# Mathematical Analysis of Algorithms.

# Exercises II. (07.05.2012)

1. (0.5pt) Show that OGF of the sequence $0, 0, \ldots, 1, m+1, \ldots, \binom{n}{m}, \ldots$ is

$$\sum_{n \geq m} \binom{n}{m} z^n = \frac{z^m}{(1-z)^{m+1}}.$$

2. (0.5pt) Show that the OGF $A(z) = \sum_{n \geq 1} a^n z^n$ for the sequence $\{a_n\}_{n \geq 0}$ with $a_n = \frac{1}{n}$ satisfies:

$$A(z) = \ln \frac{1}{1-z}$$

3. (0,5pt) Show that OGF of the sequence $0, 1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots, H_n, \ldots$ is

$$\sum_{n \geq 1} H_n z^n = \frac{1}{1-z} \ln \frac{1}{1-z}.$$

4. Find OGF for each of the following sequences:
   - (0.5pt) $\{2^{n+1}\}_{n \geq 0}$
   - (0.5pt) $\{n2^{n+1}\}_{n \geq 0}$
   - (1.0pt) $\{nH_n\}_{n \geq 0}$
   - (0.5pt) $\{n^3\}_{n \geq 0}$

5. Find $n![z^n]A(z)$ for each of the following EGFs:
   - (0.5pt) $\frac{1}{1-z} \ln \frac{1}{1-z}$
   - (0.5pt) $\left( \ln \frac{1}{1-z} \right)^2$
   - (0.5pt) $e^{z+z^2}$

6. Solve the following recurrences using OGFs.
   - (0.5pt)

$$\begin{cases} a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}, n \geq 3 \\ a_0 = 1, a_1 = 2, a_2 = 4. \end{cases}$$

   - (0.5pt)

$$\begin{cases} a_n = 2a_{n-2} - a_{n-4}, n \geq 4 \\ a_0 = a_1 = 0, a_2 = a_3 = 1. \end{cases}$$

7. (0.5pt) Show that

$$[z^n]\Big(\frac{z}{(1-z)^2}\ln\frac{1}{1-z}\Big) = n(H_n - 1)$$

8. Find the Taylor series expansion of the following functions:

   - (0.5pt) $\frac{1}{\sqrt{1-4z}}$
   - (0.5pt) $\sin z$
   - (0.5pt) $ze^z$
   - (0.5pt) $\frac{1}{1-z}\ln\frac{1}{1-z}$
   - (0.5pt) $\frac{1}{\sqrt{1-z}}\ln\frac{1}{1-z}$.

9. Show that

   - (0.5pt)

   $$\sum_{0\le k\le n}\binom{2k}{k}\binom{2n-2k}{n-k} = 4^n.$$

   - (0.5pt) What identity on binomial coefficient is implied by the convolution

   $$(1+z)^r(1-z)^s = (1-z^2)^s(1+z)^{r-s}$$

   for $r > s$?

   - (0.5pt)

   $$\sum_{0\le k\le t}\binom{t-k}{r}\binom{k}{s} = \binom{t+1}{r+s+1}.$$

10. - (0.5pt) Using EGF, find a solution to the recurrence

    $$\begin{cases} a_n = \sum_{0\le k\le n}\binom{n}{k}\frac{a_k}{2^k}, & n\ge 1 \\ a_0 = 1. \end{cases}$$

    - (0.5pt) Consider an EGF $e^{z+\frac{z^2}{2}}$ of a sequence $\{a_n\}_{n\ge 0}$. Show that $a_n$ satisfies the recurrence

    $$a_n = a_{n-1} + (n-1)a_{n-2}, \quad n\ge 2$$