Mathematical Analysis of Algorithms.

Exercises III. (18.06.2012)

1. The OGF that enumerates the binary strings with no two consecutive 0 bits is given by
$$B(z) = \frac{1+z}{1-z-z^2}.$$
   a) (0.5pt) Find a recurrence of $b_n = [z^n]B(z)$ that leads to $B(z)$.
   b) (0.5pt) Find a closed solution for b_n.

2. Let P_n be the class of plane trees on n vertices and B_n be the class of *binary* trees on n internal vertices. Notice that their OGFs satisfy
$$P(z) = z \cdot B(z).$$
This suggests that there is a combinatorial bijection between P_{n+1} and B_n.
   - (1pt) Find such a bijection.

3. The EGF of 2-regular labelled graphs is given by
$$R(z) = \frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}}.$$
   - (0.5pt) Estimate $[z^n]R(z)$ asymptotically.

4. Let t_n denote the number of r-nary trees on n vertices.
   - (0.5pt) Derive the asymptotic estimate of t_n.

5. Let X be a random variable with non-negative integer values. Show that
   a) (0.5pt)
$$\mathbb{E}(X) = \sum_{k\geq 0} \Pr(X \geq k),$$
   b) (0.5pt)
$$\mathbb{V}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

6. The BGF $Q(z,u)$ of QUICKSORT satisfies
$$\frac{\partial}{\partial z}Q(z,u) = u^2 \cdot (Q(uz,u))^2. \tag{1}$$
Let X_n denote the number of comparisons that QUICKSORT needs when the input is of size n.
   - (1pt) Using equation (1), derive $\mathbb{E}(X_n)$ and $\mathbb{V}(X_n)$.