Vanishing of cohomology groups of random simplicial complexes

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Joint work with Oliver Cooley, Nicola Del Giudice, and Philipp Sprüssel

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I. Motivation: binomial random graphs

II. High dimensional analogues

III. Main result

IV. Proof ideas

Connectedness of Binomial Random Graph

Binomial random graph $G(n, p)$

- vertex set $[\,n\,] := \{1, \dots, n\}$
- each pair of vertices is present as an edge with probability p independently

Theorem [Erdős–Rényi 59]

Let

$$p = \frac{\log n + c(n)}{n}.$$

Then

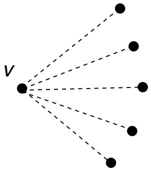
$$\mathbb{P}\Big(G(n,p) \text{ is connected} \Big) \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } c(n) \to -\infty \\ e^{-e^{-c}} & \text{if } c(n) \to c \in \mathbb{R} \\ 1 & \text{if } c(n) \to \infty \end{cases}$$

Heuristic for Threshold

Property of having no isolated vertices undergoes a phase transition at

$$p = \frac{\log n}{n}$$

$$\mathbb{E}\left(\#\, \textit{isolated vertices in } G(n,p) \right) \quad = \; n\,(1-p)^{n-1}$$



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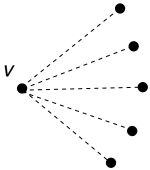

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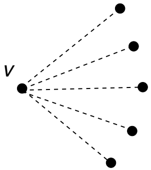
$$\sim \Theta(1)$$

if $\log n - pn = 0$



Part II

High dimensional analogues

Hypergraphs vs. Simplicial Complexes

(1) $(k+1)$**-uniform hypergraph** $H = ([n], E)$

- vertex set $[n] := \{1, \ldots, n\}$
- edge set E = set of $(k+1)$-element subsets of $[n]$, i.e. $E \subset \binom{[n]}{k+1}$

Hypergraphs vs. Simplicial Complexes

(1) Random $(k+1)$**-uniform hypergraph** $H_p = ([\,n\,], E_p)$

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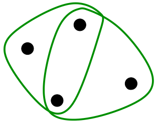
$$\mathbb{P}\Big(\text{each } (k+1)\text{-element subset of } [\,n\,] \text{ is present in } E_p\Big) = p$$



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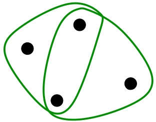

(2) k-dimensional simplicial complex

- A family X of subsets of $[\,n\,]$ is called a simplicial complex if
  - $\{v\} \in X, \quad \forall v \in [\,n\,]$
  - X is downward-closed, i.e. if $A \in X, \emptyset \neq B \subset A$, then $B \in X$
- A simplicial complex X is k-dimensional if $|A| \leq k+1, \;\; \forall A \in X$, and $A \in X$ is called k-simplex if $|A| = k+1$

Hypergraphs vs. Simplicial Complexes

(3) Random k-dimensional simplicial complexes

arising from random $(k+1)$-uniform hypergraph $H_p = ([\,n\,], E_p)$

- the 0-simplices are the singletons of $[\,n\,]$
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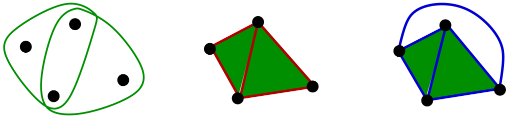
(i) $\forall j \in [k-1]$, the j-simplices are the $(j+1)$-subsets of hyperedges in E_p

$$\mathcal{G}_p = \binom{[n]}{1} \cup \ldots \cup \partial(\partial E_p) \cup \partial E_p \cup E_p$$

(ii) the full $(k-1)$-skeleton on $[n]$ is included

$$\Delta_p = \binom{[n]}{1} \cup \binom{[n]}{2} \cup \ldots \cup \binom{[n]}{k} \cup E_p$$



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[Cooley–del Giudice–K.–Sprüssel 18]

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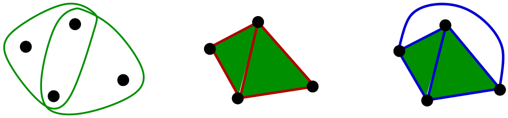
[Linial–Meshulam 06; Meshulam–Wallach 09: Kahle–Pittel 16]



Cohomology Groups

Let X be a k-dimensional simplicial complex. For each $j \in [\,k-1\,]$

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$$\delta^j f(\sigma) \quad := \quad \sum_{\tau \subset \sigma,\ |\tau| = j+1} f(\tau) \pmod 2$$

$$C^0(X) \xrightarrow{\delta^0} \ldots \to \quad C^{j-1}(X) \xrightarrow{\delta^{j-1}} C^j(X) \xrightarrow{\delta^j} C^{j+1}(X) \to \ldots \to C^k(X)$$

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- j-th cohomology group of X with coefficients in $\mathbb{F}_2$ is the quotient group

$$H^j(X;\ \mathbb{F}_2) \quad := \quad \frac{\mathrm{Ker}\ (\delta^j)}{\mathrm{Im}\ (\delta^{j-1})}$$

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We say X is $\mathbb{F}_2$-cohomologically j-connected if $H^i(X;\ \mathbb{F}_2) = 0, \quad \forall\, i \in [\,j\,]$

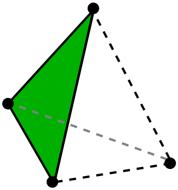
Part III

Main result

Random k-dimensional simplical complex

$$\mathcal{G}_p \;=\; \binom{[\,n\,]}{1} \;\cup\; \ldots \cup\; \partial(\partial E_p) \;\cup\; \partial E_p \;\cup\; E_p$$

- the 0-simplices are the singletons of $[\,n\,] := \{1, \ldots, n\}$
- each $(k+1)$-element subset of $[\,n\,]$ is present as a k-simplex with probability p independently
- $\forall\, j \in [\,k-1\,]$
  every $(j+1)$-element subset of k-simplices forms a j-simplex



$\mathbb{F}_2$-Cohomologically j-Connectedness

Let $k \geq 2$ and $j \in [\, k-1\,]$.

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Theorem [COOLEY–DEL GIUDICE–K.–SPRÜSSEL 18]

Let

$$p \;=\; \frac{(j+1)\log n + \log\log n + c(n)}{(k-j+1)\binom{n}{k-j}}.$$

Then

$$\mathbb{P}\Big(\, \mathcal{G}_p \text{ is } \mathbb{F}_2\text{-cohomologically } j\text{-connected}\,\Big) \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } \ c(n) \;\to\; -\infty \\ e^{-\lambda_j} & \text{if } \ c(n) \;\to\; c \in \mathbb{R} \\ 1 & \text{if } \ c(n) \;\to\; \infty \end{cases}$$

where $\lambda_j \ := \ \frac{(j+1)e^{-c}}{(k-j+1)^2\, j!}$

Non-Vanishing of Cohomology Group

Recall $C^j(\mathcal{G}_p)$ is the set of $\{0,1\}$-functions on the j-simplices in $\mathcal{G}_p$ and the coboundary operators

$$\ldots \to \quad C^{j-1}(\mathcal{G}_p) \ \xrightarrow{\delta^{j-1}} \ C^j(\mathcal{G}_p) \ \xrightarrow{\delta^j} \ C^{j+1}(\mathcal{G}_p) \ \to \ldots$$

are such that for $f \in C^j(\mathcal{G}_p)$ and $(j+1)$-simplex σ

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Observe

$$H^j(\mathcal{G}_p;\ \mathbb{F}_2) := \frac{\text{Ker }(\delta^j)}{\text{Im }(\delta^{j-1})} \neq 0 \iff \exists\, f \in \text{Ker }(\delta^j) \setminus \text{Im }(\delta^{j-1}) \subset C^j(\mathcal{G}_p)$$

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e.g. function in $C^j(\mathcal{G}_p)$ that assigns

- **even** number of 1's on the j-simplices contained in each $(j+1)$-simplex
- **odd** number of 1's on a set J of j-simplices such that every $(j-1)$-simplex is contained in even number of j-simplices in J

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e.g. a **bad** function in $C^j(\mathcal{G}_p)$ that assigns

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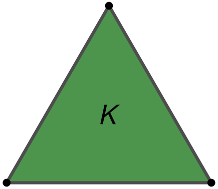
Minimal Obstruction for $k = 2$, $j = 1$

M_1 = triple (K, C, J) where K is a 2-simplex in $\mathcal{G}_p$ and



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M_1 = triple (K, C, J) where K is a 2-simplex in $\mathcal{G}_p$ and

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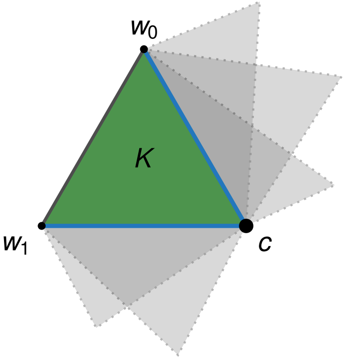

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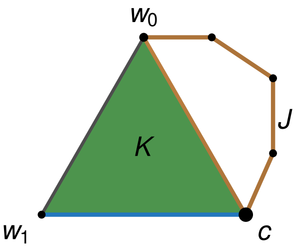

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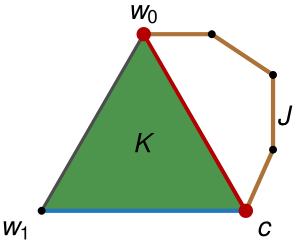
  - it contains exactly one $C \cup \{w_0\}$, $w_0 \in K \setminus C$



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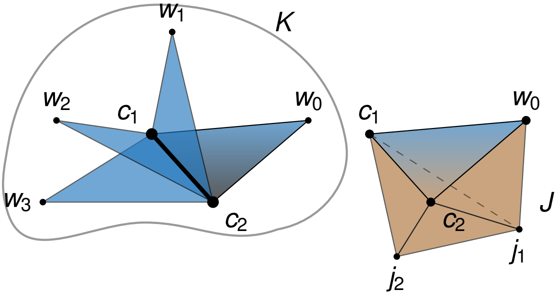

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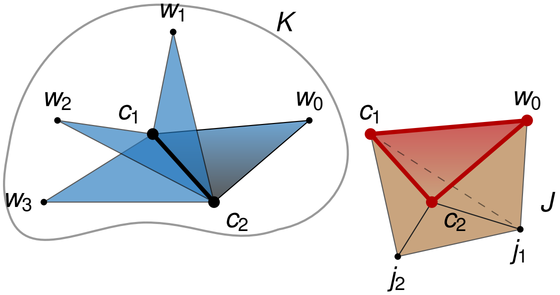

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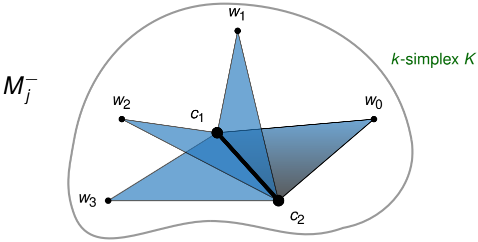
$$\mathbb{E}\left(\# M_j^-\right) = \binom{n}{k+1}\binom{k+1}{j} p\,(1-p)^{(k+1-j)\binom{n}{k-j}}$$

$$= \Theta(1)\exp\left((k+1)\log n + \log p - (k+1-j)\binom{n}{k-j}p\right)$$



Part IV

Proof ideas

Proof Ideas

- hitting time approach, relating vanishing of cohomology groups to disappearance of last minimal obstruction

  cf. disappearance of last isolated vertex [Bollobás–Thomason 85]

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- inside window: method of moments for Poisson distribution

- supercritical: $\#$ of bad functions via traversability

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  via breadth-first search: **A**lgorithm **fo**r **A**nalysis ☺