Mathematical Analysis of Algorithms.

Exercises IV. (20.06.2012)

1. Let X be a random variable with non-negative real values. Show that for any $t > 0$,
   - (0.5pt)
   $$\Pr(X \geq t \cdot \mathbb{E}(X)) \leq \frac{1}{t}.$$
2. Let Y be a random variable with real values. Show that for any $t > 0$,
   - (0.5pt)
   $$\Pr\left(|Y - \mathbb{E}(Y)| \geq t \cdot \sigma(Y)\right) \leq \frac{1}{t^2}.$$
3. (1pt) Show that for all $k \geq 1$ we have
   $$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$
4. (1pt) Show that
   $$\frac{2^{2n}}{2 \cdot \sqrt{n}} \leq \binom{2n}{n} \leq \frac{2^{2n}}{\sqrt{2n}}.$$
5. Give estimates for the sums:
   a) (0.5pt)
   $$\sum_{1 \leq k \leq n} \frac{1}{k^2 \cdot H_k};$$
   b) (0.5pt)
   $$\sum_{0 \leq k \leq n} \frac{2^k}{2^k + 1};$$
   c) (0.5pt)
   $$\sum_{0 \leq k \leq n} 2^{k^2}.$$
6. The EGF of the Bernoulli numbers is given by
   $$\sum_{n \geq 0} B_n \cdot \frac{z^n}{n!} = \frac{z}{e^z - 1}.$$
   a) (0.5pt) Deduce from this, that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$ and $B_4 = -\frac{1}{30}$.

The Bernoulli polynomial is defined as

$$B_i(x) = \sum_{0 \le k \le i} \binom{i}{k} B_k \cdot x^{i-k}.$$

b) (0.5pt) Prove that for all $i \ge 2$,

$$B_i(0) = B_i(1) = B_i$$

and for all odd $i \ge 3$

$$B_i = 0.$$