1+\log q_N)}{j_N} + (1-c)q_N 2^{-d(N-1)}. \end{aligned}$$

Hence, by (10) and (16),

$$\begin{aligned}\int_0^1 \overline{Q}_N(\theta)|S_N(\theta)|\ d\theta &= \int_0^1 \mathbb{E}\left(\overline{Q}_N(\theta)|S_N(\theta)|\Big|\mathcal{F}_{N-1}\right)\ d\theta \\ &\le \left((1-c) + 2^{d(N-1)/2}\cdot\frac{2+2\log q_N}{j_N}\right)^{1/2}\int_0^1 \overline{Q}_N\ d\theta \\ &\le \left(\sqrt{1-c}+\varepsilon\right)\int_0^1 \overline{Q}_N\ d\theta.\end{aligned} \tag{17}$$

It is obvious that for sufficiently large N

$$2^{-N}\int_0^1 \overline{Q}_N\ d\theta \le \varepsilon\int_0^1 \overline{Q}_N\ d\theta. \tag{18}$$

Combining (13), (14), (15), (17) and (18), we have proved that

$$\int_0^1 \overline{Q}_N(\theta)\left(|\overline{R}_N| + |\overline{S}_N| + 2^{-N}\right)\ d\theta \le \left(\sqrt{1-c}+3\varepsilon\right)\int_0^1 \overline{Q}_N(\theta)\ d\theta$$

for sufficiently large N. Since ε was arbitrary, this proves Lemma 1. Consequently, the proof of Theorem 1 is also complete.

4 Remarks

In [8], El Abdalaoui uses the central limit theorem (CLT) for normalized trigonometric sums, together with Bourgain's criterion (1), to prove that a certain class of rank one transformations has singular maximal spectral type. In this case the CLT holds due to the fact that by assumption the coefficients $(a_l)_{l=1}^{p_k}$ form a lacunary sequence (for each $k \geq 1$). However, without assuming any further conditions on the sequence of coefficients of spacers, the trigonometric sums in (1), which are of the form

$$\frac{1}{\sqrt{p_k}} \left| \sum_{m=1}^{p_k} e^{-2\pi i \theta \left(m h_k + \sum_{l=1}^{m} a_l^{(k)} \right)} \right|, \qquad \theta \in [0,1], \tag{19}$$

will in general not satisfy the CLT. In fact, the sequence of frequencies $(mh_k + \sum_{l=1}^m a_l^{(k)})_{m=1}^{p_k}$ in (19) may grow only linearly, while by a criterion of Erdős [10] the sequence of frequencies would have to grow at least like $e^{\sqrt{m}}$ to guarantee the CLT. However, recently several new randomized constructions of slowly growing sequences $(n_k)_{k\geq 1}$, for which the CLT for $N^{-1/2} \sum_{k=1}^N \cos 2\pi n_k x$ almost surely holds, have been presented, see [3, 4, 12]. It is likely that these results, together with El Abdalaoui's method, could yield new randomized constructions of rank one transformations which have almost surely singular maximal spectral type.

As mentioned in the introduction, several authors believe that *all* rank one transformations could have singular maximal spectral type. In [5], Bourgain notes that this question is related to the problem whether

$$\sup_{n>1} \sup_{k_1 < \dots < k_n} \frac{1}{\sqrt{n}} \left\| \sum_{m=1}^{n} e^{2\pi i k_m \theta} \right\| < 1,$$

where $\|\cdot\|$ denotes the L^1 norm on the unit interval. He proved that

$$\sup_{k_1 < \dots < k_n} \frac{1}{\sqrt{n}} \left\| \sum_{m=1}^{n} e^{2\pi i k_m \theta} \right\| \leq 1 - \frac{c \log n}{n}$$

for some constant c. Apparently, no further progress has been made since then.

References

[1] C. Aistleitner. On the law of the iterated logarithm for the discrepancy of Lacunary sequences. *Trans. Amer. Math. Soc.*, 362(11):5967–5982, 2010.

[2] C. Aistleitner and I. Berkes. On the central limit theorem for $f(n_k x)$. *Probab. Theory Related Fields*, 146(1-2):267–289, 2010.

[3] I. Berkes. A central limit theorem for trigonometric series with small gaps. *Z. Wahrsch. Verw. Gebiete*, 47(2):157–161, 1979.

[4] S. G. Bobkov and F. Götze. Concentration inequalities and limit theorems for randomized sums. *Probab. Theory Related Fields*, 137(1-2):49–81, 2007.

[5] J. Bourgain. On the spectral type of Ornstein's class one transformations. *Isr. J. Math.*, 84(1-2):53–63, 1993.

[6] R. Chacon. A geometric construction of measure preserving transformations. Proc. 5th Berkeley Symp. Math. Stat. Probab., Univ. Calif. 1965/66, 2, Part 2, 335-360, 1967.

[7] J. Choksi and M. Nadkarni. The maximal spectral type of a rank one transformation. *Can. Math. Bull.*, 37(1):29–36, 1994.

[8] E. H. El Abdalaoui. A new class of rank-one transformations with singular spectrum. *Ergodic Theory Dyn. Syst.*, 27(5):1541–1555, 2007.

[9] E. H. El Abdalaoui, F. Parreau, and A. A. Prikhodko. A new class of Ornstein transformations with singular spectrum. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(6):671–681, 2006.

[10] P. Erdős. On trigonometric sums with gaps. *Magyar Tud. Akad. Mat. Kutató Int. Közl*, 7:37–42, 1962.

[11] N. Friedman. *Introduction to ergodic theory.* Van Nostrand Reinhold Mathematical Studies. 29. New York etc.: Van Nostrand Reinhold Company IV, 143 p., 1970.

[12] K. Fukuyama. A central limit theorem and a metric discrepancy result for sequences with bounded gaps. In *Dependence in probability, analysis and number theory*, pages 233–246. Kendrick Press, Heber City, UT, 2010.

[13] S. A. Kalikow. Twofold mixing implies threefold mixing for rank one transformations. *Ergodic Theory Dyn. Syst.*, 4:237–259, 1984.

[14] A. B. Katok and A. M. Stepin. Approximations in ergodic theory. *Uspehi Mat. Nauk*, 22(5 (137)):81–106, 1967.

[15] I. Klemes. The spectral type of the staircase transformation. *Tohoku Math. J. (2)*, 48(2):247–258, 1996.

[16] I. Klemes and K. Reinhold. Rank one transformations with singular spectral type. *Isr. J. Math.*, 98:1–14, 1997.

[17] M. Nadkarni. *Spectral theory of dynamical systems.* Birkhäuser Advanced Texts. Basel: Birkhäuser. 182 p., 1998.

[18] D. S. Ornstein. On the root problem in ergodic theory. Proc. 6th Berkeley Sympos. math. Statist. Probab., Univ. Calif. 1970, 2, 347-356 (1972)., 1972.

[19] R. Salem and A. Zygmund. On lacunary trigonometric series. *Proc. Natl. Acad. Sci. USA*, 33:333–338, 1947.

[20] R. Salem and A. Zygmund. La loi du logarithme iteré pour les séries trigonométriques lacunaires. *Bull. Sci. Math., II. Ser.*, 74:209–224, 1950.