The asymptotic number of connected $d$-uniform hypergraphs*

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Abstract. For $d \geq 2$, let $H_d(n, p)$ denote a random $d$-uniform hypergraph with $n$ vertices in which each of the $\binom{n}{d}$ possible edges is present with probability $p = p(n)$ independently, and let $H_d(n, m)$ denote a uniformly distributed $d$-uniform hypergraph with $n$ vertices and $m$ edges. Let either $H = H_d(n, m)$ or $H = H_d(n, p)$, where $m/n$ and $\binom{n}{d-1}p$ need to be bounded away from $(d-1)^{-1}$ and 0 respectively. We determine the asymptotic probability that $H$ is connected. This yields the asymptotic number of connected $d$-uniform hypergraphs with given numbers of vertices and edges. We also derive a local limit theorem for the number of edges in $H_d(n, p)$, conditioned on $H_d(n, p)$ being connected.

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1 Introduction and Main Results

1.1 Phase transition and connectivity

A $d$-uniform hypergraph $H = (V, E)$ is a pair of a set $V = V(H)$ of vertices and a set $E = E(H)$ of edges $e \subset V(H)$ with $|e| = d$. The order of $H$ is the number of vertices of $H$, and the size of $H$ is the number of edges. A 2-uniform hypergraph is just a graph. We say that a vertex $v \in V(H)$ is reachable from $w \in V(H)$ if there exist edges $e_1, \ldots, e_k \in E(H)$ such that $v \in e_1, w \in e_k$ and $e_i \cap e_{i+1} \neq \emptyset$ for all $1 \leq i < k$. Reachability is an equivalence relation, and the equivalence classes are called the components of $H$. If $H$ has only a single component, then $H$ is connected. We let $\mathcal{N}(H)$ signify the maximum order (i.e., number of vertices) of a component of $H$. For all hypergraphs $H$ that we deal with the vertex set $V(H)$ will consist of integers. Therefore the subsets of $V(H)$ can be ordered lexicographically, and we call the lexicographically first component of $H$ that has order $\mathcal{N}(H)$ the largest component of $H$. In addition, we denote by $\mathcal{M}(H)$ the size (i.e., number of edges) of the largest component.

In this paper we consider two models of random $d$-uniform hypergraphs for $d \geq 2$. The random hypergraph $H_d(n, p)$ has the vertex set $V = \{1, \ldots, n\}$, and each of the $\binom{n}{d}$ possible edges is present with probability $p$ independently. Moreover, $H_d(n, m)$ is a uniformly distributed $d$-uniform hypergraph with vertex set $V = \{1, \ldots, n\}$ and with exactly $m$ edges. Finally, we say that the random hypergraph $H_d(n, p)$ satisfies a certain property $P$ with high probability (“w.h.p.”) if the probability that $P$ holds in $H_d(n, p)$ tends to 1 as $n \to \infty$; a similar terminology is used for $H_d(n, m)$.

Since the pioneering work of Erdős and Rényi [9, 10] (see also [7, 12]), the component structure of random discrete objects (e.g., graphs, hypergraphs, digraphs, ...) has been among the main subjects of probabilistic combinatorics. Erdős and Rényi [10] studied (among other things) the component structure of sparse random graphs with $O(n)$ edges. The main result is that the order $\mathcal{N}(H_d(n, m))$ of the largest component undergoes a phase transition as $2m/n \sim 1$. Let us state a more general version from Schmidt-Pruzan and Shamir [17] for $d \geq 2$. Let either $H = H_d(n, m)$ and $c = dm/n$, or $H = H_d(n, p)$ and $c = \binom{n}{d-1}p$; we refer to $c$ as the average degree of $H$. Then the result is the following.

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(i) If \( c < (d-1)^{-1} - \epsilon \) for an arbitrarily small but fixed \( \epsilon > 0 \), then \( \mathcal{N}(H) = O(\ln n) \) w.h.p.

(ii) By contrast, if \( c > (d-1)^{-1} + \epsilon \), then \( H \) contains a unique component of order \( \Omega(n) \) w.h.p., which is called the giant component. More precisely, \( \mathcal{N}(H) = (1-\rho)n + o(n) \) w.h.p. where \( \rho \) is the unique solution to the transcendental equation

\[
\rho = \exp(c(\rho^{d-1} - 1))
\]

that lies strictly between 0 and 1. Furthermore, the second largest component has order \( O(\ln n) \) w.h.p.

Using probabilistic techniques, we derived in [3] a local limit theorem for \( \mathcal{N}(H_d(n, p)) \) and in [4] local limit theorems for the joint distribution of \( \mathcal{N}(H) \) and \( \mathcal{M}(H) \) for \( H = H_d(n, m) \), or \( H = H_d(n, p) \) in the regime \( (d-1)(n_d^{-1}) p > 1 + \epsilon \), resp. \( d(d-1)m/n > 1 + \epsilon \), where \( \epsilon > 0 \) is arbitrarily small but fixed as \( n \to \infty \). Using these results, we determine in this paper the asymptotic probability that \( H \) is connected and derive a local limit theorem for the number of edges in \( H_d(n, p) \), conditioned on \( H_d(n, p) \) being connected.

These problems have been studied by a few authors. For \( d = 2 \), the asymptotic probability that \( H_2(n, p) \) is connected was first computed by Stepanov [18], Bender, Canfield, and McKay [5] were the first to compute the asymptotic probability that a random graph \( H_2(n, m) \) is connected for any ratio \( m/n \). Additionally, using their formula for the probability of \( H_2(n, m) \) being connected, Bender, Canfield, and McKay [6] inferred the probability that \( H_2(n, p) \) is connected as well as a central limit theorem for the number of edges of \( H_2(n, p) \) given that \( H_2(n, p) \) is connected. Using enumerative arguments, Pittel and Wormald [16] derived an improved version of the main result of [5] and obtained a local limit theorem that in addition to \( \mathcal{N}(H) \) and \( \mathcal{M}(H) \) also includes the order and size of the 2-core. O’Connell [15] employed the theory of large deviations in order to estimate the probability that \( H_2(n, p) \) is connected up to a factor \( \exp(o(n)) \). While this result is significantly less precise than Stepanov’s, O’Connell’s proof is simpler. In addition, van der Hofstad and Spencer [11] used a novel perspective on the branching process argument to rederive the formula of Bender, Canfield, and McKay [5] for the number of connected graphs.

In contrast to the case of graphs \((d = 2)\), little is known about the connectivity probability of random \( d \)-uniform hypergraphs with \( d > 2 \). Karoński and Łuczak [13] derived an asymptotic formula for the number of connected \( d \)-uniform hypergraphs of order \( n \) and size \( m = \frac{d}{d-1} + o(\ln n/\ln \ln n) \) via combinatorial techniques. Since the minimum number of edges necessary for connectedness is \( \frac{d}{d-1}n \), this formula addresses sparsely connected hypergraphs. Furthermore, Andriamampianina and Ravelomanana [1] extended the result from [13] to the regime \( m = \frac{n}{d-1} + o(n^{1/3}) \) via enumerative techniques. By contrast, the results of this paper concern connected hypergraphs with \( m = \frac{n}{d-1} + \Omega(n) \) edges. Thus, our results and those from [1, 13] are complementary.

1.2 Main results

The probability of connectedness. The threshold for \( H_d(n, m) \) being connected is \( m \sim \frac{n}{d} \ln n \). Hence, for \( m = O(n) \) the probability that \( H_d(n, m) \) is connected is \( o(1) \). In fact, this probability is exponentially small in \( n \). The following theorem gives an asymptotic expression for this exponentially rare event.

**Theorem 1.** Let \( d \geq 2 \) be a fixed integer. For any compact set \( \mathcal{J} \subset (d(d-1)^{-1}, \infty) \) and for any \( \delta > 0 \) there exists \( n_0 > 0 \) such that the following holds. Let \( m = m(n) \) be a sequence of integers such that \( \zeta = \zeta(n) = dm/n \in \mathcal{J} \) for all \( n \). There exists a unique number \( 0 < r = r(n) < 1 \) such that

\[
r = \exp \left( -\zeta \cdot \frac{(1-r)(1-r^{d-1})}{1-r^d} \right).
\]

Let \( \Phi_d(r, \zeta) = r^{-\zeta} (1-r^{d-1})(1-r^d)^{\frac{\zeta}{2}} \). For \( d \geq 2 \), 

\[
R_d(n, m) = \frac{1-r^d-(1-r)(d-1)\zeta r^{d-1}}{\sqrt{(1-r^d+\zeta(d-1)(r-r^{d-1}))(1-r^d)-d\zeta r(1-r^{d-1})^2}} \cdot \exp \left( \frac{(d-1)\zeta(r^2+r^{d-1}-2r^d+r^{d+2})}{2(1-r^d)} \right) \cdot \Phi_d(r, \zeta)^n,
\]
and for \( d = 2 \),
\[
R_2(n, m) = \frac{1 + r - \zeta r}{\sqrt{(1 + r)^2 - 2 \zeta r}} \cdot \exp \left( \frac{\zeta r(2 - r - r^2 + \zeta)}{2(1 + r)} \right) \cdot \Phi_2(r, \zeta)^n.
\]

Finally, let \( c_d(n, m) \) denote the probability that \( H_d(n, m) \) is connected. Then for all \( n > n_0 \) we have
\[
(1 - \delta)R_d(n, m) < c_d(n, m) < (1 + \delta)R_d(n, m).
\]

Observe that Theorem 1 yields an asymptotic formula for the number \( C_d(n, m) \) of connected \( d \)-uniform hypergraphs of given order \( n \) and size \( m \), because
\[
C_d(n, m) = \binom{n}{m} c_d(n, m).
\]

To prove Theorem 1 we shall consider a “larger” hypergraph \( H \) and for \( d \) and for \( H \)

Theorem 2. Let \( d \geq 2 \) be a fixed integer. For any compact set \( J \subset (0, \infty) \), and for any \( \delta > 0 \) there exists \( n_0 > 0 \) such that the following holds. Let \( p = p(n) \) be a sequence such that \( \zeta = \zeta(n) = \binom{n-1}{d-1}p \) \( \in J \) for all \( n \). There exists a unique \( 0 < \varrho = \varrho(n) < 1 \) such that
\[
\varrho = \exp \left( \zeta \cdot \frac{\varrho^{d-1} - 1}{(1 - \varrho)^{d-1}} \right).
\]

Let \( \Psi_d(\varrho, \zeta) = (1 - \varrho)^{\varrho^{d-1}} \cdot \varrho^{\frac{1 - \varrho^{d-1} - (1 - \varrho)^d}{2(1 - \varrho)^d}} \) for \( d \geq 2 \). Define, for \( d > 2 \),
\[
S_d(n, p) = \frac{1 - \zeta(d - 1) \left( \varrho^{\frac{1}{1 - \varrho}} \right)^{d-1}}{\varrho^{\zeta(d - 1)} \left( 1 - \varrho^{d-1} \right)^{\frac{1}{1 - \varrho}}} \cdot \exp \left( \frac{\zeta(d - 1) \varrho^{\frac{1}{1 - \varrho}} - (1 - \varrho)^d}{2(1 - \varrho)^d} \right)
\]
\[
\cdot \exp \left( \frac{\zeta(d - 1) \varrho^{\frac{1}{1 - \varrho}}}{\varrho^{\zeta(d - 1)} \left( 1 - \varrho^{d-1} \right)^{\frac{1}{1 - \varrho}}} \right) \cdot \Psi_d(\varrho, \zeta)^n,
\]

and for \( d = 2 \),
\[
S_2(n, p) = \left( 1 - \frac{\zeta}{e^{\zeta - 1}} \right) \cdot \exp \left( \frac{\zeta(2 + \zeta)}{2(e^{\zeta - 1})} \right) \cdot (1 - e^{-\zeta})^n.
\]

Finally, let \( c_d(n, p) \) denote the probability that \( H_d(n, p) \) is connected. Then for all \( n > n_0 \) we have
\[
(1 - \delta)S_d(n, p) < c_d(n, p) < (1 + \delta)S_d(n, p).
\]

Remark 3. The formulas for \( R_d(n, m) \) and \( S_d(n, p) \) for \( d \geq 2 \) given in an extended abstract version [2] of this work were incorrect.

The distribution of the number of edges in \( H_d(n, p) \) given connectedness. Interestingly, if we choose \( p = p(n) \) and \( m = m(n) \) in such a way that \( \binom{n}{d} p = m \) for each \( n \) and set \( \xi = \binom{n-1}{d-1} p = dm/n \), then the function \( \Psi_d(\varrho, \zeta) \) from Theorem 2 is strictly bigger than \( \Phi_d(r, \zeta) \) from Theorem 1. Consequently, the probability that \( H_d(n, p) \) is connected exceeds the probability that \( H_d(n, m) \) is connected by an exponential factor.
The reason for this is as follows. We can think of generating $H_d(n, p)$ as first choosing a random number $m_0$ of edges from the binomial distribution $\text{Bin}(\binom{n}{d}, p)$, and then generating a random hypergraph $H_d(n, m_0)$. The probability that $H_d(n, m_0)$ is connected increases rapidly as a function of $m_0$. Hence, $H_d(n, p)$ could “boost” its probability of being connected by including a number of edges that exceeds the expectation $\binom{n}{d}p$ of the binomial distribution considerably. Hence, once we condition on $H_d(n, p)$ being connected, the total number of edges in $H_d(n, p)$ will be significantly larger than $\binom{n}{d}p$. The following local limit theorem quantifies this phenomenon.

**Theorem 4.** Let $d \geq 2$ be a fixed integer. For any two compact sets $I \subset \mathbb{R}$, $J \subset (0, \infty)$, and for any $\delta > 0$ there exists $n_0 > 0$ such that the following holds. Suppose that $0 < p = p(n) < 1$ is a sequence such that $\zeta = \zeta(n) = \binom{n-1}{d-1}p \in J$ for all $n$. Let $0 < \rho = \rho(n) < 1$ be the unique solution to (3), and set

$$\hat{\mu} = \left[ \frac{\zeta(1 - \rho^d)}{d(1 - \rho^d)} \cdot n \right], \quad \sigma^2 = \frac{\zeta}{d(1 - \rho^d)} \left( 1 - \rho^d - \frac{\zeta d \rho(1 - \rho^{d-1})^2}{(1 - \rho)^d + \zeta(1)(\rho - \rho^{d-1})} \right) \cdot n.$$

Finally, let $|E(H_d(n, p))|$ denote the number of edges in $H_d(n, p)$. Then for all $n \geq n_0$ and all integers $y$ such that $n^{-\frac{1}{2}}y \in I$ we have

$$\frac{1 - \delta}{\sqrt{2\pi\sigma}} \exp \left( - \frac{y^2}{2\sigma^2} \right) \leq \mathbb{P} \left[ |E(H_d(n, p))| = \hat{\mu} + y \mid H_d(n, p) \text{ is connected} \right] \leq \frac{1 + \delta}{\sqrt{2\pi\sigma}} \exp \left( - \frac{y^2}{2\sigma^2} \right).$$

In the case $d = 2$ the solution to (3) is $\rho = \exp(-\zeta)$, whence the formulas from Theorem 4 simplify to

$$\hat{\mu} = \left[ \frac{\zeta}{2} \coth(\zeta/2) \cdot n \right] \quad \text{and} \quad \sigma^2 = \frac{\zeta}{2} \cdot \frac{1 - 2\zeta \exp(-\zeta) - \exp(-2\zeta)}{(1 - \exp(-\zeta))^2} \cdot n.$$

### 1.3 Techniques and Outline

In Section 2 we derive Theorem 1 from Lemma 6. The basic reason why this is possible is that given that the largest component of $H_d(n, p)$ has order $n$ and size $m$ (for suitably chosen $n > 0$), the largest component is a uniformly distributed connected hypergraph with these parameters. This observation was also exploited by Łuczak [14] to estimate the number of connected graphs up to a polynomial factor, and in [8], where an explicit relation between the numbers $c_d(n, m)$ and $\mathbb{P} \{ |N(H_d(n, m))| = n, M(H_d(n, m)) = m \}$ was derived (see Lemma 5 below). Combining this relation with Lemma 6, we obtain Theorem 1. Finally, in Sections 3 and 4 we use similar arguments to establish Theorems 2 and 4.

### 1.4 Notation

We use the “$O$-notation” to express asymptotic estimates as $n \to \infty$. Occasionally we will apply this notation to expressions that do not only depend on $n$, but also on further parameters. Suppose that $f(x_1, \ldots, x_k, n)$, $g(x_1, \ldots, x_k, n)$ are functions of $n$ and further parameters $x_i$ are from domains $D_i \subset \mathbb{R}$ ($1 \leq i \leq k$), and that $g \geq 0$. Then we say that the estimate $f(x_1, \ldots, x_k, n) = O(g(x_1, \ldots, x_k, n))$ holds uniformly in $x_1, \ldots, x_k$ if the following is true: there exist numbers $C$ and $n_0$ such that

$$|f(x_1, \ldots, x_k, n)| \leq Cg(x_1, \ldots, x_k, n) \text{ for all } n \geq n_0 \text{ and } (x_1, \ldots, x_k) \in \prod_{j=1}^k D_j.$$

Similarly, we say that $f(x_1, \ldots, x_k, n) \sim g(x_1, \ldots, x_k, n)$ holds uniformly in $x_1, \ldots, x_k$ if for any $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $n > n_0$

$$\sup_{(x_1, \ldots, x_k) \in D_1 \times \cdots \times D_k} \left| \frac{f(x_1, \ldots, x_k, n)}{g(x_1, \ldots, x_k, n)} - 1 \right| < \varepsilon.$$

We define uniformity analogously for the other Landau symbols $\Omega$, $\Theta$, etc.
2 The Probability that $H_d(n, m)$ is Connected: Proof of Theorem 1

We will derive the probability that $H_d(n, m)$ is connected (Theorem 1) from the local limit theorem for the joint distribution of the order and size of the largest component in $H_d(\nu, p)$, for suitable choice of $\nu > n$. The latter was proved by us in [3] and restated below in Lemma 6.

Let $J \subset (d(d-1)^{-1}, \infty)$ be a compact interval, and let $m(n)$ be a sequence of integers such that $\zeta = \zeta(n) = dm/n \in J$ for all $n$. The basic idea is to choose $\nu$ and $p$ in such a way that $|n - \mathbb{E}(N(H_d(\nu, p)))|$ and $|m - \mathbb{E}(M(H_d(\nu, p)))|$ are “small”, i.e., $n$ and $m$ will be “probable” outcomes of $N(H_d(\nu, p))$ and $M(H_d(\nu, p))$. Since given that $N(H_d(\nu, p)) = n$ and $M(H_d(\nu, p)) = m$, the largest component of $H_d(\nu, p)$ is a uniformly distributed connected graph of order $n$ and size $m$, we can then express the probability that $H_d(n, m)$ is connected in terms of the probability

$$\chi = \mathbb{P} [N(H_d(\nu, p)) = n, \ M(H_d(\nu, p)) = m].$$

The (somewhat technical) details of this approach were carried out in [8], where the following lemma was established.

**Lemma 5.** Suppose that $n > n_0$ for some large enough number $n_0 = n_0(J)$. Then there exist an integer $\nu = \nu(n) = \Theta(n)$ and a number $0 < p = p(n) < 1$ such that the following is true.

(i) Let $c = (\nu^{-1})^p$. Then $(d - 1)^{-1} < c = O(1)$, and letting $0 < \rho = \rho(c) < 1$ signify the solution to (1), we have

$$n = (1 - \rho)\nu, \quad m - (1 - \rho^d)\binom{\nu}{d}p = O(1).$$

(ii) The solution $r$ to (2) satisfies $|r - \rho| = o(1)$ and $|c - \frac{1 - r}{1 - \rho} \zeta| = o(1)$.

(iii) Furthermore,

$$c_d(n, m) \sim \nu \cdot 3 \cdot w \cdot \Phi_d(r, \zeta)^n$$

uniformly for $\zeta \in J$, where

$$\Phi_d(r, \zeta) = (1 - r)^{1 - \zeta} r^{d/(1 - r)} (1 - r^d)^{\zeta/d},$$

$$u = 2\pi \sqrt{r(1 - r)(1 - r^d)c/d},$$

$$v = \exp \left( \frac{d - 1)r(1 - r^d)c}{2} \right),$$

$$w = \begin{cases} 
\exp \left( \frac{c^2 r(1 + r)}{2} \right) & \text{if } d > 2, \\
\exp \left( \frac{c^2 r(1 + r)}{2} \right) & \text{if } d = 2.
\end{cases}$$

The formulas (4)–(8) are reformulated from the corresponding ones in [8] by translating the notations as follows. We exchange the roles of $\nu$ and $n$ and those of $\mu$ and $m$ respectively; $r$ and $p$ play the same role as $1 - a_1$ and $1 - \alpha_5$ respectively. The formula (5) follows from the term $(a_5(1 - a_5)(1 - a_5)/a_5)(a_5 d b_5)^{\mu} = (a_5^{1 - \zeta}(1 - a_5)(1 - a_5)/a_5(1 - a_5 d)^{\zeta/d})^{\nu}$ in (15) of [8]. Letting $\Phi_d(x, \zeta) := (1 - x)^{1 - \zeta} x^{1 + \zeta} (1 - x^d)^{-\zeta}$, we have from Lemma 12 of [8] that $\Phi_d(1 - a_5, \zeta)^\nu \sim \Phi_d(1 - a_5, \zeta)^\nu$, so we have in the current setting that $\Phi_d(\rho, \zeta)^n \sim \Phi_d(\tau, \zeta)^n$. Furthermore, (6) follows from the term $\frac{2\pi}{\sqrt{a_5(1 - a_5)b_5}} \sim u$ in (15) of [8]; (7) from the term $\exp \left[ \frac{d - 1)(1 - a_5)c(b_5 + a_5(1 - a_5)^d - 2)}{2a_5} \right] \sim v$; and (8) from the term $\exp \left[ \frac{b_5 p(1 - a_5)^d - (1 - a_5)^d)}{2a_5} \right] \sim w$.

Thus, once we know the explicit expression for

$$\chi = \mathbb{P} [N(H_d(\nu, p)) = n, \ M(H_d(\nu, p)) = m],$$

we can derive the exact asymptotic expression for $c_d(n, m)$ from (4). We can in fact compute $\chi$ explicitly using the following local limit theorem for the joint distribution of $N(H_d(\nu, p))$ and $M(H_d(\nu, p))$ from [4].
Lemma 6. Let $d \geq 2$ be a fixed integer. For any two compact sets $I \subset \mathbb{R}^2$, $J \subset ((d - 1)^{-1}, \infty)$, and for any $\delta > 0$ there exists $\nu_0 > 0$ such that the following holds. Let $p = p(\nu)$ be a sequence such that $c = c(\nu) = (\nu - 1)^p \in J$ for all $\nu$ and let $0 < \rho = \rho(\nu) < 1$ be the unique solution to (1). Further, let

$$
\sigma_N^2 = \rho \frac{(1 - \rho + c(d - 1)(\rho - \rho^{d-1}))}{(1 - c(d - 1)\rho^{d-1})^2} \cdot \nu,
$$

$$
\sigma_M^2 = c^2 \rho^d \cdot \frac{2 + c(d - 1)(\rho^{2d-2} - 2\rho^{d-1} + \rho^d) - \rho^{d-1} - \rho^d}{(1 - c(d - 1)\rho^{d-1})^2} \cdot \nu
+ (1 - \rho^d) \frac{c}{d} \cdot \nu,
$$

$$
\sigma_{NM}^2 = \frac{c}{d} \rho - c(d - 1)\rho^{d-1}(1 - \rho) \frac{1}{(1 - c(d - 1)\rho^{d-1})^2} \cdot \nu.
$$

Suppose that $\nu \geq \nu_0$ and that $n$, $m$ are integers such that

$$
x = n - (1 - \rho)\nu \quad \text{and} \quad y = m - (1 - \rho^d)\left(\frac{\nu}{d}\right)p
$$

satisfy $\nu \frac{1}{d}(x, y) \in I$. Define

$$
P(x, y) = \frac{1}{2\pi \sqrt{\sigma_N^2 \sigma_M^2 - \sigma_{NM}^2}} \cdot \exp \left( -\frac{\sigma_N^2 \sigma_M^2}{2(\sigma_N^2 \sigma_M^2 - \sigma_{NM}^2)} \left( \frac{x^2}{\sigma_N^2} - \frac{2\sigma_{NM} x y}{\sigma_N^2 \sigma_M^2} + \frac{y^2}{\sigma_M^2} \right) \right).
$$

Then we have

$$(1 - \delta) P(x, y) \leq \mathbb{P} [N(H_d(\nu, p)) = n, M(H_d(\nu, p)) = m] \leq (1 + \delta) P(x, y).
$$

Note that from (9)--(11) we have

$$
\sigma_N^2 \sigma_M^2 - \sigma_{NM}^2 = \frac{c}{d} \frac{1}{d} \frac{(1 - \rho + c(d - 1)(\rho - \rho^{d-1}))}{(1 - c(d - 1)\rho^{d-1})^2} \left( 1 - \rho^d \right) - c^2 \rho^2 (1 - \rho^{d-1})^2 \cdot \nu^2.
$$

From Lemma 5 (i) and (12), $x = 0, y = O(1)$, and from (10) $\sigma_M = \Theta(\nu)$. Thus (13)--(15) yield

$$
\chi = \mathbb{P} [N(H_d(\nu, p)) = n, M(H_d(\nu, p)) = m]
\sim \frac{1}{2\pi \sqrt{\sigma_N^2 \sigma_M^2 - \sigma_{NM}^2}}
= \frac{1 - c(d - 1)\rho^{d-1}}{2\pi \sqrt{\frac{d}{r} (1 - \rho + c(d - 1)(\rho - \rho^{d-1})) (1 - \rho^d) - c^2 \rho^2 (1 - \rho^{d-1})^2}}.
$$

(16)

Since $r \sim \rho$ and $c \sim \frac{1 - r}{1 - r^d} \zeta$ by Lemma 5 (ii), we can express $\nu \cdot \chi, u, v, w$ in (16) and (6)--(8) solely in terms of $r$ and $\zeta$:
Remark 7. While Lemma 5 was established in Coja-Oghlan, Moore, and Sanwalani [8], the exact joint limiting distribution of $\mathcal{N}(H_d(\nu, p))$ and $\mathcal{M}(H_d(\nu, p))$ (i.e. Lemma 6) was not known at that point. Therefore, Coja-Oghlan, Moore, and Sanwalani could only compute the $c_d(n, m)$ up to a constant factor. By contrast, combining Lemma 6 with Lemma 5, here we have obtained tight asymptotics for $c_d(n, m)$.

3 The Probability that $H_d(\nu, p)$ is Connected: Proof of Theorem 2

Let $\mathcal{F} \subset (0, \infty)$ be a compact set, and let $0 < p = p(n) < 1$ be a sequence such that $\zeta = \zeta(n) = \binom{n-1}{d-1}p \in \mathcal{F}$ for all $n$. All asymptotics in this section are uniform in $\zeta$.

To compute the probability $c_d(n, p)$ that a random hypergraph $H_d(n, p)$ is connected, we will establish that

$$
\mathbb{P} [\mathcal{N}(H_d(\nu, p)) = n] \sim \binom{\nu}{n} c_d(n, p) (1 - p)^{\binom{n}{d}-\binom{\nu-d}{d}} \cdot \exp \left( \frac{\zeta r (2 - r^2 + \zeta)}{2(1 + r)} \right).
$$

for a suitably chosen integer $\nu > n$. Then, we will employ the local limit theorem for $\mathcal{N}(H_d(\nu, p))$, which is implied by Lemma 6 and as well as our previous result [3] on the local limit theorem for $\mathcal{N}(H_d(n, p))$, to compute the l.h.s. of (19), so that we can just solve (19) for $c_d(n, p)$.

In order to carry this out, we use the following lemma on the component structure of $H_d(n, p)$, which is a slight variant of Theorem 5 of [8]. To obtain it, we can easily adapt the arguments of the proof of Theorem 5 of [8]. We may skip here the details, as the computations become quite technical and tedious without providing useful new insights.
Lemma 8. Let \( c = c(\nu) \) be a sequence of non-negative reals and let \( p = c(\nu)^{-1} \) and \( m = (\nu)^{-1} \) be connected hypergraphs. Then for both \( H = H_d(\nu, p) \) and \( H = H_d(\nu, \mu) \) the following holds.

(i) For any \( c_0 < (d - 1)^{-1} \) there is a number \( \nu_0 \) such that for all \( \nu > \nu_0 \) for which \( c = c(\nu) \leq c_0 \) we have

\[
\mathbb{P} [\mathcal{N}(H) \leq 300(d - 1)^2(1 - (d - 1)c_0)^{-2} \ln \nu] \geq 1 - \nu^{-100}.
\]

(ii) For any \( c_0 > (d - 1)^{-1} \) there are numbers \( \nu_0 > 0 \), \( 0 < c_0' < (d - 1)^{-1} \) such that for all \( \nu > \nu_0 \) for which \( c_0 \leq c = c(\nu) < \ln \nu / \ln \nu \) the following holds. The transcendental equation \( (1) \) has a unique solution \( 0 < \rho = \rho(\nu) < 1 \), which satisfies

\[
\rho^{d-1} c < c_0.
\]

Furthermore, with probability \( \geq 1 - \nu^{-100} \) there exists precisely one component of order \( (1-\rho)\nu + o(\nu) \) in \( H \), while all other components have order \( \leq \nu^2 \). In addition,

\[
\mathbb{E} [\mathcal{N}(H)] = (1 - \rho)\nu + o(\sqrt{\nu}).
\]

We pick \( \nu \) as follows. By Lemma 8 for each integer \( k \) such that \( c(k) = \binom{k-1}{d-1} p > (d - 1)^{-1} \) the transcendental equation \( \rho(k) = \exp(c(k)(\rho(k)^{d-1} - 1)) \) has a unique solution \( \rho(k) \) that lies strictly between 0 and 1. We let \( \nu = \max\{ k \in \mathbb{N} : (1 - \rho(k))k < n \} \). Moreover, set \( \rho = \rho(\nu) \) and \( c = c(\nu) = (\nu^{-1}) \), and let \( 0 < s < 1 \) be such that \( (1 - s)\nu = n \). We claim

\[
|n - (1 - \rho)\nu| < O(1).
\]

To see this, observe that \( (1 - \rho(\nu))\nu < n = (1 - s)\nu \leq (1 - \rho(\nu + 1))(\nu + 1) \). In order to establish (20), it suffices to show that \( |\rho(\nu + 1) - \rho(\nu)| = O(1/\nu) \), because \( n - (1 - \rho(\nu))\nu < (1 - \rho(\nu + 1))(\nu + 1) - (1 - \rho(\nu))\nu < 1 + \nu(\rho(\nu) - \rho(\nu + 1)) \). To prove this, we note that since \( \zeta = \binom{\nu}{d-1} p = \binom{(1 - s)\nu - 1}{d-1} p \),

\[
c(\nu + 1) - c(\nu) = \binom{\nu}{d-1} p - \binom{\nu - 1}{d-1} p = p \frac{(\nu - 1)}{(d - 1)} \frac{d - 1}{\nu - d + 1} = \zeta \frac{d}{(d - 1)} = O(1/\nu).
\]

This, together with Taylor series expansion, implies that \( |\rho(\nu + 1) - \rho(\nu)| = O(1/\nu) \), because \( \rho(k) = \exp(c(k)(\rho(k)^{d-1} - 1)) \) and \( \rho(k) \) is differentiable due to the implicit function theorem.

To establish (19), note that the r.h.s. is just the expected number of components of order \( n \) in \( H_d(\nu, p) \). For there are \( \binom{n}{\nu} \) ways to choose the vertex set \( C \) of such a component, and the probability that \( C \) spans a connected hypergraph is \( c_d(n, p) \). Moreover, if \( C \) is a component, then \( H_d(\nu, p) \) features no edge that connects \( C \) with \( V \setminus C \), and there are \( \binom{\nu}{d-1} - \binom{\nu-d}{d-1} \) possible edges of this type, each being present with probability \( p \) independently. Hence, we conclude that

\[
\mathbb{P} [\mathcal{N}(H_d(\nu, p)) = n] \leq \binom{\nu}{n} c_d(n, p)(1 - p)^{\binom{\nu}{d-1}} - \binom{\nu-d}{d-1} - \binom{\nu}{d-1}.
\]

(21)

On the other hand,

\[
\mathbb{P} [\mathcal{N}(H_d(\nu, p)) = n] \geq \binom{\nu}{n} c_d(n, p)(1 - p)^{\binom{\nu}{d-1}} - \binom{\nu-d}{d-1} - \binom{\nu}{d-1} \mathbb{P} [\mathcal{N}(H_d(\nu - n, p) < n)],
\]

(22)

because the r.h.s. equals the probability that \( H_d(\nu, p) \) has exactly one component of order \( n \). Furthermore, as \( |n - (1 - \rho)\nu| < O(1) \) by (20), Lemma 8 entails that

\[
\mathbb{P} [\mathcal{N}(H_d(\nu - n, p)) < n] \sim 1.
\]

Hence, combining (21) and (22), we obtain (19).

To derive an explicit formula for \( c_d(n, p) \) from (19), we need the following lemma.
Lemma 9. (i) We have \( c = \zeta (1 - s)^{1-d} \left( 1 + \frac{(d)}{2} \frac{s}{(1-s)^{\nu}} + O(\nu^{-2}) \right) \).

(ii) The transcendental equation (3) has a unique solution \( 0 < \rho < 1 \), which satisfies \( |s - \rho| = O(\nu^{-1}) \).

(iii) Letting

\[ \Psi(x) = \Psi_d(x, \zeta) := (1-x)x^{\frac{\nu}{\nu-1}} \exp \left( \frac{\zeta}{d} \cdot \frac{1-x^d - (1-x)^d}{(1-x)^d} \right), \]

we have \( \Psi(\rho)^n \sim \Psi(s)^n \).

Proof of Lemma 9. Regarding the first assertion, we note that

\[ \frac{(1-s)^{d-1}(\nu-1)}{(1-s)^{\nu-1}} = \prod_{j=1}^{d-1} \left( 1 + \frac{s}{(1-s)^{\nu-j}} \right) = 1 + \left( \frac{d}{2} \right) \frac{s}{(1-s)^{\nu}} + O(\nu^{-2}). \]  

(23)

Since \( c = \left( \frac{\nu-1}{d-1} \right)^{1/d} \) and \( n = (1-s)\nu \), (23) implies the first assertion.

In order to show the second assertion, we set

\[ \varphi_z : (0, 1) \rightarrow \mathbb{R}, \ t \mapsto \exp \left( z \frac{\rho^{d-1} - 1}{(1-t)^{d-1}} \right) \]  

for \( z > 0 \). Then \( \lim_{t \searrow 0} \varphi_z(t) = \exp(-z) > 0 \), while \( \lim_{t \nearrow 1} \varphi_z(t) = 0 \). In addition, \( \varphi_z \) is convex for any \( z > 0 \). Therefore, for each \( z > 0 \) there is a unique \( 0 < t_z < 1 \) such that \( t_z = \varphi_z(t_z) \), whence (3) in Theorem 2 has the unique solution \( 0 < \rho = t_z < 1 \). Moreover, letting \( \zeta' = (1-\rho)^{d-1}c \), we have \( \rho = t_{\zeta'} \). Thus, since \( t \mapsto t_z \) is differentiable by the implicit function theorem and \( |\zeta - \zeta'| = O(\nu^{-1}) \) by the first assertion, we conclude that \( |s - \rho| = O(\nu^{-1}) \). In addition, \( |s - \rho| = O(\nu^{-1}) \) by (20). Hence, \( |s - \rho| = O(\nu^{-1}) \), as desired.

To establish the third assertion, we compute

\[ \frac{\partial}{\partial x} \Psi(x) = (1-x)^{-d-1}x^{\frac{\nu}{\nu-1}} \exp \left( \frac{\zeta}{d} \cdot \frac{(1-x^d - (1-x)^d)}{(1-x)^d} \right) \times (1-x^d) \]  

\[ \prod_{j=1}^{d-1} \left( 1 + \frac{s}{(1-s)^{\nu-j}} \right) \]  

\[ \times \left( 1 - \frac{s}{(1-s)^{\nu-j}} \right)^{d-1} \]  

\[ \times \left( 1 - \frac{s}{(1-s)^{\nu-j}} \right)^{d-1} \]  

\[ \exp \left( \frac{\zeta}{d} \cdot \frac{(1-x^d - (1-x)^d)}{(1-x)^d} \right) \]  

\[ \times \left( 1 - \frac{s}{(1-s)^{\nu-j}} \right) \]  

(24)

As \( \varrho = \exp \left( \frac{\zeta}{d-1} \right) \), (24) entails that \( \frac{\partial}{\partial x} \Psi(\rho) = 0 \). Therefore, Taylor’s formula yields that \( \Psi(s) - \Psi(\rho) = O(s - \rho) = O(\nu^{-2}) \), because \( s - \rho = O(\nu^{-1}) \) by the second assertion. Consequently, we obtain

\[ \left( \frac{\Psi(s)}{\Psi(\rho)} \right)^{\nu} = \left( 1 + \frac{\Psi(s) - \Psi(\rho)}{\Psi(\rho)} \right)^{\nu} \exp \left( \nu \cdot \frac{\Psi(s) - \Psi(\rho)}{\Psi(\rho)} \right) = \exp(O(\nu^{-1})) \sim 1, \]

thereby completing the proof of the third assertion.

Let us continue with the proof of Theorem 2. Note that Lemma 6 implies

\[ \mathbb{P}[N(H_d(n, p)) = n] \sim \frac{1}{\sqrt{2\pi \sigma_N}} \exp \left( -\frac{(n - (1-\rho)\nu)^2}{2\sigma_N^2} \right). \]  

(25)

It follows also from our previous result [3] on the local limit theorem for \( N(H_d(n, p)) \). Since \( |s - \rho| = O(\nu^{-1}) \) by (20), we can express \( \sigma_N^2 \) (in (9)) in terms of \( s \):

\[ \sigma_N^2 = \frac{\rho (1-\rho+c(d-1)(\rho-\rho^{d-1}))}{(1-c(d-1)\rho^{d-1})^2} \cdot \nu. \]

(26)

Further, since \( n - (1-\rho)\nu < O(1) \) by (20), we have from (25) and (26)

\[ \mathbb{P}[N(H_d(n, p)) = n] \sim (2\pi)^{-\frac{1}{2}} \left( \frac{s (1-s+c(d-1)(s-s^{d-1}))}{(1-c(d-1)s^{d-1})^2} \cdot \nu \right)^{\frac{1}{2}}. \]  

(27)
Via Stirling’s formula and  \( n = (1 - s)\nu \) we can estimate the binomial coefficient

\[
{\binom{\nu}{n}} \sim \left( s^{\nu}(1-s)^{(1-s)\nu} \sqrt{2\pi s(1-s)\nu} \right)^{-1}.
\] (28)

Plugging (27) and (28) into (19), we obtain

\[
c_d(n, p) \sim \left( \frac{\nu}{n} \right)^{-1} \cdot \mathbb{P} [\mathcal{N}(H_d(\nu, p)) = n] \cdot (1-p)^{\binom{\nu}{d}} \binom{n}{\nu - d}.
\]

\[
\sim s^{\nu}(1-s)^{(1-s)\nu} \cdot \eta \cdot (1-p)^{\binom{\nu}{d}} \binom{n}{\nu - d},
\] (29)

where

\[
\eta = \frac{(1 - s)(1 - c(d - 1)s^{d-1})^2}{1 - s + c(d - 1)(s - s^{d-1})}^{1/2}.
\] (30)

Let us consider the cases \( d = 2 \) and \( d > 2 \) separately, because \( \binom{\nu}{2}p^2 = o(1) \) for \( d > 2 \), while \( \binom{\nu}{2}p^2 = \Theta(1) \) and therefore the asymptotics for \( (1-p)^{\binom{\nu}{d}} \binom{n}{\nu - d} \) behave quite differently.

**1st case: \( d = 2 \).** Note first that \( \binom{\nu-1}{2} + \binom{\nu}{2} = s(s-1)\nu^2 \), because \( n = (1-s)\nu \). Using \( p = \frac{c}{\nu - 1} \), we get

\[
(1-p)^{\binom{\nu-1}{2} + \binom{\nu}{2}} = (1-p)^{s(s-1)\nu^2}
\]

\[
\sim e^{\nu \cdot \eta \cdot (1-p)^{s(s-1)\nu^2}}
\]

\[
\sim e^{\nu \cdot \frac{c}{\nu - 1} \cdot s(s-1)((\nu - 1)(\nu + 1)} \right) + \frac{1}{2} \left( \frac{c}{\nu - 1} \right)^{2} s(s-1)\nu^2
\]

\[
\sim e^{\nu c s(1-s)(\nu + 1) + \frac{c^2}{2} s(1-s)}.
\] (31)

Moreover, (30) simplifies to \( \eta = 1 - cs \). Hence, recalling that \( \nu = (1-s)^{-1}n \) and using Lemma 9 (i)- (iii), i.e. \( c = \frac{\zeta}{1-s} \left( 1 + \frac{1}{(1-s)^{\nu}} + O(\nu^{-2}) \right) \), \( |s - \rho| = O(\nu^{-1}) \) and \( \left( (1-s)\nu^{\frac{1}{1-s}} \exp \left( \frac{s}{1-s} \right) \right)^{n} \sim \left( 1 - \rho \right)^{\frac{n}{\theta + 1}} \exp \left( \frac{\theta}{1-\theta} \right) \) \( \), we can estimate (29) as

\[
c_d(n, p) \sim s^{\nu}(1-s)^{(1-s)\nu} \cdot (1 - cs) \cdot \exp \left( cs(1-s)\nu + cs(1-s) + \frac{c^2}{2} s(1-s) \right)
\]

\[
\sim s^{\frac{n}{\theta + 1}}(1-s)^n \left( 1 - \frac{c}{1-s} \right) \exp \left( \frac{cs}{1-s} + \frac{c^2}{1-s} \right)
\]

\[
\sim s^{\frac{n}{\theta + 1}}(1-s)^n \left( 1 - \frac{cs}{1-s} \right) \exp \left( \frac{c^2}{1-s} \right)
\]

\[
\sim (\rho^{\frac{n}{\theta + 1}}(1-\rho)^n \left( 1 - \frac{c}{1-\rho} \right) \exp \left( \frac{c^2}{2(1-\rho)} \right)
\] (32)

Finally, for \( d = 2 \) the unique solution to (3) is just \( \rho = \exp(-\zeta) \), so we have \( \frac{c}{\nu - 1} = \frac{1}{e\xi - 1} \). Plugging these into (32), we obtain

\[
c_d(n, p) \sim (1 - e^{-\zeta})^n \left( 1 - \frac{\zeta}{e\xi - 1} \right) \exp \left( \frac{\zeta^2}{2(e\xi - 1)} \right) \]

(33)

as desired.
2nd case: $d > 2$. For $0 < \alpha < 1$, using

$$
\alpha^d \left( \frac{\alpha \nu}{d} \right)^{-1} \left( \frac{\nu}{d} \right) = \prod_{i=0}^{d-1} \frac{\alpha (\nu - i)}{\alpha \nu - i} = \prod_{i=0}^{d-1} \left( 1 + \frac{(1 - \alpha) i}{\alpha \nu - i} \right) = 1 + \frac{1 - \alpha}{\alpha \nu} \left( \frac{d}{2} \right) + O(\nu^{-2}),
$$

and $n = (1 - s) \nu$, we estimate

$$
\begin{aligned}
\binom{n}{d}^{-1} \left( \frac{\nu}{d} \right) + \binom{\nu - n}{d}^{-1} \left( \frac{\nu}{d} \right) &= \binom{(1 - s) \nu}{d}^{-1} \left( \frac{\nu}{d} \right) + \binom{s \nu}{d}^{-1} \left( \frac{\nu}{d} \right) \\
&= (1 - s)^d \left( 1 - \frac{s}{(1 - s) \nu} \left( \frac{d}{2} \right) + O(\nu^{-2}) \right) + s^d \left( 1 - \frac{1 - s}{s \nu} \left( \frac{d}{2} \right) + O(\nu^{-2}) \right) \\
&= (1 - s)^d + s^d - \frac{1}{\nu} \left( \frac{d}{2} \right) (s(1 - s)^{d-1} + (1 - s)s^{d-1}) + O(\nu^{-2})
\end{aligned}
$$

and thus we have

$$
\begin{aligned}
\binom{n}{d} + \binom{\nu - n}{d} - \binom{\nu}{d} &= \binom{(1 - s) \nu}{d} \left( (1 - s)^d + s^d - 1 \right) \\
&= \binom{(1 - s) \nu}{d} \left( (1 - s)^d + s^d - 1 \right) + O(\nu^{-d-2}).
\end{aligned}
$$

Because $\binom{\nu - 1}{d-1} p = c = \Theta(1)$, we have $\binom{\nu}{d} p^2 = o(1)$ for $d > 2$, and hence

$$
(1 - p) \binom{\nu}{d} ((1 - s)^d + s^{d-1}) \sim \exp \left( -p \binom{\nu}{d} \left( (1 - s)^d + s^d - 1 \right) \right) = \exp \left( \frac{c \nu}{d} \left( 1 - s^d - (1 - s)^d \right) \right)
$$

and

$$
(1 - p) \binom{\nu - 1}{d-1} \binom{s(1 - s)^{d-1} + (1 - s)s^{d-1}}{d-1} \sim \exp \left( p \binom{\nu}{d} \frac{1}{\nu} \left( \frac{d}{2} \right) (s(1 - s)^{d-1} + (1 - s)s^{d-1}) \right) \\
= \exp \left( p \binom{\nu - 1}{d-1} \frac{d}{2} \left( (1 - s)^{d-1} + (1 - s)s^{d-1} \right) \right) \\
= \exp \left( \frac{c(d - 1)}{2} \left( (1 - s)^{d-1} + (1 - s)s^{d-1} \right) \right).
$$

Putting (34)–(36) together, we get

$$
(1 - p) \binom{\nu}{d} + \binom{\nu - 1}{d-1} \binom{\nu}{d} \binom{s(1 - s)^{d-1} + (1 - s)s^{d-1}}{d-1} \sim \exp \left( \frac{c \nu}{d} (1 - s^d - (1 - s)^d) + \frac{c(d - 1)}{2} \left( (1 - s)^{d-1} + (1 - s)s^{d-1} \right) \right).
$$

Before proceeding further computations toward the asymptotic estimation of $c_d(n, p)$, we note that taking $d = 2$ in the estimate (37) yields $(1 - p) \binom{\nu}{2} + \binom{\nu - 1}{1} \binom{\nu}{2} \sim \exp (c s(1 - s)(\nu + 1))$, which differs by a factor $\exp(\frac{c}{2} s(1 - s))$ from the estimate (31), the reason being that $\binom{\nu}{2} p^2 = o(1)$ for $d > 2$, while $\binom{\nu}{2} p^2 = \Theta(1)$. This in turn results in an extra factor $\exp(\frac{c}{2} g(1 - g))$ in the estimate (32) of $c_2(n, p)$, in comparison to the estimate of $c_d(n, p)$ when taking $d = 2$ in (41).
We now return to the computation of (37). Using
\[ c = \zeta(1 - s)^{1-d} \left( 1 + \frac{d}{2} \frac{s}{(1-s)^\nu} + O(\nu^{-2}) \right) \]
by Lemma 9 (i) and recalling that \( \nu = (1 - s)^{-1} n \),
\[ \frac{cv}{d} = \frac{\zeta n}{d(1-s)^d} + \frac{\zeta(d-1)s}{2(1-s)^d} + O(n^{-1}), \]
and thus
\[
\begin{align*}
\frac{cv}{d} (1-s^d - (1-s)^d) &+ \frac{c(d-1)}{2} ((1-s)s^{d-1} + s(1-s)^{d-1}) \\
= \frac{\zeta n}{d(1-s)^d} (1-s^d - (1-s)^d) &+ \frac{\zeta(d-1)s}{2(1-s)^d} (1-s^d - (1-s)^d) \\
&+ \frac{\zeta n}{d(1-s)^d} (1-s^d - (1-s)^d) &+ \frac{\zeta(d-1)s}{2(1-s)^d} (1-s^d - (1-s)^d) \\
&+ \frac{\zeta(d-1)s}{2} \left( \left( \frac{s}{1-s} \right)^{d-2} + 1 \right) + O(n^{-1}). \tag{38}
\end{align*}
\]
Using this, we can restate (37) as
\[
(1-p)^{\binom{1}{2}} \cdot \binom{\binom{1}{1} - \binom{1}{2}}{1} \\
\sim \exp \left( \frac{\zeta(1-s^d - (1-s)^d) n}{d(1-s)^d} + \frac{\zeta(d-1)s(1-s^d - (1-s)^d)}{2(1-s)^d} \right) \\
\cdot \exp \left( \frac{\zeta(d-1)s}{2} \left( \left( \frac{s}{1-s} \right)^{d-2} + 1 \right) \right). \tag{39}
\]
Due to the same reasons, we estimate (30) as
\[
\begin{align*}
\eta &= \left( \frac{(1-s)(1-c(d-1)s^{d-1})^2}{1-s+c(d-1)(s-s^{d-1})} \right)^{1/2} \\
&= (1-c(d-1)s^{d-1}) (1+ c(d-1)(1-s)^{-1}(s-s^{d-1}))^{-1/2} \\
&= \left( 1 - \zeta(d-1) \left( \frac{s}{1-s} \right)^{d-1} + O(n^{-1}) \right) \\
&\quad \cdot \left( 1 + \frac{\zeta(d-1)(s-s^{d-1})}{(1-s)^d} + O(n^{-1}) \right)^{-1/2} \\
&= \left( 1 - \zeta(d-1) \left( \frac{s}{1-s} \right)^{d-1} \right) \\
&\quad \cdot \left( 1 + \frac{\zeta(d-1)(s-s^{d-1})}{(1-s)^d} \right)^{-1/2} + O(n^{-1}). \tag{40}
\end{align*}
\]
Plugging (39) and (40) into (29) and recalling \( \nu = (1 - s)^{-1} n \), we obtain
\[
\begin{align*}
c_d(n, p) &\sim s^{\nu}\left(1-s\right)^{(1-s)^\nu}(1-p)^{\binom{\nu}{d} + \binom{\nu}{d-1}} \cdot \eta \\
&\sim s^{\nu} \left(1-s\right)^n \exp \left( \frac{\zeta(1-s^d - (1-s)^d) n}{d(1-s)^d} \right) \\
&\quad \cdot \exp \left( \frac{\zeta(d-1)s}{2} \left( \left( \frac{s}{1-s} \right)^{d-2} + 1 \right) \right) \cdot \left( 1 + \frac{\zeta(d-1)(s-s^{d-1})}{(1-s)^d} \right)^{-1/2} + O(n^{-1}).
\end{align*}
\]
From (9) and (15) we have
\[ y = \exp \left[ \frac{\zeta(d-1)s(1-s^d-1-s)^d}{2(1-s)^d} + \frac{\zeta(d-1)s}{2} \left( \frac{s}{1-s} \right)^{d-2} + 1 \right] \]
\[ \cdot \left( 1 - \zeta(d-1) \left( \frac{s}{1-s} \right)^{d-1} \right) \left( 1 + \frac{\zeta(d-1)(s-s^{d-1})^{-1}}{(1-s)^d} \right). \]

Finally, using Lemma 9 (ii)–(iii), i.e. \(|s - \theta| = O(\nu^{-1})\) and
\[ \left( s^{\nu^{-1}}(1-s) \exp \left( \frac{\zeta(1-s^d-1-s^d)}{d(1-s)^d} \right) \right)^n \sim \left( \theta^{\nu^{-1}}(1-\theta) \exp \left( \frac{\zeta(1-\theta^d-1-\theta^d)}{d(1-\theta)^d} \right) \right)^n, \]
we estimate (41) as
\[ c_d(n, p) \sim \left( (1-\theta) \theta^{\nu^{-1}} \exp \left( \frac{\zeta(1-\theta^d-1-\theta^d)}{d(1-\theta)^d} \right) \right)^n \]
\[ \cdot \exp \left( \frac{\zeta(d-1)(1-\theta^d-1-\theta^d)}{2(1-\theta)^d} + \frac{\zeta(d-1)\theta}{2} \left( \frac{\theta}{1-\theta} \right)^{d-2} + 1 \right) \]
\[ \times \left( 1 - \zeta(d-1) \left( \frac{\theta}{1-\theta} \right)^{d-1} \right) \left( 1 + \frac{\zeta(d-1)(\theta-\theta^{d-1})^{-1}}{(1-\theta)^d} \right)^{-1/2}, \tag{41} \]
which is exactly the formula stated in Theorem 2.

\[ \square \]

4 The Conditional Edge Distribution: Proof of Theorem 4

Let \( J \subset (0, \infty) \) and \( \mathcal{I} \subset \mathbb{R} \) be compact sets, and let \( 0 < p = p(n) < 1 \) be a sequence such that \( \zeta = \zeta(n) = (n^{-1})p \in J \) for all \( n \). All asymptotics in this section are uniform in \( \zeta \).

To compute the limiting distribution of the number of edges of \( H_d(n, p) \) given that this random hypergraph is connected, we choose \( \nu > n \) as in Section 3. Thus, letting \( c = (n^{-1})p \), we know from Section 3 that \( c > (d-1)^{-1} \), and that the solution \( 0 < \nu < 1 \) to (1) satisfies \((1-\rho) \nu \leq n < (1-\rho) \nu + O(1)\). Now, we investigate the random hypergraph \( H_d(n, p) \) given that \( \mathcal{N}(H_d(n, p)) = n \). Then the largest component of \( H_d(n, p) \) is a random hypergraph \( H_d(n, p) \) given that \( H_d(n, p) \) is connected. Therefore,
\[ \mathbb{P} [ \|E(H_d(n, p))\| = m \mid H_d(n, p) \text{ is connected}] = \mathbb{P} [ \mathcal{M}(H_d(n, p)) = m \mid \mathcal{N}(H_d(n, p)) = n] \]
\[ = \frac{\mathbb{P} [ \mathcal{M}(H_d(n, p)) = m, \mathcal{N}(H_d(n, p)) = n]}{\mathbb{P} [ \mathcal{N}(H_d(n, p)) = n]} \tag{42} \]

Furthermore, as \(|n - (1-\rho) \nu| < O(1)\) by (20), we can apply Lemma 6 to get an explicit expression for the r.h.s. of (42). Namely, using (13) with \( x = O(1) \), for any integer \( m \) such that \( \nu^{-1} y \in \mathcal{I} \) and \( y = m - (1-\rho^d)(n^{-1})p \) satisfying \( \nu^{-1} y \in \mathcal{I} \) we obtain
\[ \mathbb{P} [ \|E(H_d(n, p))\| = m \mid H_d(n, p) \text{ is connected}] \approx \frac{1}{\sqrt{2\pi}} \cdot \left( \frac{\sigma_N^2}{\sigma_N^2 - \sigma_{N,M}^2} \right)^{1/2} \exp \left( \frac{-\sigma_N^2}{2(\sigma_N^2 - \sigma_{N,M}^2)} \cdot y^2 \right). \tag{43} \]

From (9) and (15) we have
\[ \sigma_N^2 = \rho \left( 1 - \rho + c(d-1)(\rho - \rho^{d-1}) \right) \cdot \nu, \]
\[ \sigma_N^2 - \sigma_{N,M}^2 = \frac{\rho c (1 - \rho + c(d-1)(\rho - \rho^{d-1})) (1-\rho^d) - d c^2 \rho (1-\rho^{d-1})^2}{d (1-c(d-1)\rho^{d-1})^2} \cdot \nu^2. \]
Thus we have
\[
\frac{\sigma_N^2}{\sigma^2_N - \sigma^2_{N,M}} = \frac{d(1 - \rho + c(d-1)(\rho - \rho^{d-1}))}{c((1 - \rho + c(d-1)(\rho - \rho^{d-1}))(1 - \rho^d) - dcp(1 - \rho^{d-1})^2)} \cdot \frac{1}{\nu} 
\]
\[
= \frac{d}{c\nu} \left( 1 - \rho^d - \frac{dcp(1 - \rho^{d-1})^2}{1 + c(d-1)(\rho - \rho^{d-1})} \right)^{-1}. 
\]
(44)

In order to reformulate (44) in terms of \( n, \zeta, \) and the solution \( \varrho \) to (3), we just observe that \(|c - \zeta(1 - \rho)^{1-d}| = O(\nu^{-1})\) and \(|\rho - \varrho| = O(\nu^{-1})\) by Lemma 9, and that \(|\nu - (1 - \rho)^{-1}n| = O(\nu^{-1})\). Using these we obtain
\[
\left( \frac{\sigma_N^2}{\sigma^2_N - \sigma^2_{N,M}} \right)^{-1} = \frac{c\nu}{d} \left( 1 - \rho^d - \frac{dcp(1 - \rho^{d-1})^2}{1 + c(d-1)(\rho - \rho^{d-1})} \right) 
\]
\[
\sim \frac{\zeta\nu}{(1 - \rho)^d} \left( 1 - \rho^d - \frac{d\zeta(1 - \rho)^{1-d}c(1 - \rho^{d-1})^2}{1 + c(d-1)(\rho - \rho^{d-1})} \right)^{-1} 
\]
\[
= \frac{\zeta}{(1 - \rho)^d} \left( 1 - \rho^d - \frac{d\zeta c(1 - \rho^{d-1})^2}{(1 - \rho)^d + \zeta(d-1)(\rho - \rho^{d-1})} \right) \cdot n 
\]
\[
\sim \frac{\zeta}{(1 - \rho)^d} \left( 1 - \rho^d - \frac{d\zeta c(1 - \rho^{d-1})^2}{(1 - \rho)^d + (d-1)\zeta(\rho - \rho^{d-1})} \right) \cdot n 
\]
(45)

and
\[
(1 - \rho^d)\left( \frac{\nu}{d} \right)^p = (1 - \rho^d)\frac{\nu^c}{d(1 - \rho)} \sim (1 - \rho^d)\frac{\nu}{d(1 - \rho)}^c \sim (1 - \rho^d)\frac{\nu}{d(1 - \rho)}^c \frac{n}{d(1 - \rho)}^c(1 - \rho)^{1-d} = \frac{\zeta(1 - \rho^d)}{d(1 - \rho)^d} \cdot n. 
\]

Plugging (45) into (43) we have
\[
\mathbb{P} \left[ |E(H_d(n, p))| = m \mid H_d(n, p) \text{ is connected} \right] \sim \frac{1}{\sqrt{2\pi\hat{\sigma}}} \exp \left( - \frac{y^2}{2\hat{\sigma}^2} \right), 
\]
as desired.

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**References**