Mathematical Analysis of Algorithms

Exercises I. (for the exercise session on 26.03.2012)

1. (1.5pt) Consider a sequence $\{D_n\}_{n\geq 1}$ satisfying the recurrence

$$\begin{cases} D_n = D_{\lfloor \frac{n}{2} \rfloor} + 1, n \geq 2, \\ D_1 = 2. \end{cases}$$

   Show that $D_n = \lfloor \log n \rfloor + 2$. Here $\log n = \log_2 n$.

2. (0.5pt) For all $b \in \mathbb{N}$ with $b \geq 2$, show that

$$\sum_{k=1}^{n} \lfloor \log_b k \rfloor = (n+1)\lfloor \log_b n \rfloor - \frac{b^{\lfloor \log_b n \rfloor + 1} - b}{b-1}.$$

3. (0.5pt) Show that for $n \geq 1$,

$$\sum_{k=1}^{n} a_n = na_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k).$$

4. (0.5pt) Show that

$$\sum_{k=1}^{n} H_k = (n+1)H_n - n,$$

   where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ is the harmonic number.

5. (1.5pt) Find initial conditions a_0, a_1 and a_2 for which the growth rate of the solution to the recurrence

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, n \geq 3$$

   is

   - constant
   - exponential and
   - fluctuating in sign.

6. (1.0pt) Find a way to solve a higher-order linear recurrence of the term

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + r, \; n \geq k,$$

   where $r, c_1, c_2, \ldots, c_k$ are constants independent of n.

7. (1.0pt) Solve the recurrence:

$$\begin{cases} n(n-1)a_n = (n-1)a_{n-1} + a_{n-2}, n \geq 2, \\ a_0 = a_1 = 1. \end{cases}$$