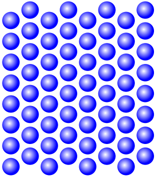
Phase Transitions in Random Discrete Structures

Mihyun Kang

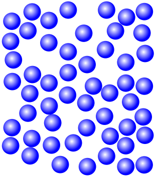
Institut für Optimierung und Diskrete Mathematik

Phase Transition in Thermodynamics

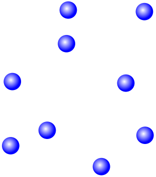
The phase transition deals with a sudden change in the properties of an asymptotically large structure by altering critical parameters.



dense



short range order

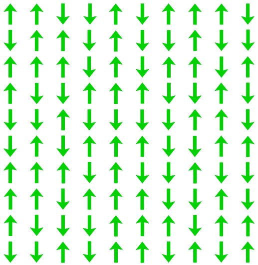


sparse, irregular

Phase Transition in Statistical Physics

Ising model (mathematical model of ferromagnetism)

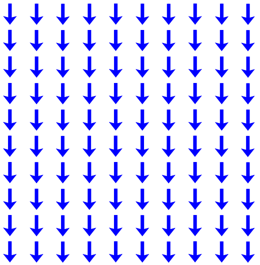
(up or down) Spins are arranged in lattice which interact with nearest neighbours



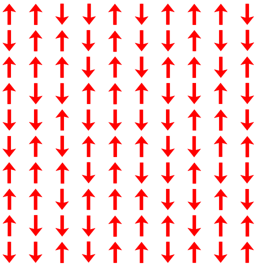
Phase Transition in Statistical Physics

Ising model (mathematical model of ferromagnetism)

(up or down) Spins are arranged in lattice which interact with nearest neighbours



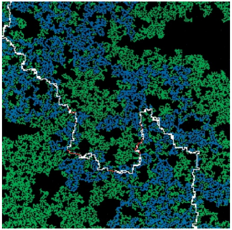
Ordered phase at low temperatures



Disordered phase at high temperatures

Percolation in Physics, Materials Science and Geography

the passage of fluid or gas going through porous or disordered media



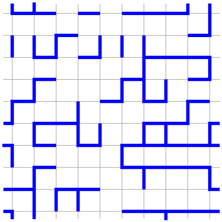



Percolation in Physics, Materials Science and Geography

Mathematical models of percolation

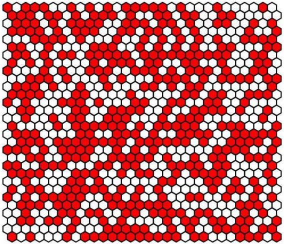
- Bond percolation: each bond (or edge) is either open with prob. p or closed with prob. $1 - p$, independently
- Site percolation: each site (or vertex) is either occupied with prob. p or empty with prob. $1 - p$, independently

$p < p_c$



Bond Percolation on Square Lattice

$p > p_c$

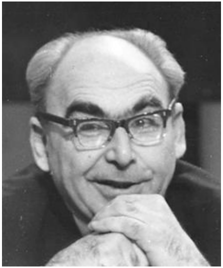


Site Percolation on Hexagonal Lattice

Erdős–Rényi Random Graphs

- $G(n, p)$: each edge of the complete graph K_n is open with probability p, independently of each other
- $G(n, m)$: a graph sampled uniformly at random among all graphs on n vertices and m edges

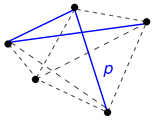
Paul Erdős (1913 – 1996)

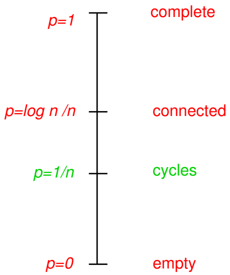


Alfréd Rényi (1921 – 1970)

Erdős–Rényi Random Graphs

- $G(n,p)$: each edge of the complete graph K_n is open with probability p, independently of each other
- $G(n,m)$: a graph sampled uniformly at random among all graphs on n vertices and m edges



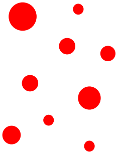


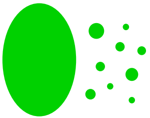
Phase Transition

Binomial random graph $G(n, p)$ [ERDŐS–RÉNYI 60]

Let $p = t/n$ for a constant $t > 0$.

- If $t < 1$, with probability tending to 1 as $n \to \infty$ (whp) all the components have $O(\log n)$ vertices.
- If $t > 1$, whp there is a unique largest component of order $\Theta(n)$, while every other component has $O(\log n)$ vertices.





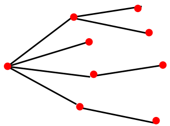
▷ Component exposure via breath-first search and Galton-Watson tree

Galton-Watson Tree

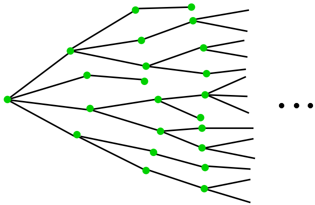
Branching Process

The number of children is given by i.i.d. random variable $\sim \mathrm{Po(t)}$.

- If $t < 1$, the process dies out with probability 1.
- If $t > 1$, with positive probability the process continues forever.



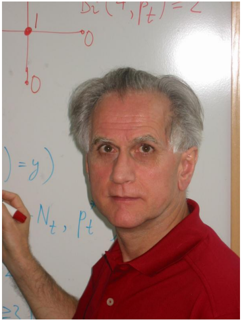
„small" component in $G(n, p)$



„giant" component of order ρn in $G(n, p)$ where $1 - \rho = e^{-t\rho}$

Critical Phase

How big is the largest component in $G(n,p)$, when $pn = 1 + \varepsilon$ for $\varepsilon = o(1)$?



Béla Bollobás

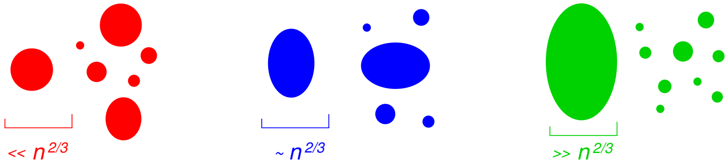
Tomasz Łuczak

Critical Phase

How big is the largest component in $G(n,p)$, when $pn = 1 + \varepsilon$ for $\varepsilon = o(1)$?

[Bollobás 84; Łuczak 90; Łuczak–Pittel–Wierman 94]

- If $\varepsilon\, n^{1/3} \to -\infty$, whp all components are of order $o(n^{2/3})$.
- If $\varepsilon\, n^{1/3} \to \lambda$, whp the largest component is of order $\Theta(n^{2/3})$.
- If $\varepsilon\, n^{1/3} \to \infty$, whp $\exists$ a unique component of order $(1 + o(1))\, 2\varepsilon n$.



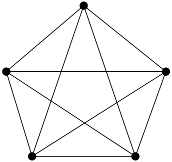
▷ Uniform random graph $G(n,m)$: $\quad m = n/2 + s, \quad s\, n^{-2/3} = \varepsilon\, n^{1/3}$

Planar Graphs

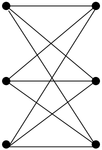
Planar graphs

A planar graph is a graph that *can* be embedded in the plane (without crossing edges).

non–planar



K_5



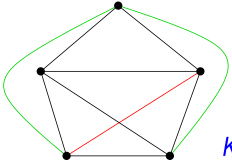
$K_{3,3}$

Planar Graphs

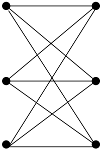
Planar graphs

A planar graph is a graph that *can* be embedded in the plane (without crossing edges).

non–planar



K_5

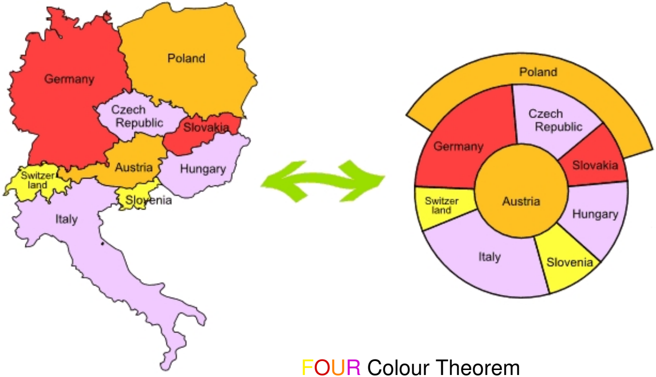


$K_{3,3}$

Planar Graphs

Planar graphs

A planar graph is a graph that *can* be embedded in the plane (without crossing edges).

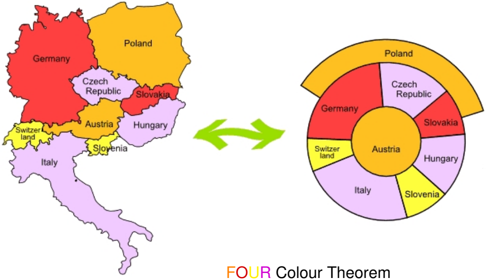


FOUR Colour Theorem

Random Planar Graphs

Planar graphs

A planar graph is a graph that *can* be embedded in the plane (without crossing edges).



FOUR Colour Theorem

Random planar graphs

Let $P(n, m)$ be a uniform random planar graph with n vertices and m edges.

Phase Transition in Random Planar Graphs

Let $L(n)$ denote the number of vertices in the largest component in $P(n, m)$.

Two critical periods [K.– ŁUCZAK 12]

- Let $m = n/2 + s$.

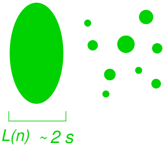
  If $s\, n^{-2/3} \to -\infty$, whp $L(n) \ll n^{2/3}$.

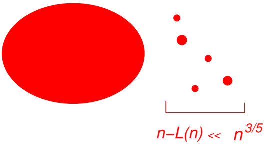
  If $s\, n^{-2/3} \to \infty$, whp $L(n) = (2 + o(1))s \gg n^{2/3}$.

- Let $m = n + r$.

  If $r\, n^{-3/5} \to -\infty$, whp $n - L(n) \gg n^{3/5}$.

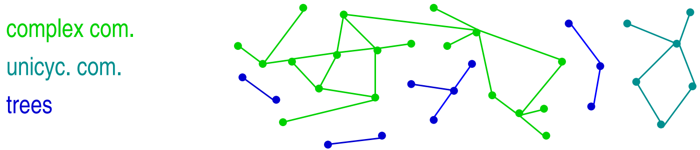
  If $r\, n^{-3/5} \to \infty$, whp $n - L(n) = \Theta(n^{3/2} r^{-3/2}) \ll n^{3/5}$.





Random Planar Graphs

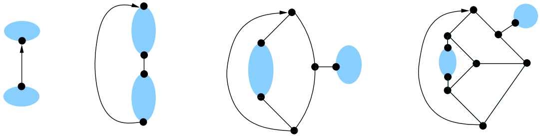
Look into internal structure $\Rightarrow$ Kernel of complex components



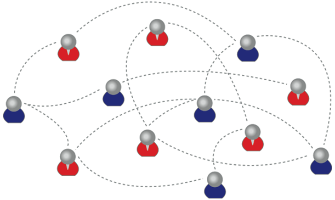
Typical kernel

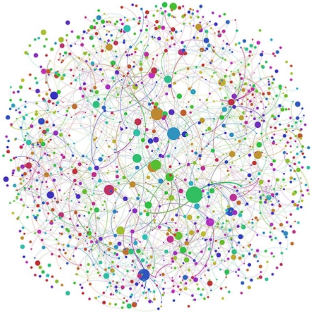
[K.– ŁUCZAK 12]

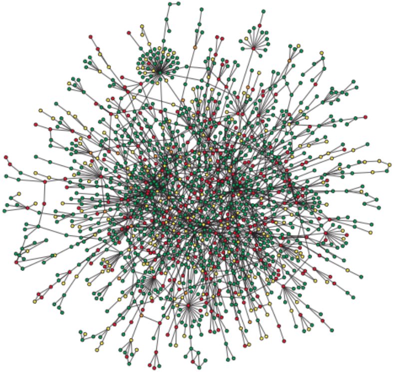
▷ Cubic planar weighted multigraphs
through singularity analysis of generating functions

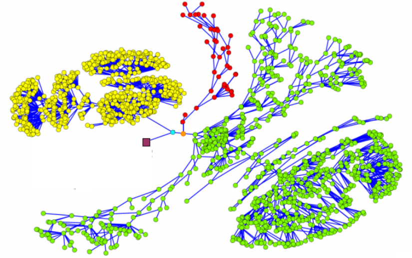


Complex Networks









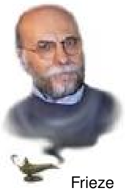
Random Graph Processes

A random graph process is a Markov process defined on the set of graphs of interest; in each step one or several edges are added according to some rule.

▷ Achlioptas process, Bohman-Frieze process: power of two choices

Achlioptas

Bohman



Frieze

In each step, two random edges are present

- if the first edge would join two isolated vertices, it is added to a graph
- otherwise the second edge is added

▷ it delays the appearance of the giant component [BOHMAN-FRIEZE 01]

Bohman-Frieze Process

Phase Transition

[Spencer–Wormald 07; Janson–Spencer 10+]

- Susceptibility (= average component size): let $t = 2 \# \text{ edges } /n$.

$$S(t) = \frac{1}{n}\sum_{1\le i\le n}|C(v_i)| = \frac{1}{n}\sum_{1\le k\le n} k\, X_k(t,n).$$

Here $X_k(t,n)$ is the number of vertices in components of size k at time t.

- Differential equations method: $\exists$ a deterministic function $x_k(t)$ s.t. whp

$$\frac{X_k(t,n)}{n} \sim x_k(t)$$

Janson

Spencer

Wormald

Critical Phase in Bohman-Frieze Process

Let t_c be the critical point of the phase transition and $t = t_c + \epsilon$ for ϵ small.

Finer behaviour

[K.–PERKINS–SPENCER 12]

- The size of the second largest component at time t is whp $\Theta(\epsilon^{-2} \log n)$.
- Vertices in small components: $\exists$ constants $a, b > 0$ s.t.

$$x_k(t) \sim a\, k^{-3/2} \exp\left(-\epsilon^2 k\, b\right).$$

Quasi-linear partial differential equation

[K.–PERKINS–SPENCER 12]

- Susceptibility (= average component size): $S(t) \sim \sum_{k \geq 1} k\, x_k(t)$
- The moment generating function $G(t,z) = \sum_{k \geq 1} x_k(t)\, z^k$ satisfies

$$\frac{\partial G(t,z)}{\partial t} - z(G(t,z) - 1)\frac{\partial G(t,z)}{\partial z} = 0, \quad G(0,z) = z.$$

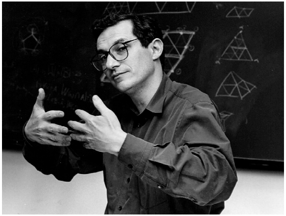
Concluding Remarks

Ubiquitous phase transitions in
thermodynamics, statistical physics, percolation, random graphs, . . .

Cut-off phenomenon of random walks

- How many riffle shuffles are necessary and sufficient to approximately randomise n cards? [DIACONIS 92]

Riffle shuffle



Persi Diaconis

Concluding Remarks

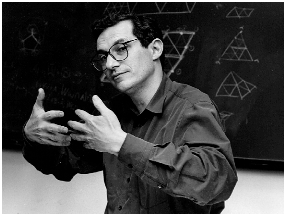
Ubiquitous phase transitions in

thermodynamics, statistical physics, percolation, random graphs, . . .

Cut-off phenomenon of random walks

- How many riffle shuffles are necessary and sufficient to approximately randomise n cards? $\frac{3}{2} \log_2 n$ [DIACONIS 92]

Riffle shuffle



Persi Diaconis