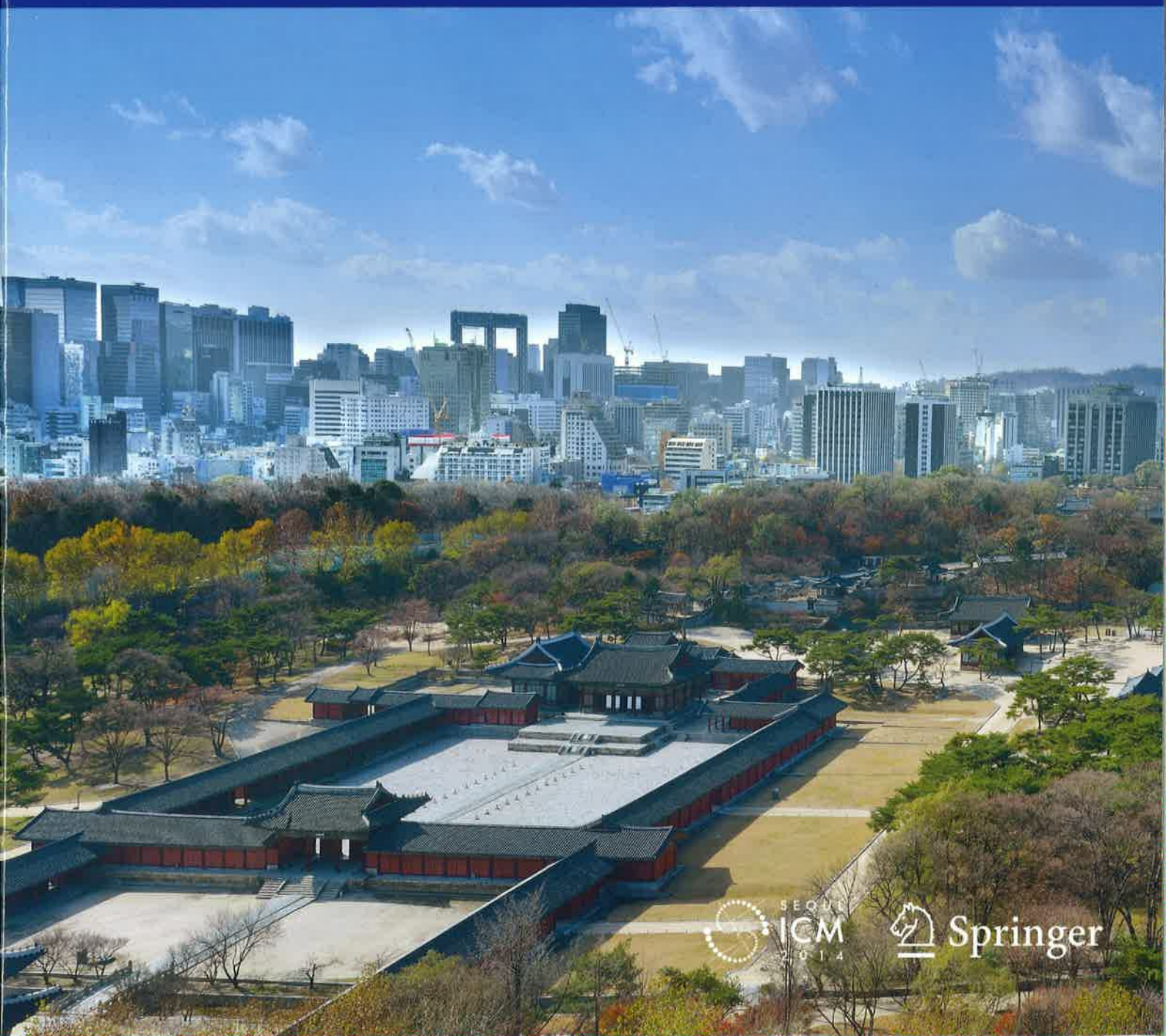


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Random Graphs: From Nature to Society to the Brain

The theory of random graphs deals with asymptotic properties of graphs equipped with a certain probability distribution, for example, how the component structure of a uniform random graph evolves as the number of edges increases. Since the foundation of the theory of random graphs by Erdős and Rényi five decades ago, various random graph models have been introduced and studied. Random graph theory has meanwhile found its way into other sciences as a rich source of models describing fundamental aspects of a broad range of complex phenomena.

1. Erdős–Rényi Random Graphs

The Beginning. Erdős and Rényi initiated the theory of random graphs in their paper entitled “On random graphs I” published in 1959, in which they addressed, among other things, the questions of the probability of a random graph being connected, and the probability that the largest component of a random graph covers almost all vertices. One year later, Erdős and Rényi discovered that a random graph undergoes a drastic change in the number of vertices in the largest component when the number of edges is around half the number of vertices, i.e., there is a phase transition in the evolution of a random graph.

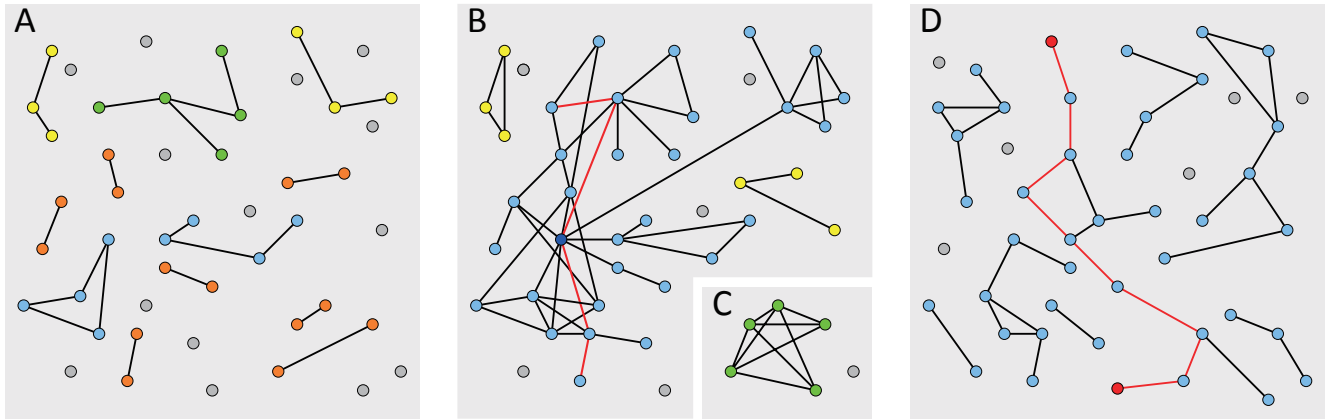
There are three random graph models hidden under the name of the Erdős–Rényi random graph. The uniform random graph $G(n, m)$ is a graph chosen uniformly at

random from the set of all graphs with vertex set $[n] = \{1, \dots, n\}$ and m edges, for an integer $0 \leq m \leq \binom{n}{2}$. The binomial random graph $G(n, p)$ is a graph with vertex set $[n]$ in which each pair of vertices is joined by an edge independently with probability p , for $p \in [0, 1]$. These two models are essentially equivalent when the parameters are appropriately selected, i.e., when $m = p \binom{n}{2}$. Finally, the Erdős–Rényi process $\{G_n(m) : m = 0, \dots, \binom{n}{2}\}$ describes creation of a random graph: it begins with a graph $G_n(0)$ with n isolated vertices and no edges, and in each step $1 \leq m \leq \binom{n}{2}$ a new random edge is added to an evolving graph $G_n(m-1)$ to obtain a new graph $G_n(m)$. After m edges are added, the resulting graph $G_n(m)$ of the Erdős–Rényi process is distributed like the uniform random graph $G(n, m)$.

Phase Transition. To describe the phase transition in the Erdős–Rényi random graph, we consider the uniform random graph $G(n, m)$ with positive constant average vertex degree $t = 2m/n$. If t is smaller than the critical value one, then *whp* (with high probability, meaning with probability tending to one as the number of vertices $n \rightarrow \infty$) the graph consists of isolated trees and unicyclic components and the largest component is a tree of order $O(\log n)$; such components are called ‘small’. On the other hand, if t is larger than one, then *whp* there exists a unique largest component of

linear order (called the ‘giant’ component), while all but the giant component are trees or unicyclic components of order $O(\log n)$. In other words, all but the giant component are ‘small’. The order of the giant component is *whp* equal to $\rho n + o(n)$, where $\rho = \rho(t) \in (0, 1)$ is uniquely defined by the equation $1 - \rho = e^{-t\rho}$, and is the same as the survival probability of the associated Galton–Watson branching process. If t is equal to one, then *whp* the order of the largest component is $\Theta(n^{2/3})$.

In short, the order of the largest component in the Erdős–Rényi random graph changes from logarithmic to sublinear and then to linear order in the number of vertices when the average vertex degree passes through one (for example from 0.99 to 1.01) as more edges are added. Erdős and Rényi described this phenomenon as a ‘double jump’ and considered it to be one of the most striking facts concerning random graphs. However, Bollobás showed that there is in fact no jump, but a smooth phase transition with three different phases: the subcritical phase in which *whp* there are many large components of almost equal order; the critical phase in which *whp* there are a few large components of the same order up to constant factor; and the supercritical phase which is characterised by the fact that *whp* there is a unique largest component that is much larger than the second largest component.



A When the number of edges is small, the random graph consists of many small components

B As more edges are added, a **giant component** emerges (blue).

D **Planar graph** can be drawn in plane so that its edges do not cross.

The **degree of a vertex** is the number of adjacent edges. The dark blue vertex in B has the largest degree 9.

The **order of a component** is the number of vertices it contains. The blue component in B has order 30, the coloured components in A have order 4 (blue), 3 (yellow), 2 (orange) and 1 (grey).

The graph **diameter** is the longest minimum distance between any two vertices. The largest component in A has a diameter three, in B four, in C two, and in D eight.

In a "**small-world**" network, the typical shortest distance between any two vertices in the large component is "short", i. e., the largest component has a small diameter (marked by a red line in B and D).

High **clustering** means that any neighbours of a vertex are also very likely to be connected, as in the green component in C.

Real-world networks often have a small **diameter** and exhibit high **clustering**.

Figure 1. Random graphs and their structure.

New Random Graph Models. Since the seminal work of Erdős–Rényi, many other random graph models have been proposed and studied. Examples include random hypergraphs, random directed graphs, inhomogeneous random graphs, random planar graphs, and random graph processes.

One of the most natural generalisations of the Erdős–Rényi random graph is a random graph with a given vertex degree sequence (an example of an *inhomogeneous* random graph), whose phase transition has been well studied. A very general model of an inhomogeneous random graph was introduced by Bollobás, Janson, and Riordan: the vertices are classified into different types, and the probability of forming an edge depends on the types of its vertices. In particular, the edges are added independently and the number of edges is linear in the number of vertices. By relating their model to multi-type branching processes, the authors determined the critical condition for the phase transition and the order of the largest component. Many random graphs suggested as models for real-world networks are examples of inhomogeneous random graphs.

A graph is called *planar* if it can be embedded in the plane without crossing edges. One of the most well-known results

about planar graphs is the Four Colour Theorem, which states roughly that given any separation of a plane into regions, say countries, the countries can be coloured using at most four colours so that no two countries sharing a common border have the same colour. Random planar graphs have received considerable attention since McDiarmid, Steger (Invited Speaker at Seoul ICM 2014), and Welsh derived their important asymptotic properties and Giménez and Noy (Invited Speaker at Seoul ICM 2014) determined the exact asymptotic number of planar graphs. Kang and Łuczak showed that there are two critical periods in the evolution of a random planar graph.

Important variants of the Erdős–Rényi process are random graph processes based on the ‘power’ of two choices, known as Achlioptas processes: in each step two potential edges are chosen randomly, and using a certain rule, only one of them is selected and added to the evolving graph. One natural question is whether there is a simple rule of edge choice that allows the control of the critical time for the emergence of the giant component. The product rule was suggested as the most promising to delay the critical time: given two potential edges, only the edge which minimises the product of the orders of the

components of its endvertices is added to the graph. Bohman and Frieze (Plenary Speaker at Seoul ICM 2014) showed shortly after the product rule was proposed that the critical time can be delayed by a much simpler rule, now known as the Bohman–Frieze process: given two potential edges, the first edge is added to the graph only if it would join two isolated vertices; otherwise the second edge is added. Krivelevich (Invited Speaker at Seoul ICM 2014), Loh, and Sudakov determined thresholds for the avoidance of small subgraphs in Achlioptas processes.

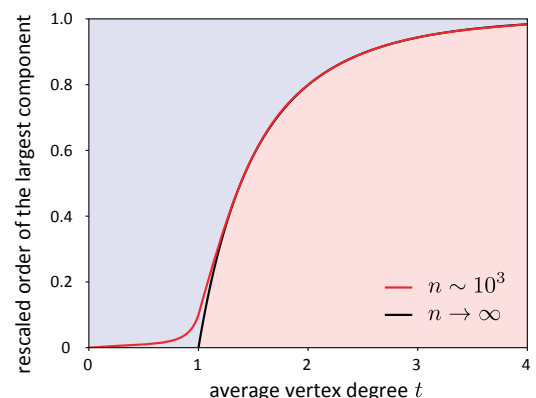


Figure 2. Phase transition in the Erdős–Rényi random graph.

2. Related Areas and Applications

Random graphs are closely related to other random discrete structures such as random

surfaces, random maps, random matrices, random satisfiability problems, Ising and Potts models, and percolation. They are useful for modelling, analysis, and solving of structural and algorithmic problems arising in mathematics, theoretical computer science, statistical mechanics, natural sciences, and even in social sciences.

Life Sciences. The recent boom in life sciences has generated huge amounts of data: genomes of whole organisms have been sequenced; proteins and patterns of their interactions have been identified; and metabolic networks relating the biochemical reactions have been mapped. Detailed analysis of genes, protein interaction, and metabolic networks using the methods of graph theory is expected to help us to understand the properties of the network that are determined by its large-scale structure rather than by the physical or chemical details of individual interactions. How does a combination of mutations of different proteins increase the risk of a disease, how does the topology of a protein network affect its robustness against random mutations, or how may the network have evolved over time?

Man-made Networks. Large complex structures found in nature are often formed by individual uncorrelated interactions between a large number of their constituents, therefore it is not surprising that random graphs provide good models for many of them. Although man-made structures are a result of a rational design (or we wish so), they often exhibit sufficient ‘randomness’, either because their structure developed over time (like the World-Wide-Web), or because they copy a pre-existing network (for example, airport network, electricity network) or are subject to random or other constraints (for example, geography). Random graphs with well-chosen characteristics can thus serve as useful models for the evolution and the function of many complex artificial structures.

Furthermore, new insights can be gained by studying the effects of interactions between two or more networks. If the interaction is such that the removal of a vertex in one network can eliminate vertices in the other network, the chain of subsequent vertex removals can lead to a large-scale fragmentation of both networks. A well-known example of such a collapse of coupled networks is an electrical blackout in Italy in 2003, when a shut-down of power stations caused failure of the internet

network controlling the power grid. Studies of interdependent networks can help to design networks resistant to this type of failure.

Social Sciences. Analysis of human communication by means of the theory of graphs is a useful tool in social sciences. One of the oldest and best-known examples is the ‘six degree of separation’ phenomenon described decades ago: any two people can be connected by a chain of acquaintances on average six-persons-long. This means that, similarly to the Erdős–Rényi random graph, the graph of human acquaintanceship has a small diameter; it is a ‘small-world network.’ However, unlike the Erdős–Rényi random graph, it has a high clustering coefficient. Many real-world networks have this structure, for example, the scientific co-authorship network, known to mathematicians by the concept of the Erdős number.

The structure of the network determines the efficiency of spreading of information or diseases over it. The spread of infectious diseases is closely related to the pattern of human mobility. Modelling human travel as diffusion on the Erdős–Rényi random graph does not take into account the geographical restrictions which, unlike the case of communication over the internet, certainly play a strong role here. On the other hand, a random walk on a regular lattice neglects the effect of long jumps enabled by air travel. The analysis of data on the circulation of bank notes collected by volunteers using tracking websites revealed a power-law decay of distribution of travelling distances, a consequence of the long jumps. The presence of long-distance jumps means that the disease spreading across this network is a fast, superdiffusive process — an important fact to consider when designing strategies to handle epidemics. Interestingly, the spread of plague in Europe in the 14th century did not follow this trend, suggesting the absence of the long-range links and therefore a different structure of human contacts at the time.

The Brain. The theory of graphs can help us to understand also perhaps the most complex structure found in nature: the brain. The network of interconnected neurons in the brain has been modelled as a random graph with different properties (for example, scale-free network). The way the neurons are connected and interact with each other largely determines the brain functionality. The experimental studies of spontaneous

brain activity — the analysis of electrical spikes produced by neurons, and of the size and duration of avalanches of this neuronal activity — often produce results that can be described by power-law distributions. This has led to a hypothesis that the brain, viewed as a network of neurons, may be operating at a critical state, a state close to a phase transition. This is very interesting, as various network properties related to the communication efficiency, dynamic range of response, etc., are optimised at the critical point.

This brief overview aims to give a flavour of how the field of random graphs has evolved over the last 50 years from the first definition of a random graph to a rich mathematical theory with applications across many scientific disciplines. On one hand, the theory of random graphs has proven to be appropriate for the description and analysis of complex structures arising everywhere from nature to society, even the brain. On the other hand, diverse applications continue to motivate and inform the study of random graphs. The expansion of random graph theory and its applications shows us again how much abstract mathematical ideas can teach us about the ‘real world’.

