Pursuing the Giant in Random Graph Processes

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Abstract

We study the evolution of random graph processes that are based on the paradigm of the power of multiple choices. The processes we consider begin with an empty graph on n vertices. In each subsequent step a set with a specific number $\ell \geq 2$ of random vertices is presented, and we may select any edge among them to be included in the graph. For example, if $\ell = 2$ this corresponds to the classical Erdős-Rényi (ER) process. A striking characteristic of the ER process is the *phase transition* with respect to the distribution of its component sizes. This distribution undergoes a drastic change when the number of edges is around $n/2$; at this point the so-called *giant component* emerges, which contains a linear fraction of the vertices.

In this paper we address the component-size distribution of a general family rules. We determine the typical size of the giant component shortly after the phase transition in all these processes and provide bounds for the size distribution of small components. In particular, it has been conjectured by various authors that these processes have many similarities with the ER process, for example that the giant component grows with a constant "rate". Our results confirm this conjecture.

On the technical side, we develop a novel method for the analysis of the component size distribution based on *partial differential equations* (PDEs). We develop a novel analytic framework that allows us to study the solutions of a fairly general class of quasi-linear PDEs, the so-called family of *Smoluchowski's coagulation equations*, that have several far-reaching applications in the study of large systems consisting of interacting particles. Finally, our family of rules allows us to "approximate" formally any general size rule by a sequence of appropriately defined bounded size rules, where the given size-bound increases gradually. Thus, our results open an avenue for the future research on general random graph processes.

1 Introduction

An important topic in the analysis of algorithms is the investigation of the so-called *paradigm of the power of multiple choices*. In this paper we study processes of graph evolution that involve this paradigm. The general setting that we consider is as follows. We begin with a graph G_0 with n vertices and no edges. In order to obtain G_m, $m \geq 1$, we are presented a set S_m that contains a specific number $\ell \geq 2$ of randomly selected vertices, and we may choose any edge with both endpoints in S_m, which is then added to G_{m-1}. If $\ell = 2$ (or if we choose randomly an edge from $\binom{S_m}{2}$) then we of course obtain the well-known Erdős-Rényi process [16]. The mathematical properties of this process are very well-understood, and today we have a very precise picture of the underlying dynamics and critical phenomena, see e.g. [9, 22, 2]. On the other hand, for general "rules" that dictate how to select one of the presented edges, our current knowledge is comparatively rather limited.

A particular striking feature in the evolution of the Erdős-Rényi process is the *phase transition* that occurs around $m = n/2$. More precisely, if we write $m = tn$, then for $t < 1/2$, the number L_1 of vertices in the largest connected component is whp[1] of order $\log n$, whereas for $t > 1/2$, L_1 is linear in n. Thus, there is a dramatic change in the connectivity structure of the resulting graph, and the so-called *giant* component emerges after the critical time $t_c = 1/2$. The study of more general rules, as described in the previous paragraph, is motivated by the following question posed by Dimitris Achlioptas: suppose that in each round the edges come in pairs and we are allowed to choose one of them. Is there an (online) algorithm that *changes* the main characteristics of the Erdős-Rényi process, and in particular the point in time of the phase transition?

Among all possible rules that can be defined in our framework the most natural ones are the so-called *size rules*, whose choices depend only on the sizes of the components containing the vertices in S_m. During the last 15 years the evolution of such processes has received considerable attention [7, 8, 40, 38, 24, 6, 39], and in particular *bounded*-size rules were studied intensely. These rules have the additional property that they treat all sizes larger than some absolute bound K in the same way. A particular example is the *Bohman-Frieze rule* [7, 40] that is presented in each round four vertices v_1, v_2, v_3, v_4 and chooses the edge v_1v_2 only if both v_1 and v_2 are isolated and v_3v_4 otherwise. An example of a non-bounded rule is the so-called *product rule* given by Achlioptas: v_1v_2 is added to the graph if the product of the corresponding component sizes is smaller than the product of the component sizes that contain v_3 and v_4.

The bottom line of the current research is that all "reasonable" size rules seem to have many qualitative similarities with the Erdős-Rényi process; in physics jargon, this means that they belong to the same universality class. For example, if we consider only bounded-size rules, then it is known [40, 39] that the random graph undergoes a phase transition at a rule-dependent time t_c: if $m = tn$ and $t < t_c$, then whp the largest component contains $O(\log n)$ vertices, and if $t > t_c$, then again L_1 is linear in n. However, prior to our work, no detailed information about the distribution of components, for example the size of the giant for $t > t_c$, was available. More crucially, for general size rules, with only a few exceptions [39], we cannot even tell with certainty where and whether a phase transition occurs. Understanding such processes, in which there are significant and long-term dependencies between the appearance of specific edges, requires the development of new tools and methods.

[1] *with high probability*, i.e., with probability tending to 1 as $n \to \infty$

In this paper we address the component-size distribution of a general family of bounded size rules. Our contribution is three-fold: first, we are able to pin down the typical size of the giant component shortly after the phase transition in *all* these processes, and in addition we can give bounds for the size distribution of "small" components. Our results further confirm that these processes have many similarities with the Erdős-Renyi process, in the sense that the giant component grows with a comparable "rate". Secondly, in order to obtain such a precise description we develop a novel method based on the analysis of random processes via *partial differential equations*. While it is not very difficult to derive the equation in the specific setting, finding a solution and describing its properties is a fundamental and hard problem. Here we develop an appropriate analytic framework that allows us to extract the relevant properties, and we believe that this methodology will have applications in many other problems, also outside of computer science and mathematics, as well. Finally, our family of rules allows us to "approximate" formally any general size rule by a sequence of appropriately defined bounded size rules, where the given bound increases gradually. Thus, our results open an avenue for the future research on general random graph processes.

1.1 Main Results

A particular property of the phase transition in the Erdős-Renyi process is that the giant component grows "smoothly". More precisely, if $t = 1/2 + \varepsilon$ for $\varepsilon > 0$ and we write $L_1(t)$ for the number of vertices in the giant component after tn random edges were added, then whp $L_1(t)/n = 2\varepsilon + O(\varepsilon^2)$. Such a transition is called *continuous*. In addition to this result, the continuity of similar transitions in a much more general class of models, where not all edges are included with the same probability, is also known [10].

The interest in graph processes that exploit the power of choice was triggered immensely in 2009, when Achlioptas, D'Souza and Spencer [1] announced that already a simple rule – namely the product rule mentioned previously – seemed to behave vastly different from the Erdős-Rényi process, i.e. to cause a discontinuous transition. The claims in [1] were supported by convincing experimental evaluation, and in a very short period of time dozens of papers were published in which similar phenomena for related rules could be observed. However, Riordan and Warnke [38] proved that this is not the case for *any* rule that can choose among a bounded number of edges; their result shows that the continuity of the phase transition is such a fundamental property of the Erdős-Rényi model that is not distorted by (essentially) any rule that offers the possibility to choose among several edges in each step.

Although in [38] the continuity of the phase transition is established, a particular intriguing question remains open. What can we say about the number of vertices in the giant component? Apart from the Erdős-Rényi case, the answer is also known for the Bohman-Frieze rule. In [24] it was shown that if we denote by $t_c \approx 0.588$ the critical time of that process, then whp the number $L_1(t)$ of vertices in the giant component after tn edges were added satisfies for $\varepsilon > 0$

$$\lim_{n\to\infty} n^{-1} L_1(t_c + \varepsilon) = \gamma\varepsilon + O(\varepsilon^{4/3}), \qquad \text{where} \qquad \gamma \approx 2.463.$$

This bears a striking qualitative similarity to the estimate $n^{-1}L_1(1/2+\varepsilon) = 2\varepsilon+O(\varepsilon^2)$ for the Erdős-Rényi process, where the "rate" with which the giant component grows is constant. The obvious question is now whether this is just a coincidence, or if it is, like the continuity of the phase transition, a fundamental property of the Erdős-Rényi model that is stable enough to persist even if we consider several different rules. Our first main result addresses this question and shows that a large class of bounded-size rules exhibits exactly the same behavior.

Before we present our findings let us introduce some notation and the family of rules that we will study. Suppose that G is a graph with vertex set $\{1,\dots,n\} =: [n]$. The rules that we consider consist of a rule-specific function

$$R : G \times [n]^\ell \to \big\{\{i,j\} : 1 \le i < j \le \ell\big\}$$

that has the following interpretation. Suppose that we are given the graph G and the vertices $v_1,\dots,v_\ell \in [n]$. Then if we write $R(G, v_1,\dots,v_\ell) = \{i,j\}$, the edge that is selected by R to be added to G is $\{v_i, v_j\}$. Any such rule (i.e. mapping R) defines a random graph process in the natural way, where we start with an empty graph and in every succeeding round we present the rule ℓ randomly and independently selected vertices from the vertex set of G.

Of course, this definition captures the notion of a general rule that may base its decision on any property of G and of the presented vertices. Here we restrict our focus on specific R that have the following properties. For a vertex $v \in [n]$ we denote by $c_G(v)$ the size of the component in which v is contained. Similarly, for a sequence of vertices $\overline{v} = (v_1,\dots,v_\ell)$ we write $c_G(\overline{v}) = (c_G(v_1),\dots,c_G(v_\ell))$. Moreover, let $\ell \ge 2, K \ge 0$ be fixed integers. An (ℓ,K)-*bounded* rule R that is presented G and $v_1,\dots,v_\ell$ has the following characteristics:

- it bases its decision *only* on $c_G(v_1),\dots,c_G(v_{\ell-2})$.
- if *any* of $c_G(v_1),\dots,c_G(v_{\ell-2})$ exceeds K it selects $\{v_{\ell-1}, v_\ell\}$.

In words, R implements an arbitrary choice mechanism among the vertices $v_1,\dots,v_{\ell-2}$ provided that all these vertices are contained in components of size $\le K$. Otherwise, it picks a random edge.

Note that the Erdős-Rényi process and the Bohman-Frieze process belong to the class of (ℓ,K)-bounded rules. Indeed, Erdős-Rényi is the only $(2,0)$ process, and Bohman-Frieze is a $(4,1)$ process that selects $\{v_1,v_2\}$ if and only if $c_G(\overline{v}) = (1,1,1,1)$. Two remarks are in place here. First of all, our main motivation for studying such rules is that they provide a concise way of defining a notion of "approximation" for size rules. Indeed, consider for example the product rule. Then we can define a sequence of $(4+2,k)$-bounded rules, where $k \ge 0$, and the rule applies the product criterion to $v_1,\dots,v_4$ unless one of the corresponding components has size $\ge k$. In that case, it selects $\{v_5,v_6\}$, i.e., a random edge. In other words, the behavior of the bounded rule differs only if large components are selected, and we would expect that for large k the behavior resembles qualitatively very much the behavior of the size rule. We comment more on this issue in Section 5, where we also discuss possible implications for our results. Second, the family of (ℓ,K)-bounded rules is rather broad, and it contains also rules that are not bounded size in the sense of Spencer and Wormald [40]; moreover, our rules are more general that the so-called 'first-edge algorithms' defined in [8]. Finally, we believe that our methods can be extended to cover also more general cases, but we leave this for the moment as an open problem.

Let $L_1(R; m, n)$ denote the size of the largest component in the graph if a rule R is applied m times (and the initial graph contains n vertices and no edge). For any (ℓ,K)-bounded rule it follows from previous results [39] that there is a phase transition at some critical (rule-dependent) time that we denote by t_R: if $m = tn$ and $t < t_R$, then whp $L_1(R; m, n) = O(\log n)$. On the other hand, if $t > t_R$, then whp $L_1(R; m, n) = \Omega(n)$. Our results allow us to reprove this fact with a completely different method. In addition to that, we show an universality phenomenon regarding the rate at which the giant component grows for $t > t_R$.

Theorem 1.1. *Let $\ell \geq 2, K \geq 0$, and let R be an (ℓ, K)-bounded rule. Then there is a constant $c_R > 0$ such that for any $\varepsilon > 0$ whp*

$$\lim_{n\to\infty} n^{-1} L_1(R;\, (t_R + \varepsilon)n, n) = c_R\varepsilon + O(\varepsilon^2).$$

In words, for any (ℓ, K)-bounded rule the giant component grows at a linear rate, establishing that with respect to this parameter all these rules belong to the same universality class. Theorem 1.1 thus describes the evolution of the size of the giant component with high precision for a whole *class* of rules. Moreover, our proof allows us to compute the value of c_R for any such rule. Indeed, c_R is the value of some function that is specified by a small system of differential equations at a particular point, see Sections 3 and 4. This is of course not as explicit as we would like; however, even the critical point t_R is only given implicitly, so asking for an explicit expression would be too much to hope for.

Apart from the size of the giant component, we also obtain further information about the distribution of the component sizes. For a graph G with vertex set $[n]$ and an integer $k \geq 1$ let us write

$$X_k(G) = \big|\{v \in [n] : c_G(v) = k\}\big| \tag{1.1}$$

for the number of vertices in components of size k. Equivalently, $n^{-1}X_k$ is the probability that a random vertex is contained in a component of size k. Let us write $X_k(\mathsf{ER};\, t)$ if G is the Erdős-Renyi graph with $m = tn$ edges. It is well-known [23, 26] (see also Section 2) that whp

$$n^{-1}X_k(\mathsf{ER};\, t) = (1 + o(1))\frac{k^{i-1}}{k!}(2te^{-2t})^k.$$

From this, a simple and slightly tedious calculation shows that for $t = 1/2 + \varepsilon$ (where ε may now be also negative) we obtain whp

$$\lim_{k\to\infty} \frac{\log n^{-1}X_k(\mathsf{ER};\, t)}{k} = -\varepsilon^2/2 + O(\varepsilon^3).$$

In [26] it was forecast that a similar qualitative behavior should hold for the Bohman-Frieze process. In the spirit of Theorem 1.1 we show also for this parameter the a similar behavior for all (ℓ, K)-bounded rules.

Theorem 1.2. *Let $\ell \geq 2, K \geq 0$, and let R be an (ℓ, K)-bounded rule. Let $X_k(R;\, t)$ be the number of vertices in components of size k after tn edges were added. Then there is a constant $d_R > 0$ and an $\varepsilon_R > 0$ such that for any $|\varepsilon| < \varepsilon_R$ whp*

$$\limsup_{k\to\infty} \frac{\log n^{-1}X_k(R;\, t_R + \varepsilon)}{k} = -d_R\varepsilon^2 + O(\varepsilon^3).$$

Proof Techniques As already mentioned in the previous section, for every (ℓ, K)-bounded rule a phase transition occurs at some critical time that we denote by t_R. We will argue in the next section that t_R is given by the solution to the equation $S(t_R) = 0$, where S is defined by the ordinary differential equation (ODE)

$$S'(t) = -\overline{h}(t, z) - g(t)S(t)^2 \qquad \text{and} \qquad S(0) = 1. \tag{1.2}$$

Here $\overline{h}$ and g are non-negative rule-dependent functions, but we omit the dependence on R to avoid notational cluttering. The function S corresponds to the inverse of the so-called

'susceptibility', which is the expected size of the component of a randomly selected vertex in the graph in which tn edges were added. The ODE in (1.2) is in perfect agreement with the corresponding ODE for bounded-size rules derived in [40].

In order to achieve our main results we go further and actually track simultaneously the *actual* component size distribution of the random graph generated by repeated application of R. Indeed, let as before $X_k(R;\, t)$ denote the number of vertices in components of size k after $m = tn$ edges were added. Then we argue, see Section 2, that whp the sequence $n^{-1}X_k$ converges as $n \to \infty$ to a (deterministic) function $x_k(R;\, t)$ satisfying a certain ODE, which depends only on $x_1, \dots, x_k$. In order to study the whole distribution we define the bivariate generating function

$$D(t, z) = \sum_{k \geq 1} x_k(R;\, t) z^k$$

that encodes concisely all the desired quantities. The ODE's for the x_k's then imply that D satisfies a non-homogeneous quasi-linear partial differential equation (PDE)

$$D_t + zg(t)(1 - D)D_z = h(t, z) \quad \text{with initial condition} \quad D(0, z) = z, \tag{1.3}$$

where D_x stands for the partial derivative with respect to the variable x, and the function g is the same as in (1.2) and $h_z = \overline{h}$ in (1.2). There is a vast literature on treatment of such equations. Indeed, they arise in the context of *stochastic models for coalescence* in physics, chemistry and cosmology, see [2] for an excellent survey. In an abstract setting, in these models n clusters with (possibly different) masses move through space. Any two such clusters with masses i and j merge at a specific rate, typically denoted by $K(i, j)$, to a new cluster with mass $i + j$. This gives rise to a PDE similar to the one in (1.3). However, this PDE is so general that to our knowledge no complete treatment can be found in the literature; only some special cases are understood, for example the case $K(i, j) = ij$ that corresponds to the Erdős-Rényi process. Our main technical contribution is the analysis of the solutions of the equation (1.3) in a rather general setting. Particularly, it turns out that there is a critical point (that, of course, coincides with the one given by (1.2) for the special case $z = 1$), around which the PDE admits *two* solutions, which meet at a so-called *double point*, see Sections 3.1 and 4. We obtain a precise description of the local behavior of these solutions around that critical point, which then enables us to infer both claimed results. Thus, our contribution may find several applications in stochastic models for coalescence.

Let us remark at this point that the use of PDE's to study random graph processes with various degrees of dependencies is not entirely new. As far as we know, it appeared for the first time in [37] in the context of "triangle percolation" on random graphs. Moreover, it appears in several places in the context of analytic combinatorics, see e.g. [17, 29]. However, in all these applications the considered PDE's are either solvable explicitly or the properties of the solutions can be comparatively easily, which simplifies the subsequent analysis significantly. Finally, in [26] this PDE was studied for the special case of the Bohman-Frieze process.

1.2 Further Related Work

In the classical setting the paradigm of the power of choice was investigated in the context of various balls-into-bins processes that play a crucial role in the study of load balancing problems and in resource allocation. In their seminal paper, Azar, Broder, Karlin and Upfal [3] analyzed the following natural occupancy problem: suppose that we place sequentially n

balls into n bins, where for each ball we select two bins randomly and then assign it into the bin that is less full at the time of placement. Then the *maximum load*, i.e., the number of balls in the fullest bin, is exponentially smaller compared to the case where the balls have "no choice" and are just placed randomly. The paper [3] has triggered a long line of research (see e.g. [33, 41, 13, 5, 36, 4] for a non-exhaustive list) and the main ideas have found applications in many different areas, for example in the development of more effective hashing schemes [11, 35, 14], in particular Cuckoo Hashing [34, 19, 12, 20, 21].

On the other hand, in the context of random graphs, quite a few papers have attempted to settle many open problems about processes that are inspired by the paradigm of choice. Many of these models are called 'Achlioptas processes', since Achlioptas, inspired by the celebrated result on the power of two choices [3], suggested to investigate generalized versions of the classical Erdős-Rényi setting, where multiple edges are presented in each step. Except of the papers mentioned previously, there are results concerning the appearance of subgraphs [32, 18, 31, 30, 27], hamiltonicity [28], and connectivity [25].

2 Component Size Distribution & Derivation of Main Result

2.1 Component Sizes & Differential Equations

For a natural number n we write $[n] = \{1, \dots, n\}$. Given an (ℓ, K)-bounded rule R and $\tau \geq 0$ we let $G_{\tau n}(R)$ be the graph after $m = \lfloor \tau n \rfloor$ edges are added, where for $\tau = 0$ it is the graph with n vertices and no edge. We write $X_k(R; \tau)$ for the number of vertices in components of size k in $G_{\tau n}(R)$. We will omit from now on the explicit dependence on the specific rule R under consideration. Applying Wormald's differential equation method (Theorem 5.1 in [42]), in [40] it was shown that for any $(4, K)$-bounded rule, there exist deterministic functions $x_k = x_k(\tau)$, where $1 \leq k \leq K$, such that whp for any $\tau \geq 0$

$$n^{-1} X_k(\tau) = x_k(\tau) + o(1),$$

where the family $(x_k(\tau))_{1 \leq k \leq K}$ is the solution of a system of ODEs. We extend this result to any (ℓ, K)-bounded rule for any $\ell \geq 2$ and, crucially, for *all* components sizes k. The aim of this section is to give a concise description of the resulting (infinite) system of ODEs, see Lemma 2.1, which is the basic object that we study in the rest of this paper.

In order to derive the equations for the dynamics of the random variable X_k we begin with studying the typical evolution, i.e., the expected change of X_k, in one round. Since it will be useful in the sequel, we first demonstrate the procedure in the case $K = 0$, i.e., the Erdős-Rényi case, where the rule picks in every step a random edge $\{v_{\ell-1}, v_\ell\}$. We note first that X_1 decreases by one exactly when only one of $v_{\ell-1}, v_\ell$ is an isolated vertex in $G_{\tau n}$; this event occurs with probability $2n^{-1}X_1(\tau)\big(1 - n^{-1}X_1(\tau)\big)$. Moreover, X_1 decreases by two if both $v_{\ell-1}$ and v_ℓ are isolated, which happens with probability $n^{-2}X_1(\tau)(X_1(\tau) - 1)$. In other words, X_1 changes by adding an edge in expectation by

$$\mathbb{E}[X_1(\tau + n^{-1}) - X_1(\tau) \,|\, G_{\tau n}] = -2n^{-1}X_1(\tau) + O(n^{-1}). \tag{2.1}$$

For component sizes $k \geq 2$ we can use a similar reasoning. The quantity X_k decreases by k exactly when only one of $v_{\ell-1}, v_\ell$ is contained in a component of size k in $G_{\tau n}$, which happens with probability $2n^{-1}X_k(\tau)\big(1 - n^{-1}X_k(\tau)\big)$. Additionally, X_k decreases by $2k$ if both $v_{\ell-1}$ and v_ℓ are in distinct components of size k, which happens with probability $n^{-2}X_k(\tau)(X_k(\tau) -$

k). On the other hand, X_k can also increase. This happens if $v_{\ell-1}, v_\ell$ join two components of sizes j and $k-j$ for some $1 \le j < k$. Thus

$$\mathbb{E}[X_k(\tau + n^{-1}) - X_k(\tau) \,|\, G_{\tau n}] = -2k\frac{X_k(\tau)}{n} + k \sum_{1 \le j < k} \frac{X_j(\tau)}{n}\frac{X_{k-j}(\tau)}{n} + O(kn^{-1}). \quad (2.2)$$

The big-O term here includes the contribution of events with small probability, i.e. the case where both $v_{\ell-1}, v_\ell$ are contained in the same component. The system of equations (2.1) and (2.2) describes the expected change of X_k in the Erdős-Renyi case ($K = 0$).

Let us now consider general $K \ge 1$. Suppose we are given the graph $G = G_{\tau n}$ generated by the rule R after τn edges were added, and the randomly selected vertices $v_1, \dots, v_\ell$. In order to compute $\mathbb{E}[X_k(\tau + n^{-1}) - X_k(\tau) \,|\, G_{\tau n}]$ we shall consider two cases separately: first, when $v_1, \dots, v_{\ell-2}$ are contained in components of size $\le K$ – in which case we say that R picks a *non-Erdős-Rényi* (non-ER) edge – and second, when $\{v_{\ell-1}, v_\ell\}$ is selected (in which case we say that R picks an ER edge).

Assume that an ER edge is chosen; the probability of this event is

$$G(\tau) = 1 - n^{-\ell+2}(X_1(\tau) + \cdots + X_K(\tau))^{\ell-2}. \quad (2.3)$$

In this case X_k follows exactly the same trajectory as decipted in Equations (2.1) and (2.2), since $v_{\ell-1}$ and v_ℓ are (unconditionally) random vertices. Thus, for any $k \ge 1$

$$\mathbb{E}[X_k(\tau + n^{-1}) - X_k(\tau) \,|\, G_{\tau n} \wedge \text{ER edge}] = -2k\frac{X_k(\tau)}{n} + k \sum_{1 \le j < k} \frac{X_j(\tau)}{n}\frac{X_{k-j}(\tau)}{n} + O(kn^{-1}). \quad (2.4)$$

The case in which R chooses a non-ER edge is significantly more involved. The probability for this event equals $1 - G(\tau) = n^{-\ell+2}(X_1(\tau) + \cdots + X_K(\tau))^{\ell-2}$. To capture how the decision of R influences the expected change of X_k, we introduce a (rule-dependent) auxiliary random variable $Q(G;\, i, j)$, where $1 \le i, j \le K$, which denotes the probability that $R(G;\, v_1, \dots, v_\ell) = \{x, y\} \ne \{\ell - 1, \ell\}$ such that $c_G(v_x) = i$ and $c_G(v_y) = j$, i.e., the randomly chosen vertices $v_1, \dots, v_{\ell-2}$ are such that the rule merges a component of size i with a component of size j in G. Let us write $R^{-1}(i, j)$ for the set of $(\ell-2)$-tuples $(c_1, \dots, c_{\ell-2}) \in [K]^{\ell-2}$, such that for any graph G with the property that the vertices $v_1, \dots, v_{\ell-2}$ satisfy $c_G(v_i) = c_i$ for $1 \le i \le \ell - 2$ and $v_{\ell-1}, v_\ell$ are arbitrary we have that $R(G;\, v_1, \dots, v_{\ell-2}, v_{\ell-1}, v_\ell) = \{x, y\}$ and $c_x = i$ and $c_y = j$. Note that R^{-1} is well-defined and in particular dependent only on the sequence of component sizes and not on any other property of G, since R is a size rule. With this notation it follows that

$$Q(G_{\tau n};\, i, j) = \sum_{(c_1, \dots, c_{\ell-2}) \in R^{-1}(i,j)} \prod_{s=1}^{\ell-2} \frac{X_{c_s}(\tau)}{n}. \quad (2.5)$$

Note that not necessarily $Q(G;\, i, j) = Q(G;\, j, i)$, since R could be asymmetric in terms of component sizes. In the sequel we omit the explicit dependence of G on Q.

Let us first study the expected change in the number of isolated vertices. X_1 decreases by one whenever the rule merges a component of size one with a component of size $2 \le j \le K$). Moreover, it decreases by two with probability $2Q(1, 1)$. Thus we have

$$\mathbb{E}[X_1(\tau + n^{-1}) - X_1(\tau) \mid G_{\tau n} \wedge \text{non-ER edge}] = -\frac{1}{1 - G(\tau)} \sum_{1 \le j \le K} (Q(1, j) + Q(j, 1)) + O(K^3 n^{-1}), \quad (2.6)$$

where the error term originates from rare events in which two or more out of the randomly selected vertices $v_1, \dots, v_\ell$ are in the same component of $G_{\tau n}$. For $2 \le k \le K$, X_k decreases by k when the rule merges a component of size k with a component of size $1 \le j \le K$, $j \neq k$, and it decreases by $2k$ if $j = k$. Moreover, X_k increases by k if the rule merges a component of size i with a component of size j with $i + j = k$, where $1 \le i, j \le K$. Thus

$$\begin{aligned}&\mathbb{E}[X_k(\tau + n^{-1}) - X_k(\tau) \mid G_{\tau n} \wedge \text{non-ER edge}]\\ &\quad = \frac{1}{1 - G(\tau)} \left(-k \sum_{1 \le j \le K} (Q(k,j) + Q(j,k)) + k \sum_{i+j=k} Q(i,j) \right) + O(K^3 n^{-1}). \end{aligned} \tag{2.7}$$

Next, for $K < k \le 2K$, X_k cannot decrease if an non-ER edge was selected. However, it can increase by k if the total number of vertices in the selected components is k. In this case we obtain as above

$$\begin{aligned}&\mathbb{E}[X_k(\tau + n^{-1}) - X_k(\tau) \mid G_{\tau n} \wedge \text{non-ER edge}]\\ &\quad = \frac{k}{1 - G(\tau)} \sum_{\substack{1 \le i,j \le K \\ i+j=k}} (Q(k,j) + Q(j,k)) + O(K^3 n^{-1}). \end{aligned} \tag{2.8}$$

Finally, for $k > 2K$ the expected change of X_k conditional on a non-ER edge is 0, since in a non-ER edge both involved component sizes are at most K.

Summing up, the system (2.3)–(2.8) of equations specifies the expected change of the quantities X_k, where $k \ge 1$, if an edge is added to $G_{\tau n}$. Define the auxiliary functions

$$q(i, j; x_1, \dots, x_K) = \sum_{(c_1, \dots, c_{\ell-2}) \in R^{-1}(i,j)} \prod_{s=1}^{\ell-2} x_{c_s} \tag{2.9}$$

and $f_k(x_1, \dots, x_k), g(x_1, \dots, x_K)$ where, at $(x_1, \dots, x_k)$

$$f_k = k \left(-\mathbf{1}[1 \le k \le K] \sum_{1 \le j \le K} (q(k,j) + q(j,k)) + \sum_{i+j=k} q(i,j) \mathbf{1}[1 \le i, j \le K] \right) \tag{2.10}$$

and

$$g = 1 - (x_1 + \cdots + x_K)^{\ell-2}. \tag{2.11}$$

Note that $f_k = 0$ for $k > 2K$. If we write $\mathbf{X}(\tau)$ for the vector $(X_1(\tau), \dots, X_K(\tau))$ then the system (2.3)–(2.8) implies for $k \ge 1$ that

$$\begin{aligned}&\mathbb{E}[X_k(\tau + n^{-1}) - X_k(\tau) \mid G_{\tau n}]\\ &\quad = f_k\left(\frac{\mathbf{X}(\tau)}{n}\right) + g\left(\frac{\mathbf{X}(\tau)}{n}\right) k \left(-\frac{2X_k(\tau)}{n} + \sum_{i+j=k} \frac{X_i(\tau)}{n} \frac{X_j(\tau)}{n} \right) + O\Big(\frac{K^3 + k}{n}\Big). \end{aligned} \tag{2.12}$$

Moreover, for any $k \ge 1$ the difference $X_k(\tau + n^{-1}) - X_k(\tau)$ is bounded, and the functions on the right-hand side of (2.12) are continuous and satisfy a Lipschitz condition. Following

the general principle of the differential equations method (see Theorem 5.1 in [42]) we infer for any $k \geq 1$, if we choose the x_k's such that they satisfy the ODE

$$x_k' = f_k(x_1, \dots, x_K) + g(x_1, \dots, x_K)k \left(-2x_k + \sum_{i+j=k} x_i x_j \right) \tag{2.13}$$

then we obtain the following result.

Lemma 2.1. *Let R be an (ℓ, K)-bounded rule. Then the system (2.13) with initial conditions $x_1(0) = 1$ and $x_k(0) = 0$ for $k \geq 2$ has a unique solution. Moreover, for any $k \geq 1$ we have that whp*

$$n^{-1} X_k(\tau) = x_k(\tau) + o(1).$$

Let us illustrate the conclusion with a few examples.

The Erdős-Rényi process. As already mentioned, this process is given by the unique $(2, 0)$-bounded rule. We have $f_k = 0$ for all $k \geq 1$, and thus

$$x_k' = -2kx_k + k \sum_{i+j=k} x_i x_j.$$

Let us remark already at this point that this system can be solved explicitly; we obtain that

$$x_k(\tau) = \frac{1}{2\tau} \frac{k^{k-1}}{k!} (2\tau e^{-2\tau})^k. \tag{2.14}$$

The Bohman-Frieze process. This is the unique $(4, 1)$-bounded rule with

$$R(G;\, v_1, v_2, v_3, v_4) = \begin{cases} \{1, 2\}, \text{ if } (c_G(v_1), c_G(v_2), c_G(v_3), c_G(v_4)) = (1, 1, 1, 1) \\ \{3, 4\}, \text{ otherwise} \end{cases}.$$

In words, R selects the edge $v_1 v_2$ only if v_1, v_2 are isolated in G. Thus $R^{-1}(1, 1) = (1, 1)$ and by applying (2.9) we infer that $q(1, 1) = x_1^2$, and $q(i, j) = 0$ otherwise. Using (2.10) we further obtain that $f_1 = -2q(1, 1)$ and $f_2 = 2q(1, 1)$. Altogether we obtain from (2.13) the system

$$x_1' = -2x_1^2 - 2(1 - x_1^2)x_1, \qquad x_2' = 2x_1^2 + 2(1 - x_1^2)(-2x_2 + x_1^2),$$

and for $k \geq 3$

$$x_k' = (1 - x_1^2) \cdot k \left(-2x_k + \sum_{i+j=k} x_i x_j \right).$$

The K-Product-Rule. This family of rules are bounded versions of Achlioptas's original product rule, where in the formal limit $K \to \infty$ all choices coincide. More precisely, the K-Product-Rule is a $(6, K)$-bounded rule such that if we write $\overline{v} = (v_1, \dots, v_6)$ and $c_G(\overline{v}) = (c_G(v_1), c_G(v_2), c_G(v_3), c_G(v_4), c_G(v_5), c_G(v_6))$, then

$$R(G;\, \overline{v}) = \begin{cases} \{1, 2\}, \text{ if } c_G(v_1)c_G(v_2) < c_G(v_3)c_G(v_4) \text{ and } c_G(\overline{v}) \in [K]^4 \times [n]^2 \\ \{3, 4\}, \text{ if } c_G(v_1)c_G(v_2) \geq c_G(v_3)c_G(v_4) \text{ and } c_G(\overline{v}) \in [K]^4 \times [n]^2 \\ \{5, 6\}, \text{ otherwise} \end{cases}.$$

So, these rules 'mimic' the product rule for component sizes up to K, and otherwise choose a random edge. From the definition it follows that for $1 \leq i, j \leq K$ we have

$$R^{-1}(i,j) = \big\{(i,j,i',j') \in [K]^4 : ij < i'j'\big\} \cup \big\{(i',j',i,j) \in [K]^4 : ij \leq i'j'\big\}.$$

Together with Equations (2.9)-(2.13) this specifies fully the resulting system of ODEs, where

$$q(i,j) = x_i x_j \cdot \sum_{(i,j,i',j') \in R^{-1}(i,j)} x_{i'} x_{j'} \cdot \sum_{(i',j',i,j) \in R^{-1}(i,j)} x_{i'} x_{j'}.$$

2.2 Derivation of Main Results

In this section we prove Theorems 1.1 and 1.2. Let R be an (ℓ, K)-bounded rule and let $(x_k(\tau))_{k \geq 1}$ denote the system of ODEs from Lemma 2.1. Define a formal power series by

$$D(\tau, z) = \sum_{k \geq 1} x_k(\tau) z^k.$$

Letting D_x stand for the partial derivative with respect to the variable x we obtain the following statement.

Lemma 2.2. *In the situation of this section we have $D(0,z) = z$ and D satisfies the PDE*

$$D_\tau + 2zg(\tau)(1-D)D_z = h(\tau, z),$$

where h is given in (2.17) *and $g(\tau) = g(x_1(\tau), \ldots, x_K(\tau))$, where g is given in* (2.11).

Proof. The initial condition is implied by the fact $x_1(0) = 1$ and $x_k(0) = 0$ for $k \geq 2$, c.f. Lemma 2.1. To see the second claim, first note that

$$D_\tau = \sum_{k \geq 1} x_k'(\tau) z^k \quad \text{and} \quad D_z = \sum_{k \geq 1} k x_k(\tau) z^{k-1}. \tag{2.15}$$

Using (2.13) we thus obtain

$$\begin{aligned} D_\tau = \sum_{k \geq 1} x_k'(\tau) z^k &= \sum_{k \geq 1} f_k(x_1, \ldots, x_K) z^k \\ &\quad + g(x_1, \ldots, x_K) \left(-2 \sum_{k \geq 1} k x_k z^k + \sum_{k \geq 2} k \sum_{1 \leq j < k} x_j x_{k-j} z^k \right). \end{aligned} \tag{2.16}$$

Note that

$$-2 \sum_{k \geq 1} k x_k z^k + \sum_{k \geq 2} k \sum_{1 \leq j < k} x_j x_{k-j} z^k = -2zD_z + z \frac{\partial}{\partial z} D^2 = -2z(1-D)D_z.$$

Moreover, by using (2.10) and the fact that $f_k = 0$ for $k > 2K$ we obtain that

$$h(\tau, z) = \sum_{k \geq 1} f_k(x_1, \ldots, x_K) z^k = \sum_{1 \leq i,j \leq K} (-iq(i,j) z^i - jq(i,j) z^j + (i+j) q(i,j) z^{i+j}). \tag{2.17}$$

The proof is completed by plugging this into (2.16). ∎

Before we proceed with studying the solutions to (2.15) we need the following crucial properties of the functions g and h.

Corollary 2.3. *In the situation of this section*

$$g(\tau) > 0 \text{ for } \tau > 0 \text{ and } \int_0^\infty g(\tau)\, d\tau > \frac{1}{2},$$

and

$$h(\tau, 1) = 0, \qquad h_z(\tau, 1),\ h_{zz}(\tau, 1) \geq 0 \text{ for } \tau > 0.$$

Proof. The statements for h follow immediately from the explicit expression given in (2.17) and from the fact $q(i, j) \geq 0$ for all $1 \leq i, j \leq K$. To see that $g(\tau) > 0$ when $\tau > 0$, let us write $y = x_1 + \cdots + x_K$, so that $g = 1 - y^{\ell-2}$. Then, by the sake of contradiction, let t be the infimum over all $\tau > 0$ such that $g(\tau) = 0$, and assume that $t > 0$. By applying (2.10), for all $\tau < t$

$$0 = y'(\tau) \leq -(K+1) \sum_{1 \leq i,j \leq K} q(i, j)\mathbf{1}[i + j > K]$$

So $q(i, j) = 0$ *for all* $1 \leq i, j \leq K$ such that $i + j > K$, and in particular, by (2.9), for all $0 < \tau \leq t$

$$\prod_{s=1}^{\ell-2} x_{c_s}(\tau) = 0 \quad \text{for all} \quad (c_1, \ldots, c_{\ell-2}) \in \bigcup_{\substack{1 \leq i,j \leq K \\ i+j>K}} R^{-1}(i, j). \tag{2.18}$$

Next, let us note that for all $\tau > 0$ we have that $0 < x(\tau) < 1$. Indeed, (2.14) implies that for the Erdős-Rényi process we have whp that the limiting fraction of isolated vertices is

$$x_1(\mathsf{ER};\, \tau) = e^{-2\tau}.$$

However, for any (ℓ, K)-bounded rule we have that $x_1(\tau) \geq x_1(\mathsf{ER};\, \ell\tau)$, since the graph generated after adding τn edges is a subgraph of the Erdős-Rényi process with $\ell\tau n$ edges, namely all edges that are presented to the rule.

Since $x_1(\tau) \neq 0$, let us consider the terms in (2.18) in which $i = 1, j = K$. It follows that $x_K(\tau) = 0$ for all $0 \leq \tau \leq t$. Define the function $y_2 = x_1 + \cdots + x_{K-1}$. It follows that

$$y_2'(\tau) \leq -K \sum_{1 \leq i,j \leq K} q(i, j)\mathbf{1}[i + j > K - 1]$$

With the same reasoning as before we infer that $x_{K-1}(\tau) = 0$ for all $0 \leq \tau \leq t$. Moreover, by iterating this argument for the partial sum $x_1 + \cdots + x_i$, for any $2 \leq i \leq K - 2$, we obtain similarly that $x_2 = \cdots = x_K = 0$ for all $0 \leq \tau \leq t$. But then, since $x_1 < 1$, we have $g > 0$, a contradiction.

Finally, to see that $\int_0^\infty g(\tau)d\tau > 1/2$ note that this integral actually diverges. Indeed, $g(t)$ is non-decreasing (since the fraction of vertices in components with size less than K is non-increasing). Moreover, if $g(t) > \varepsilon$ for some $t > 0$, then this implies that $g(\tau) \geq g(t)$ for all $\tau \geq t$. The claim follows. ∎

Theorem 2.4. *Suppose that $g(\tau)$ and $h(z, \tau)$ are twice differentiable functions that are defined for $\tau \geq 0$ and for complex z in a neighborhood of $z = 1$ that satisfy the conclusions of*

Corollary 2.3, Furthermore, let $\tau_c > 0$ be defined as the smallest positive zero of the solution $S(\tau)$ of the ODE $S' = -2g(\tau) - h_z(\tau, 1)S^2$, $S(0) = 1$.

Then there exist constants $c \neq 0$, $d \neq 0$ such that the solution $D = D(\tau, z)$ of the PDE

$$D_\tau + 2zg(\tau)(1 - D)D_z = h(\tau, z)$$

with boundary condition $D(0, z) = z$ has the following properties:

1. **Behaviour for** $z < 1$*: There exists constants $\eta_1 > 0, \eta_2 > 0$ such that the solution $D(\tau, z)$ exists for $0 \leq \tau \leq \tau_c + \eta_1$ and $1 - \tau_2 \leq z < 1$ and we have*

$$\lim_{z \to 1-} D(\tau, z) = \begin{cases} 1 & \text{for } 0 \leq \tau \leq \tau_c, \\ 1 - c(\tau - \tau_c) + O((\tau - \tau_c)^2) & \text{for } \tau_c \leq \tau \leq \tau_c + \eta_1. \end{cases}$$

2. **Squareroot singularity***: There exist constants $\eta_3 > 0$, $\eta_4 > 0$ and functions $\rho(\tau)$, $a(z, \tau)$, $b(z, \tau)$ for $\tau \in [\tau_c - \eta_3, \tau_c + \eta_3]$ and $|z - \rho(\tau)| < \eta_4$ with the following properties:*

   - *$\rho(\tau)$ is the radius of convergence of the function $z \mapsto D(z, \tau)$, it is nonzero and satisfies*
   $$\rho(\tau) = 1 + d(\tau - \tau_c)^2 + O((\tau - \tau_c)^3).$$
   - *The functions $a(z, \tau)$, $b(z, \tau)$ are non-zero and the mappings $z \mapsto a(\tau, z)$ and $z \mapsto b(\tau, z)$ are analytic for $|z - \rho(\tau)| < \eta_4$.*
   - *The function $z \mapsto D(\tau, z)$ is represented by*
   $$D(\tau, z) = a(\tau, z) - b(\tau, z)\sqrt{1 - \frac{z}{\rho(\tau)}}$$
   for $\tau \in [\tau_c - \eta_3, \tau_c + \eta_3]$ and $|z - \rho(\tau)| < \eta_4$ with $\arg(z - \rho(\tau)) \neq 0$.

It is now easy to deduce Theorems 1.1 and 1.2 from Theorem 2.4.

Proof of Theorem 1.1. We know from [39, Theorem 3] that whp

$$\lim n^{-1} L_1(R; (\tau)n, n) = 1 - \sum_{i \geq 1} x_i(\tau).$$

Furthermore, from the definition we readily infer that

$$\lim_{z \to 1-} D(\tau, z) = \sum_{i \geq 1} x_i(\tau).$$

Theorem 1.1 follows. Since $\sum_{i \geq 1} x_i(\tau) \leq 1$ it is clear that $c > 0$. ∎

Proof of Theorem 2.4. The radius of convergence of the function $z \mapsto D(\tau, z)$ is given by

$$\rho(\tau) = \left(\limsup_{k \to \infty} x_k(\tau)^{1/k} \right)^{-1}$$

which implies that

$$\limsup_{k \to \infty} \frac{\log x_k(\tau)}{k} = -\log \rho(\tau) = -d(\tau - \tau_c)^2 + O((\tau - \tau_c)^3).$$

Since $0 < x_k(\tau) < 1$ it is clear that $d > 0$. ∎

3 The Erdős-Rényi Case

3.1 First Approach – A Direct Solution

Recall that the PDE for the Erdős-Rényi process is given by

$$D_\tau + 2zD_z(1-D) = 0, \quad D(0,z) = z$$

As already mentioned, this PDE has an explicit solution. Indeed, by using (2.14) we readily obtain that

$$D(\tau, z) = \frac{1}{2\tau} T(2\tau e^{-2\tau} z) \tag{3.1}$$

and $T(x) = \sum_{k\geq 1} k^{k-1}x^k/k!$ is the tree function that satisfies

$$T = xe^T. \tag{3.2}$$

This directly implies that $D = D(\tau, z)$ satisfies the equation

$$D = ze^{2\tau(D-1)}. \tag{3.3}$$

It is well known, see e.g. [15], that the function $T = T(x)$ has a critical point at $x = 1/e$, where (3.2) has a quadratic branch point, that is, we have two solutions that behave like

$$T(x) = 1 \pm \sqrt{2}\sqrt{1-ex} + O(1-ex). \tag{3.4}$$

Set $P(x,T) = xe^T - T$. Then the behavior of T follows from the properties

$$P(x_0,T_0) = 0, \quad P_T(x_0,T_0) = 0, \quad P_x(x_0,T_0) \neq 0, \quad P_{TT}(x_0,T_0) \neq 0,$$

where $x_0 = 1/e$ and $T_0 = 1$, see [15] for the details. A heuristic approach to derive the expression for T is to consider the Taylor-series expansion for P. We have approximately

$$0 = P(x,T) = P_x(x_0,T_0)(x-x_0) + P_{TT}(x_0,T_0)(T-T_0)^2.$$

Solving this equation gives (3.4). If we try a similar approach to solve (3.3) for $D(\tau, z)$ we recover a critical point at $\tau_0 = 1/2$, $z_0 = 1$, and $D_0 = 1$. More precisely, if we set $Q(\tau, z, D) = ze^{2\tau(D-1)} - D$ then we have

$$Q(\tau_0,z_0,D_0) = 0, \quad Q_D(\tau_0,z_0,D_0) = 0, \quad Q_\tau(\tau_0,z_0,D_0) = 0, \quad Q_z(\tau_0,z_0,D_0) = 1 \neq 0,$$

and the second derivatives evaluate to

$$Q_{\tau\tau}(\tau_0,z_0,D_0) = 0, \quad Q_{\tau D}(\tau_0,z_0,D_0) = 2 \neq 0, \quad Q_{DD}(\tau_0,z_0,D_0) = 1 \neq 0.$$

Actually this leads to a so-called *double point* of the equation $Q = 0$ if $z = z_0 = 1$ and to hyperbola-like solutions if $z \neq z_0$ (but close to z_0).

In order to give an intuition for the behavior of the solution we look at the Taylor series expansion, where we again neglect the higher order terms. We have

$$Q_z(z-1) + Q_{\tau D}(\tau - 1/2)(D-1) + \frac{1}{2}Q_{DD}(D-1)^2 = 0.$$

If $z = z_0 = 1$ then we have two (linear) solutions

$$D = 1 \quad \text{and} \quad D = 1 - \frac{2Q_{\tau D}}{Q_{DD}}(\tau - 1/2), \tag{3.5}$$

whereas if $z \neq 1$ we obtain two branches of a hyperbola with asymptotes given by (3.5). For example, if $z < 1$ they can be computed explicitly by

$$D = 1 - \frac{Q_{\tau D}}{Q_{DD}}(\tau - 1/2) \pm \sqrt{\frac{Q_{\tau D}^2}{Q_{DD}^2}(\tau - 1/2)^2 - \frac{2Q_z}{Q_{DD}}(z-1)}.$$

Figure 1 shows this behaviour.

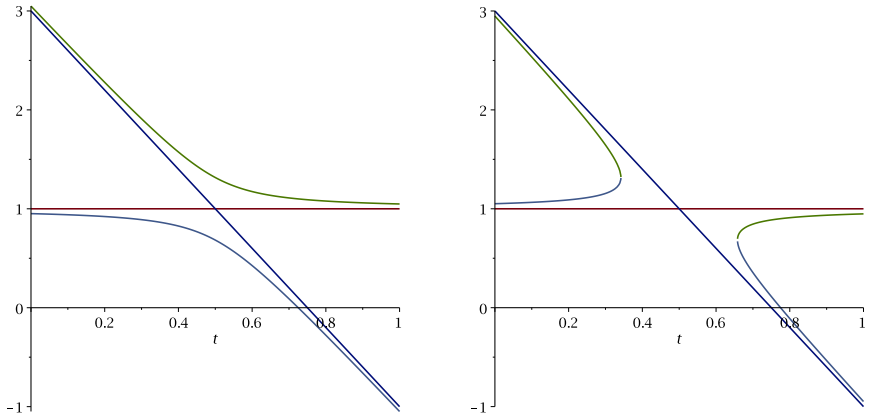

Figure 1: Double point ($z = 1$) and hyperbola solution ($z = 0.95$, $z = 1.05$)

Actually this is an explanation from a different point of view why the function $D(\tau, z)$ behaves *nicely* in τ for all (positive) $z < 1$ and why we observe a quareroot singularity in τ for $z > 1$.

In general we just need the conditions

$$\begin{aligned} Q(\tau_0, z_0, D_0) &= 0, \\ Q_D(\tau_0, z_0, D_0) &= 0, \\ Q_\tau(\tau_0, z_0, D_0) &= 0, \\ Q_z(\tau_0, z_0, D_0) &\neq 0, \\ Q_{DD}(\tau_0, z_0, D_0) &\neq 0, \\ (Q_{\tau\tau}Q_{DD} - Q_{\tau D}^2)(\tau_0, z_0, D_0) &\neq 0 \end{aligned}$$

in order to observe a double point for $z = z_0$ and hyperbola like solutions for $z \neq z_0$. Note that the condition $Q_{DD} \neq 0$ might be disregarded since it only assures that there is no horizontal solution.

3.2 Second Approach – Method of Characteristics

The PDE for any (ℓ, K)-bounded rule is of the form

$$D_\tau + 2zg(\tau)(1-D)D_z = h(\tau, z), \quad D(0,z) = z.$$

For this general PDE, it seems impossible to obtain an explicit solution of the form (3.1). In order to circumvent this difficulty, we explore a different approach that is based on a general method for solving non-linear PDEs, the so-called *method of characteristics*. Before we apply it in order to solve the general equation in Section 4 will illustrate the procedure in the simple Erdős-Rényi case first.

To begin with, we recall quickly how the solution (3.1) can be found with the help of the method of characteristics. The differential equation (3.1) is quasi-linear. We introduce D as a third variable and solve the PDE

$$f_\tau + 2z(1-D)f_z = 0 \tag{3.6}$$

for a function $f = f(\tau, z, D)$, from which we can reconstruct $D = D(\tau, z)$ by solving the equation $f(\tau, z, D) = \text{const}$.

The first step is to introduce a system of ODE's for functions $\overline{\tau} = \overline{\tau}(t)$, $\overline{z} = \overline{z}(t)$, and $\overline{D} = \overline{D}(t)$ of the form

$$\dot{\overline{\tau}} = 1, \quad \dot{\overline{z}} = 2\,\overline{z}\,(1-\overline{D}), \quad \dot{\overline{D}} = 0.$$

The next step is to eliminate the *time variable* t by considering $\tilde{z} = \tilde{z}(\tau)$ and $\tilde{D} = \tilde{D}(\tau)$ that satisfy the differential equation

$$\tilde{z}' = \frac{d\tilde{z}}{d\tau} 2\,\tilde{z}\,(1-\tilde{D}), \qquad \tilde{D}' = \frac{d\tilde{D}}{d\tau} = 0$$

and solve this under the initial conditions $\tilde{z}(0) = c_1$ and $\tilde{D}(0) = c_2$. In this case this is easily done in an explicit way and we obtain

$$\begin{aligned} \tilde{z} &= \tilde{z}(\tau, c_1, c_2) = c_1 e^{2\tau(1-c_2)}, \\ \tilde{D} &= \tilde{D}(\tau, c_1, c_2) = c_2. \end{aligned}$$

Next we have to invert this functions to function $\tilde{c}_1(\tau, z, D)$ and $\tilde{c}_2(\rho, z, D)$. Again this can be done explicitly:

$$\tilde{c}_1 = \tilde{c}_1(\tau, z, D) = ze^{2\tau(D-1)}, \qquad \tilde{c}_2 = \tilde{c}_2(\tau, z, D) = D$$

that are precisely the *characteristics* of the PDF (3.6). Hence $f = f(\tau, z, D)$ is given by

$$f(\tau, z, D) = G(\tilde{c}_1(\tau, z, D), \tilde{c}_2(\tau, z, D)) = G(ze^{2\tau(D-1)}, D),$$

where G is an arbitrary function satisfying certain differentiability assumptions). Finally, we can go back to the solution of the original PDE (3.1) by solving the equation

$$f(\tau, z, D) = G(\tilde{c}_1(\tau, z, D), \tilde{c}_2(\tau, z, D)) = \text{const}.$$

This means that we can rewrite this implicit equation (by assuming that $G_{c_2} \neq 0$)

$$\tilde{c}_1(\tau, z, D) = H(\tilde{c}_2(\tau, z, D)) \quad \text{or} \quad D = H(ze^{2\tau(D-1)})$$

for some function H (or the other way round). The solution $D = D(\tau, z)$ of this implict equation is then a solution of (3.1). In order to specify the (up to now) unknown function H we can use the initial condition $D(0, z) = z$. In this special example this means that $z = H(z)$ and consequently we derive the equation

$$\tilde{c}_1(\tau, z, D) = \tilde{c}_2(\tau, z, D) \quad \text{or} \quad D = ze^{2\tau(D-1)}.$$

Actually we can observe this relation directly from the ODE systems from the above. Since we have started with $\tilde{z}(0, c_1, c_2) = c_1$ and $\tilde{D}(0, c_1, c_2) = c_2$ and $\tilde{c}_1(\tau, z, D)$ and $\tilde{c}_2(\rho, z, D)$ are the inverse functions of $\tilde{z}(\tau, c_1, c_2)$ and $\tilde{D}(\tau, c_1, c_2)$ it follows that $\tilde{c}_1(0, z, D) = z$ and $\tilde{c}_2(0, z, D) = D$ and consequently

$$z = H(D).$$

Of course, the solution of this equation is $D(0, z)$ and since we assume that $D(0, z) = z$ it follows (whithout any calculation) that $H(z) = z$.

As mentioned in Section 3.1 the equation $D = ze^{2\tau(D-1)}$ is a direct translation of $D = (1/2\tau)T(2\tau ze^{-2\tau})$. We just have to multiply by 2τ and substibute $x = 2\tau ze^{-2\tau}$ and $T = 2\tau D$.

4 Proof of Theorem 2.4

In this section we discuss the general PDE

$$D_\tau + 2zg(\tau)(1 - D)D_z = h(\tau, z), \tag{4.1}$$

where $g(\tau)$ and $h(\tau, z)$ satisfy the conditions of Theorem 2.4 and we have the boundary conditions $D(0, z) = z$.

We will show that the solution of the PDE has in principal the same properties as corresponding solution in the Erdös-Rényi case. Actually we show by the method of characteristics that there exists a function $Q(\tau, z, D)$, where the solution $D = D(\tau, z)$ of the equation $Q = 0$ is solution of the above PDE. Furthermore we will identify in general a point $\tau_c > 0$ (which is precisely the smallest positive zero of the solution $S(\tau)$ of the ODE $S' = -2g(\tau) - h_z(\tau, 1)S^2$ with $S(0) = 1$), such that we have for $z_0 = 1$ and $D_0 = 1$ the properties

$$\begin{aligned}
Q(\tau_c, z_0, D_0) &= 0, \\
Q_D(\tau_c, z_0, D_0) &= 0, \\
Q_\tau(\tau_c, z_0, D_0) &= 0, \\
Q_z(\tau_c, z_0, D_0) &\neq 0, \\
Q_{\tau\tau}(\tau_c, z_0, D_0) &= 0, \\
Q_{\tau D}(\tau_c, z_0, D_0) &\neq 0, \\
Q_{DD}(\tau_c, z_0, D_0) &\neq 0.
\end{aligned}$$

Actually all assertions of Theorem 2.4 follow from these properties.

1. **Behaviour for** $z < 1$: As mentioned at the end of Section 3 the conditions $Q = Q_D = Q_\tau = 0$, $Q_z \neq 0$, $Q_{\tau D} \neq 0$, and $Q_{DD} = 0$ ensure that we have a double point at $(\tau, D) = (\tau_c, 1)$ when $z = 1$ is fixed. In particular the two branches of the solution $Q(\tau, 1, D) = 0$ are given by

$$D_1(\tau, 1) = 1 + O((\tau - \tau_c)^2) \quad \text{and} \quad D_2(\tau, 1) = 1 - c(\tau - \tau_c) + O((\tau - \tau_c)^2),$$

where

$$c = \frac{2Q_{\tau D}}{Q_{DD}} \neq 0.$$

Since $Q_z \neq 0$ we observe that when $z \neq 1$ (but sufficiently close to 1) there are hyperbola like solutions for τ sufficiently close to τ_c. Actually if $z \to 1-$ one of the hyperbola arcs has the limit $D_1 = 1$ for $\tau \leq \tau_c$ and the limit $D_2 = 1 - c(\tau - \tau_c) + O((\tau - \tau_c)^2)$ for $\tau \geq \tau_c$. (The other hyperbola arc has limit $D_2 = 1 - c(\tau - \tau_c) + O((\tau - \tau_c)^2)$ for $\tau \leq \tau_c$ and limit $D_1 = 1$ for $\tau \geq \tau_c$.)

Furthermore, if $0 < z < 1$ the function $D(\tau, z) = \sum_{i \geq 1} x_i(\tau) z^i$ exists for all $\tau \geq 0$ and we also have $\lim_{z \to 1-} D(\tau, z) = 1$ for $\tau \leq \tau_c$. This means that the first hyperbola has to coincide with the this function.

2. **Squareroot singularity**: We now use the properties

$$Q(\tau_c, z_0, D_0) = Q_D(\tau_c, z_0, D_0) = 0$$
$$Q_z(\tau_c, z_0, D_0) \neq 0,\ Q_{\tau D}(\tau_c, z_0, D_0) \neq 0,\ Q_{DD}(\tau_c, z_0, D_0) \neq 0.$$

Now suppose that $\tau = \tau_c$ is fixed and we are searching for the solution of the function $z \mapsto D(\tau_c, z)$. Then the above mentioned properties of the derivatives of Q are precisely those that are needed to show that all solutions $D(\tau_0, z)$ have a squareroot singularity at $z = z_0 = 1$ of the form

$$D(\tau_c, z) = a(z) \pm b(z)\sqrt{1 - z/z_0},$$

where a and b are analytic at $z = z_0 = 1$ and $b(z_0) \neq 0$, see [15]. Without loss of generality we can assume that $D(\tau_c, z) = \sum_{i \geq 1} x_k(\tau_c) z^i = a(z) - b(z)\sqrt{1 - z/z_0}$. Next we show that we have the same kind of singular behaviour if τ is close to τ_c. Let $z_0(\tau)$ and $D_0(\tau)$ be the solutions of the system of equations

$$Q(\tau, z_0(\tau), D_0(\tau)) = 0,$$
$$Q_D(\tau, z_0(\tau), D_0(\tau)) = 0.$$

Of course there is a solution for $\tau = \tau_c$: $z_0(\tau_c) = 1$, $D_0(\tau_c) = 1$. The functional determinant is given by

$$\begin{vmatrix} Q_z & Q_D \\ Q_{zD} & Q_{DD} \end{vmatrix} = \begin{vmatrix} Q_z & 0 \\ Q_{zD} & Q_{DD} \end{vmatrix} = Q_z Q_{DD} \neq 0.$$

Hence, be the implicit function theorem the functions $z_0(\tau)$ and $D_0(\tau)$ are uniquely defined for τ close to τ_c. Furthermore we also have

$$Q_z(\tau, z_0(\tau), D_0(\tau)) \neq 0,$$
$$Q_{\tau D}(\tau, z_0(\tau), D_0(\tau)) \neq 0,$$
$$Q_{DD}(\tau, z_0(\tau), D_0(\tau)) \neq 0$$

if τ is sufficiently close to τ_c. Summing up, if τ is sufficiently close to τ_c then we have

$$D(\tau_c, z) = a(\tau, z) - b(\tau, z)\sqrt{1 - z/z_0(\tau)},$$

where a and b are analytic and $b(\tau, z_0(\tau)) \neq 0$.

If $\tau = \tau_c$ then we already know that $D(\tau_c, z)$ is regular for $|z| < 1 = z_0$. Hence, z_0 is certainly the radius of convergence of the mapping $z \mapsto D(\tau_c, z)$. If τ is close to τ_c then $z_0(\tau)$ is certainly a singularity of the mapping $z \mapsto D(\tau_c, z)$. Since $x_i(\tau) \geq 0$ the radius of convergence of $D(\tau, z)$ coincides with the smallest singularity on the positive real line. Consequently $z_0(\tau) = \rho(\tau)$ is the radius of convergence.

Finally we describe the local behaviour of $z_0(\tau)$. Since $z_0(\tau)$ (and $D_0(\tau)$) satisfy the equation $Q(\tau, z_0(\tau), D_0(\tau)) = Q_D(\tau, z_0(\tau), D_0(\tau)) = 0$ we derive by implicit differentiation that

$$z_0'(\tau_c) = -\frac{Q_\tau(\tau_c, 1, 1)}{Q_z(\tau_c, 1, 1)} = 0$$

and

$$z_0''(\tau_c) = \frac{Q_{\tau D}(\tau_c, 1, 1)^2}{Q_z(\tau_c, 1, 1) Q_{DD}(\tau_c, 1, 1)} \neq 0.$$

Hence, if we set $d = z_0''(\tau_c)/2$ we are done.

4.1 Method of Characteristics for (4.1)

In order to apply the method of characteristics for the PDE (4.1) we have to consider first the linear PDE

$$f_\tau + 2zg(\tau) f_z (1 - D) + h(\tau, z) f_D = 0.$$

for the function $f = f(\tau, z.D)$. We introduce a system of ODE's for functions $\overline{\tau} = \overline{\tau}(t)$, $\overline{z} = \overline{z}(t)$, and $\overline{D} = \overline{D}(t)$ of the form

$$\dot{\overline{\tau}} = 1, \quad \dot{\overline{z}} = 2\,\overline{z}\,g(\overline{\tau})\,(1 - \overline{D}), \quad \dot{\overline{D}} = h(\overline{\tau}, \overline{z}).$$

and eliminate the variable t by considering $\tilde{z} = \tilde{z}(\tau)$ and $\tilde{D} = \tilde{D}(\tau)$ that satisfy the differential equations

$$\tilde{z}' = \frac{d\tilde{z}}{d\tau} = 2\,\tilde{z}\,g(\tau)\,(1 - \tilde{D}), \quad \tilde{D}' = \frac{d\tilde{D}}{d\tau} = h(\tau, \tilde{z})$$

and solve this again under the initial conditions $\tilde{z}(0) = c_1$ and $\tilde{D}(0) = c_2$. This leads to functions $\tilde{z}(\tau, c_1, c_2)$ and $\tilde{D}(\tau, c_1, c_2)$. If we invert them to obtain the functions $\tilde{c}_1(\tau, z, D)$ and $\tilde{c}_2(\tau, z, D)$ we arrive at the characteristics of the linear PDF for the function f, that is

$$f = f(\tau, z, D) = G(\tilde{c}_1(\tau, z, D), \tilde{c}_2(\tau, z, D))$$

for some function G. The solution of the original PDE (4.1) is then given by the equation

$$\tilde{c}_1(\tau, z, D) = H(\tilde{c}_2(\tau, z, D))$$

for some function H (or the other way round). As in the Erdős-Rényе case we (obviously) have $\tilde{c}_1(0, z, D) = z$ and $\tilde{c}_2(0, z, D) = D$ and thus $z = H(D)$. By assumption $D(0, z) = z$ and thus $H(z) = z$. This leads to the general relation

$$\tilde{c}_1(\tau, z, D) = \tilde{c}_2(\tau, z, D). \tag{4.2}$$

Of course we have to justify that we can invert the functions $\tilde{z}(\tau, c_1, c_2)$ and $\tilde{D}(\tau, c_1, c_2)$. The functional matrix is given by

$$\begin{pmatrix} \tilde{z}_{c_1} & \tilde{z}_{c_2} \\ \tilde{D}_{c_1} & \tilde{D}_{c_2} \end{pmatrix}.$$

In particular we have (by assumption)

$$\begin{pmatrix} \tilde{z}_{c_1}(0) & \tilde{z}_{c_2}(0) \\ \tilde{D}_{c_1}(0) & \tilde{D}_{c_2}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\tilde{z}(\tau, 1, 1) = \tilde{D}(\tau, 1, 1) = 1$$

Thus, if c_1 and c_2 are sufficiently close to 1 the functions $\tilde{z}$ and $\tilde{D}$ will stay close to 1.

Now, given the function $\tilde{z}(\tau, c_1, c_2)$ and $\tilde{D}(\tau, c_1, c_2)$, the functions $\tilde{z}_{c_1}, \tilde{z}_{c_2}, \tilde{D}_{c_1}, \tilde{D}_{c_2}$ are solutions of the system of differential equations

$$\begin{aligned} \tilde{z}'_{c_1} &= 2\,\tilde{z}_{c_1}\, g(\tau)\,(1-\tilde{D}) - 2\,\tilde{z}\, g(\tau)\,\tilde{D}_{c_1}, \\ \tilde{z}'_{c_2} &= 2\,\tilde{z}_{c_2}\, g(\tau)\,(1-\tilde{D}) - 2\,\tilde{z}\, g(\tau)\,\tilde{D}_{c_2}, \\ \tilde{D}'_{c_1} &= h_z(\tau, \tilde{z})\tilde{z}_{c_1} \\ \tilde{D}'_{c_2} &= h_z(\tau, \tilde{z})\tilde{z}_{c_2}. \end{aligned}$$

Hence the factor $(1-\tilde{D}) \approx 0$ makes the system almost homogeneous. Suppose for a moment that we can omit the inhomogeneous part. Then $(\tilde{z}_{c_1}, \tilde{D}_{c_1})$ is the solution with inital condition $(\tilde{z}_{c_1}(0), \tilde{D}_{c_1}(0)) = (1, 0)$ and $(\tilde{z}_{c_2}, \tilde{D}_{c_2})$ is the solution (of the same homogeneous linear system of differential equations) with inital condition $(\tilde{z}_{c_2}(0), \tilde{D}_{c_2}(0)) = (0, 1)$, that is, $(\tilde{z}_{c_1}, \tilde{D}_{c_1})$ and $(\tilde{z}_{c_2}, \tilde{D}_{c_2})$ are fundamental solutions. In particular it follows that the Wronskian determinant

$$W(\tau) = \det \begin{pmatrix} \tilde{z}_{c_1} & \tilde{z}_{c_2} \\ \tilde{D}_{c_1} & \tilde{D}_{c_2} \end{pmatrix}$$

is non-zero for all τ. By continuity we get the same property for any given interval for τ provided that $\tilde{z}(0) = c_1$ and $\tilde{D}(0) = c_2$ are sufficiently close to 1. Hence, it is possible to invert these functions and so the functions $\tilde{c}_1(\tau, z, D)$ and $\tilde{c}_2(\tau, z, D)$ exist for any given interval for τ if z and D are sufficently close to 1.

Finally if we set

$$Q(\tau, z, D) = \tilde{c}_1(\tau, z, D) - \tilde{c}_2(\tau, z, D).$$

then the solution $D(\tau, z)$ of the PDF (4.1) with $D(z) = z$ satisfies $Q(\tau, z, D(\tau, z)) = 0$. It is therefore our goal to show that there exists $\tau_c > 0$ such that the above properties for the derivatives of Q are satisfied (for $(\tau, z, D) = (\tau_c, 1, 1)$).

4.2 Existence of the Critical Point

The next goal is to show that there exits $\tau_c > 0$ with $Q_D(\tau_c, z_0, D_0) = Q_D(\tau_c, 1, 1) = 0$, that is, we have to find $\tau_c > 0$ with $\tilde{c}_{1,D}(\tau_c, 1, 1) = \tilde{c}_{2,D}(\tau_c, 1, 1)$. Since $\tilde{z}$ and $\tilde{D}$ are the inverse functions of $\tilde{c}_1$ and $\tilde{c}_2$ (and the functional determinant is non-zero) we have $\tilde{c}_{1,D}(\tau, 1, 1) = \tilde{c}_{2,D}(\tau, 1, 1)$ if and only if

$$\tilde{z}_{c_1}(\tau, 1, 1) = -\tilde{z}_{c_2}(\tau, 1, 1).$$

Let $Z = Z(\tau)$ denote the sum $Z(\tau) = \tilde{z}_{c_1}(\tau, 1, 1) + \tilde{z}_{c_2}(\tau, 1, 1)$. and $T = T(\tau)$ the sum $T(\tau) = \tilde{D}_{c_1}(\tau, 1, 1) + \tilde{D}_{c_2}(\tau, 1, 1)$. Then these functions satisfy the system of differential equations

$$\begin{aligned} Z' &= -2g(\tau)T, \\ T' &= h_z(\tau, 1)Z. \end{aligned}$$

with initial conditions $Z(0) = 1$ and $D(0) = 1$. Since we know that $g(\tau) \geq 0$ and $h_z(\tau, 1) \geq 0$ it follows that $Z(\tau)$ is decreasing and $T(\tau)$ is increasing for small $\tau \geq 0$. More precisely, as long as $Z(\tau) \geq 0$ it follows that $T(\tau)$ is increasing and $T(\tau) \geq T(0) = 1 \geq 0$. Thus, $Z(\tau)$ is decreasing. Again by $T(\tau) \geq 1$ and the assumption

$$\int_0^\infty g(\tau)\, d\tau > \frac{1}{2} \tag{4.3}$$

it follows that $Z(\tau)$ has a (smallest) zero τ_c which is also simple. Summing up, if condition (4.3) holds then there exists a unique $\tau_c > 0$ with $Q_D(\tau_c, 1, 1) = 0$.

Next we show that τ_c is precisely the point where the susceptibility $D_z(\tau, 1) = \sum_{i \geq 1} i x_i(\tau)$ diverges. By differentiating the PDE (4.1) with respect to z and setting $z = 1$ we obtain

$$D_{\tau,z}(\tau, 1) - 2g(\tau)D_z(\tau, 1)^2 = h_z(\tau, 1)$$

and consequently, $D_z(\tau, 1)$ satisfies the ODE

$$H' = 2g(\tau)H^2 + h_z(\tau, 1), \qquad H(0) = 1.$$

Furthermore the reciprocal $S = 1/H$ is the solution of the ODE

$$S' = -h_z(\tau, 1)S^2 - 2g(\tau), \qquad S(0) = 1.$$

It is immediate that S is strictly decreasing and has a unique smallest zero. We now compare this solution with the (above) system of differential equations Z, T. It directly follows that the quotient Z/T satisfies the ODE

$$\left(\frac{Z}{T}\right)' = \frac{Z'T - ZT'}{T^2} = \frac{-2g(\tau)T^2 - h_z(\tau, 1)Z^2}{T^2} = -2g(\tau) - h_z(\tau, 1)\left(\frac{Z}{T}\right)^2.$$

Since $Z(0)/T(0) = 1$ it follows that

$$S(\tau) = \frac{Z(\tau)}{T(\tau)}.$$

Hence the smallest positive zero of S equals τ_c and this is precisely the point, where $D_z(\tau, 1) = 1/S(\tau)$ diverges.

4.3 Properties of the Critical Point

We have already shown that $Q(\tau_c, 1, 1) = Q_D(\tau_c, 1, 1) = 0$ In the next step we show that the condition $Q_\tau(\tau_c, 1, 1) = 0$ is then automatically satisfied. From $\tilde{z}(\tau, \tilde{c}_1(\tau, z, D), \tilde{c}_2(\tau, z, D)) = z$ we get

$$\tilde{z}_\tau + \tilde{z}_{c_1}\tilde{c}_{1,\tau} + \tilde{z}_{c_2}\tilde{c}_{2,\tau} = 0.$$

However, since $\tilde{z}(\tau, 1, 1) = 1$ we obtain $\tilde{z}_\tau(\tau, 1, 1) = 0$. Furthermore, since we have $\tilde{z}_{c_1} = -\tilde{z}_{c_2}$ at the critial point $(\tau_c, 1, 1)$ (and also $\tilde{z}_{c_2} \neq 0$ since it starts at 0 as is decreasing) we automatically get

$$Q_\tau(\tau_c, 1, 1) = \tilde{c}_{1,\tau}(\tau_c, 1, 1) - \tilde{c}_{2,\tau}(\tau_c, 1, 1) = 0.$$

The next step is to show that $Q_z(\tau_c, 1, 1) \neq 0$. This is equivalent to $\tilde{c}_{1,z}(\tau_c, 1, 1) - \tilde{c}_{2,z}(\tau_c, 1, 1) \neq 0$ and also equivalent to $\tilde{D}_{c_1}(\tau_c, 1, 1) + \tilde{D}_{c_2}(\tau_c, 1, 1) \neq 0$. Actually the last condition is just $T(\tau_c) \neq 0$, however, this has been proved implicitly when we have constructed the zero τ_c of $Z(\tau)$. Actually we have $T(\tau_c) \geq 1$. We note, however, that $\tilde{c}_1(\tau, 1, 1) = \tilde{c}_2(\tau, 1, 1) = 1$. Consequently we observe also from this property $\tilde{c}_{1,\tau}(\tau_c, 1, 1) = \tilde{c}_{2,\tau}(\tau_c, 1, 1) = 0$.

Finally, we have to deal with the second derivatives. We start with the relation

$$\tilde{z}\left(\tau, c_1(\tau, z, D), c_2(\tau, z, D)\right) = z, \tag{4.4}$$

$$\tilde{D}\left(\tau, c_1(\tau, z, D), c_2(\tau, z, D)\right) = D. \tag{4.5}$$

and recall that $Q(\tau, z, D) = c_1(\tau, z, D) - c_2(\tau, z, D)$. Furthermore we have $\tilde{z}(\tau, 1, 1) = \tilde{D}(\tau, 1, 1) = 1$ which implies that we also have $c_1(\tau, 1, 1) = c_1(\tau, 1, 1) = 1$ (for $\tau \geq 0$). In particular this implies that

$$\tilde{z}_\tau(\tau, 1, 1) = \tilde{D}_\tau(\tau, 1, 1) = c_{1,\tau}(\tau, 1, 1) = c_{2,\tau}(\tau, 1, 1) = 0$$

and similarly for higher derivatives with respect to τ. Note that also gives

$$Q_\tau(\tau, 1, 1) = Q_{\tau\tau}(\tau, 1, 1) = 0.$$

Hence it suffices to show that $Q_{DD}(\tau_c, 1, 1) \neq 0$ and $Q_{\tau D}(\tau_c, 1, 1) \neq 0$. By taking derivatives with respect to D in (4.4) we obtain

$$\tilde{z}_{c_1} c_{1,D} + \tilde{z}_{c_2} c_{2,D} = 0. \tag{4.6}$$

From the above considerations we know already that

$$\tilde{z}_{c_1}(\tau_c, 1, 1) = -\tilde{z}_{c_2}(\tau_c, 1, 1) \neq 0$$

and

$$c_{1,D}(\tau_c, 1, 1) = c_{2,D}(\tau_c, 1, 1) \neq 0.$$

Now by taking the second derivative we obtain

$$\tilde{z}_{c_1c_1} c_{1,D}^2 + 2\tilde{z}_{c_1c_2} c_{1,D} c_{2,D} + \tilde{z}_{c_2c_2} c_{2,D}^2 + \tilde{z}_{c_1} c_{1,DD} + \tilde{z}_{c_2} c_{2,DD} = 0$$

and consequently

$$Q_{DD}(\tau_c, 1, 1) = -\frac{\left(\tilde{z}_{c_1c_1} + 2\tilde{z}_{c_1c_2} + \tilde{z}_{c_2c_2}\right) c_{1,D}^2}{\tilde{z}_{c_1}}(\tau_c, 1, 1).$$

Hence, in order to show that $Q_{DD} \neq 0$ we just have to show that $\tilde{z}_{c_1c_1} + 2\tilde{z}_{c_1c_2} + \tilde{z}_{c_2c_2} \neq 0$.

In passing we can handle $Q_{\tau D}$. By taking the derivative with respect to τ in (4.6) we get

$$\left(\tilde{z}'_{c_1} + \tilde{z}_{c_1c_1}c_{1,\tau} + \tilde{z}_{c_1c_2}c_{2,\tau}\right) c_{1,D} + \left(\tilde{z}'_{c_2} + \tilde{z}_{c_1c_2}c_{1,\tau} + \tilde{z}_{c_2c_2}c_{2,\tau}\right) c_{2,D} + \tilde{z}_{c_1}c_{1,\tau D} + \tilde{z}_{c_2}c_{2,\tau D} = 0$$

or (since $\tilde{z}'_{c_1} + \tilde{z}'_{c_2} = Z' = -2g(\tau)T$)

$$-2g(\tau_c)T, c_{1,D} + \tilde{z}_{c_1}Q_{\tau D} =,$$

which, after rearranging yields

$$Q_{\tau D}(\tau_c, 1, 1) = \frac{2g(\tau_c)T(\tau_c)c_{1,D}(\tau_c, 1, 1)}{\tilde{z}_{c_1}(\tau_c, 1, 1)} \neq 0.$$

It remains to study the function $\tilde{z}_{c_1c_1} + 2\tilde{z}_{c_1c_2} + \tilde{z}_{c_2c_2}$. For this purpose we consider the second derivatives of the differential equations

$$\begin{aligned}
\tilde{z}'_{c_1} &= 2\tilde{z}_{c_1}g(\tau)(1-\tilde{D}) - 2\tilde{z}g(\tau)\tilde{D}_{c_1},\\
\tilde{z}'_{c_2} &= 2\tilde{z}_{c_2}g(\tau)(1-\tilde{D}) - 2\tilde{z}g(\tau)\tilde{D}_{c_2},\\
\tilde{D}'_{c_1} &= h_z(\tau,\tilde{z})\,\tilde{z}_{c_1},\\
\tilde{D}'_{c_2} &= h_z(\tau,\tilde{z})\,\tilde{z}_{c_2},
\end{aligned}$$

which leads to (for $c_1 = c_2 = \tilde{z} = \tilde{D} = 1$)

$$\begin{aligned}
\tilde{z}'_{c_1c_1} &= 2\tilde{z}_{c_1c_1}g(\tau)(1-\tilde{D}) - 4\tilde{z}_{c_1}g(\tau)\tilde{D}_{c_1} - 2\tilde{z}g(\tau)\tilde{D}_{c_1c_1},\\
\tilde{z}'_{c_1c_2} &= 2\tilde{z}_{c_1c_2}g(\tau)(1-\tilde{D}) - 2\tilde{z}_{c_1}g(\tau)\tilde{D}_{c_2} - 2\tilde{z}_{c_2}g(\tau)\tilde{D}_{c_1} - 2\tilde{z}g(\tau)\tilde{D}_{c_1c_2},\\
\tilde{z}'_{c_2c_2} &= 2\tilde{z}_{c_2c_2}g(\tau)(1-\tilde{D}) - 4\tilde{z}_{c_2}g(\tau)\tilde{D}_{c_2} - 2\tilde{z}g(\tau)\tilde{D}_{c_2c_2},\\
\tilde{D}'_{c_1c_1} &= h_{zz}(\tau,\tilde{z})\,\tilde{z}^2_{c_1} + h_z(\tau,\tilde{z})\,\tilde{z}_{c_1,c_1},\\
\tilde{D}'_{c_1c_2} &= h_{zz}(\tau,\tilde{z})\,\tilde{z}_{c_1}\tilde{z}_{c_2} + h_z(\tau,\tilde{z})\,\tilde{z}_{c_1,c_2},\\
\tilde{D}'_{c_2c_2} &= h_{zz}(\tau,\tilde{z})\,\tilde{z}^2_{c_2} + h_z(\tau,\tilde{z})\,\tilde{z}_{c_2,c_2}
\end{aligned}$$

and by setting $c_1 = c_2 = \tilde{z} = \tilde{D} = 1$ to

$$\begin{aligned}
\tilde{z}'_{c_1c_1} &= -4\tilde{z}_{c_1}g(\tau)\tilde{D}_{c_1} - 2g(\tau)\tilde{D}_{c_1c_1},\\
\tilde{z}'_{c_1c_2} &= -2\tilde{z}_{c_1}g(\tau)\tilde{D}_{c_2} - 2\tilde{z}_{c_2}g(\tau)\tilde{D}_{c_1} - 2g(\tau)\tilde{D}_{c_1c_2},\\
\tilde{z}'_{c_2c_2} &= -4\tilde{z}_{c_2}g(\tau)\tilde{D}_{c_2} - 2g(\tau)\tilde{D}_{c_2c_2},\\
\tilde{D}'_{c_1c_1} &= h_{zz}(\tau,1)\,\tilde{z}^2_{c_1} + h_z(\tau,1)\,\tilde{z}_{c_1,c_1},\\
\tilde{D}'_{c_1c_2} &= h_{zz}(\tau,1)\,\tilde{z}_{c_1}\tilde{z}_{c_2} + h_z(\tau,1)\,\tilde{z}_{c_1,c_2},\\
\tilde{D}'_{c_2c_2} &= h_{zz}(\tau,1)\,\tilde{z}^2_{c_2} + h_z(\tau,1)\,\tilde{z}_{c_2,c_2}.
\end{aligned}$$

Now set $Z_2 = \tilde{z}_{c_1c_1} + 2\tilde{z}_{c_1c_2} + \tilde{z}_{c_2c_2}$ and $T_2 = \tilde{D}_{c_1c_1} + 2\tilde{D}_{c_1c_2} + \tilde{D}_{c_2c_2}$. With the help of this notation we obtain

$$\begin{aligned}
Z'_2 &= -2g(\tau)T_2 - 4g(\tau)TZ,\\
T'_2 &= h_z(\tau,1)Z_2 + h_{zz}(\tau,1)Z^2.
\end{aligned}$$

with $Z_2(0) = T_2(0) = 0$.

Next we set $Z_3(\tau) = Z(\tau) - Z_2(\tau)$ and $T_3(\tau) = T(\tau) - T_2(\tau)$. These functions satisfy

$$\begin{aligned} Z_3' &= -2g(\tau)T_3 + 4g(\tau)TZ, \\ T_3' &= h_z(\tau,1)Z_3 - h_{zz}(\tau,1)Z^2. \end{aligned}$$

with $Z_3(0) = T_3(0) = 1$ or

$$\begin{aligned} Z_3(\tau) &= 1 - \int_0^\tau 2g(t)T_3(t)\,dt + \int_0^\tau 4g(t)T(t)Z(t)\,dt, \\ T_3(\tau) &= 1 + \int_0^\tau h_z(t,1)Z_3(t)\,dt - \int_0^\tau h_{zz}(t,1)Z(t)^2\,dt. \end{aligned}$$

Recall that

$$\begin{aligned} Z(\tau) &= 1 - \int_0^\tau 2g(t)T(t)\,dt, \\ T(\tau) &= 1 + \int_0^\tau h_z(t,1)Z(t)\,dt. \end{aligned}$$

It is now immediate to derive

$$Z_3(\tau) \geq Z(\tau) \quad \text{and} \quad T_3(\tau) \leq T(\tau), \quad 0 \leq \tau \leq \tau_c.$$

Actually we have

$$Z_3(\tau) > Z(\tau) \quad \text{and} \quad T_3(\tau) < T(\tau), \quad 0 < \tau \leq \tau_c.$$

We just remark that

$$Z_3(\tau) = 1 - \int_0^\tau 2g(t)T_3(t)\,dt + \int_0^\tau 4g(t)T(t)Z(t)\,dt, \geq 1 - \int_0^\tau 2g(t)T(t)\,dt = Z(\tau).$$

Clearly $Z_3(\tau_c) = Z(\tau_c) - Z_2(\tau_c) > Z(\tau_c)$ implies $Z_2(\tau_c) \neq 0$ and consequently we arrive at the desired claim $Q_{DD}(\tau_c, 1, 1) \neq 0$.

5 Size-Rules & Outlook

As already indicated in the introduction, an important motivation for studying the class of (ℓ, K)-bounded rules is that they provide a concise way of defining a notion of approximation for general size rules. Indeed, consider for example the product rule, which is presented two random edges and chooses the one that "locally" minimizes the susceptibility, i.e., the product of the sizes of the components that are merged. In Section 2.1 we described how this rule can be approximated with a sequence of $(6, K)$-bounded rules. For such rule with parameter K the choice is made according to the product rule, *unless* the component containing any of the first four selected vertices contains more than K vertices. In the latter case we join the fifth and the sixth vertex, i.e., two randomly selected vertices. In Figure 2 simulation results for some values of K are presented.

Our main result implies all these rules undergo a phase transition, i.e., there is a rule dependent critical point in time t_K such that the $(6, K)$-bounded product rule PR_K has a

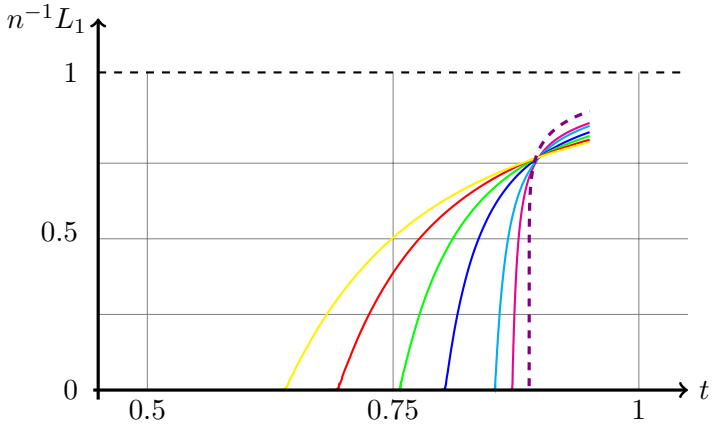

Figure 2: Successive approximations for the product rule (dashed line) with $K = 3$ (yellow), 5, 10, 20, 75, 200, where $n = 5 \times 10^8$.

giant component if tn edges are added, where $t > t_K$. Moreover, we obtain that there are constants $(c_K)_{K\geq 1} > 0$ such that for $\varepsilon > 0$

$$\lim_{n\to\infty} n^{-1}L_1(\mathsf{PR}_K;\, (t_K + \varepsilon)n) = c_K\varepsilon + O(\varepsilon^2).$$

A tantalizing conjecture is now that the sequence of critical times $(t_K)_{K\geq 1}$ has a limit t_{PR} and moreover, this limit should coincide with the critical time of the product rule. Our explicit description of the critical points should in principle allow to conclude that the limit indeed exists, but currently we have not studied this question in sufficient detail to give an affirmative answer.

Secondly and more importantly, our results may be used to understand the phenomenon of 'explosive percolation' [1] via the study of the sequence of the rates c_K. It seems very plausible that this is a diverging sequence, thus providing a partial explanation of why one could think that the product rule undergoes a discontinuous transition: the rate at which the giant component initially grows is *unbounded*.

Our results also provide a much stronger conjecture about the actual component size distribution for times t close to t_K. Indeed, the square-root expansion of D given in Theorem 2.4 strongly suggests that

$$x_k(\tau) \sim k^{-3/2}\alpha(\tau)^k \quad \text{for } k \geq 1.$$

This is much more quantitative compared to our estimate in Theorem 1.2; however, we are currently not able to establish the validity of the expansion in Theorem 2.4 in a sufficiently large domain.

Finally, our methods are currently not mature enough to study the size of the giant component in any rule at point in time much larger than the critical time, i.e., to provide a "fine" description at times $t_R + \varepsilon$ for any $\varepsilon > 0$. In principle, all this information is encoded in the PDE that we study. However, it is an open problem to extract the behavior more explicitly.

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