Recent developments in phase transitions and critical phenomena

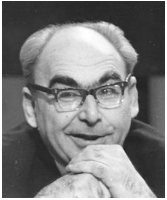
54 years since the seminal work of Erdős and Rényi

Mihyun Kang

TU Graz

The Beginning: “Asymptotic Statistical Properties”

Erdős (1913 – 1996)



Rényi (1921 – 1970)

On random graphs I.

Dedicated to O. Varga, at the occasion of his 50^{th} birthday.

By P. ERDŐS and A. RÉNYI (Budapest).

Let us consider a “random graph” $\Gamma_{n,N}$ having n possible (labelled) vertices and N edges; in other words, let us choose at random (with equal probabilities) one of the $\binom{\binom{n}{2}}{N}$ possible graphs which can be formed from the n (labelled) vertices $P_1, P_2, \dots, P_n$ by selecting N edges from the $\binom{n}{2}$ possible edges $\widehat{P_iP_j}$ $(1 \leqq i < j \leqq n)$. Thus the effective number of vertices of $\Gamma_{n,N}$ may be less than n, as some points P_i may be not connected in $\Gamma_{n,N}$ with any other point P_j; we shall call such points P_i *isolated* points. We consider the isolated points also as belonging to $\Gamma_{n,N}$. $\Gamma_{n,N}$ is called completely connected if it effectively contains all points $P_1, P_2, \dots, P_n$ (i. e. if it has no isolated points) and is connected in the ordinary sense. In the present paper we consider asymptotic statistical properties of random graphs for $n \to +\infty$. We shall deal with the following questions:

1. What is the probability of $\Gamma_{n,N}$ being completely connected?
2. What is the probability that the greatest connected component (sub-graph) of $\Gamma_{n,N}$ should have effectively $n-k$ points? $(k=0, 1, \dots)$.
3. What is the probability that $\Gamma_{n,N}$ should consist of exactly $k+1$ connected components? $(k=0, 1, \dots)$.
4. *If the edges of a graph with n vertices are chosen successively so that after each step every edge which has not yet been chosen has the same probability to be chosen as the next, and if we continue this process until the graph becomes completely connected, what is the probability that the number of necessary steps ν will be equal to a given number l?*

"Growth of Greatest Component"

ON THE EVOLUTION OF RANDOM GRAPHS

by

P. ERDŐS and A. RÉNYI

Dedicated to Professor P. Turán at his 50th birthday.

§ 9. **On the growth of the greatest component**

We prove in this § (see Theorem 9b) that the size of the greatest component of $\Gamma_{n,N(n)}$ is for $N(n) \sim cn$ with $c > 1/2$ with probability tending to 1 approximately $G(c)n$ where

$$G(c) = 1 - \frac{x(c)}{2c} \tag{9.1}$$

and $x(c)$ is defined by (6.4). (The curve $y = G(c)$ is shown on Fig. 2b).

Thus by Theorem 6 for $N(n) \sim cn$ with $c > 1/2$ almost all points of $\Gamma_{n,N(n)}$ (i. e. all but $o(n)$ points) belong either to some small component which is a tree (of size at most $1/\alpha\,(\log n - \frac{5}{2}\log\log n) + O(1)$ where $\alpha = 2c - 1 - \log 2c$ by Theorem 7a) or to the single "giant" component of the size $\sim G(c)n$.

Thus the situation can be summarized as follows: the largest component of $\Gamma_{n,N(n)}$ is of order $\log n$ for $\frac{N(n)}{n} \sim c < 1/2$, of order $n^{2/3}$ for $\frac{N(n)}{n} \sim \frac{1}{2}$ and of order n for $\frac{N(n)}{n} \sim c > 1/2$. This double "jump" of the size of the largest component when $\frac{N(n)}{n}$ passes the value $1/2$ is one of the most striking facts concerning random graphs. We prove first the following

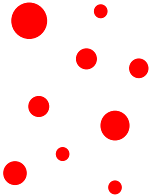
The Phase Transition

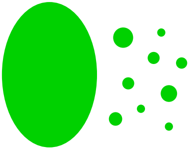
Erdős-Rényi random graph $G(n, m)$ [ERDŐS–RÉNYI 60]

$L(m) = \#$ vertices in the largest component after m edges are added

$m = c \cdot n/2, \quad c > 0$

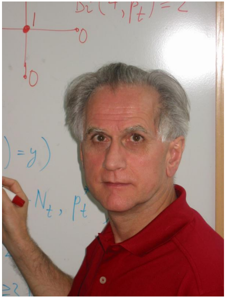
- If $c < 1$, whp $\quad L(m) = O(\log n)$
- If $c > 1$, whp $\quad L(m) = \Theta(n)$





Critical Phenomenon

How big is the largest component, when $m = n/2 + s,\ s = o(n)$?



Béla Bollobás

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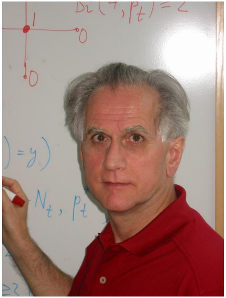

Béla Bollobás

Tomasz Łuczak

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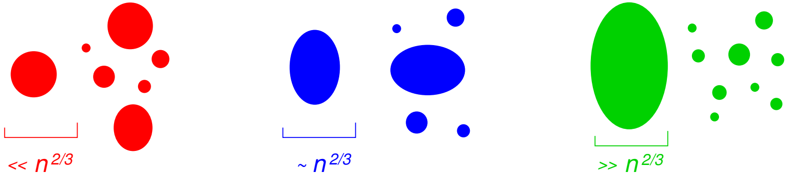
[Bollobás 84; Łuczak 90; Janson–Knuth–Łuczak–Pittel 93; Bollobás–Riordan 13+]

- If $s\, n^{-2/3} \to -\infty$, whp $\qquad L(m) = o(n^{2/3})$
- If $s\, n^{-2/3} \to \lambda$, a constant, whp $\qquad L(m) = \Theta(n^{2/3})$
- If $s\, n^{-2/3} \to \infty$, whp $\qquad L(m) = (4 + o(1))\, s$



Random Planar Graph

Let $L(m)$ denote the number of vertices in the largest component in $P(n,m)$.

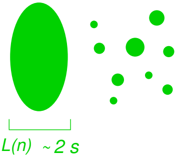
Two critical periods [K.– ŁUCZAK 12]

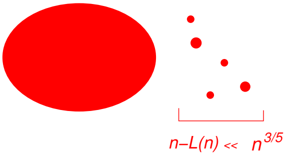
- Let $m = n/2 + s$.

  If $n^{2/3} \ll s \ll n$, whp $\qquad L(m) = (2+o(1))\, s$

- Let $m = n + r$.

  If $n^{3/5} \ll r \ll n^{2/3}$, whp $\qquad n - L(m) = \Theta(n^{3/2} r^{-3/2}) \ll n^{3/5}$





Achlioptas Processes

Power of two choices

[ACHLIOPTAS 00]

- In each step, two potential edges are present:

  one of them is chosen according to a given rule and added to a graph.

Achlioptas

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[ACHLIOPTAS 00]

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  one of them is chosen according to a given rule and added to a graph.

Bohman-Frieze process delays the giant

[BOHMAN-FRIEZE 01]

- If the first edge joins two isolated vertices, it is added to a graph;
  otherwise the second edge is added.

Achlioptas

Bohman

Frieze

Bohman-Frieze Process

Phase transition

[SPENCER–WORMALD 07; JANSON–SPENCER 12]

- Susceptibility: let $t = \#$ edges $/n$

$$S(t) \;=\; \frac{1}{n}\sum\nolimits_{v\in[n]} |C(v)| = \frac{1}{n}\sum\nolimits_{i\geq 1} i\, X_i(t,n)$$

$X_i(t,n) \;=\; \#$ vertices in components of size i at time t


Janson


Spencer


Wormald

Bohman-Frieze Process

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$X_i(t,n) = \#$ vertices in components of size i at time t

- Differential equations method: $\exists$ a deterministic function $x_i(t)$ s.t. whp

$$\frac{X_i(t,n)}{n} = x_i(t) + o(1)$$

Variant of Smoluchowski's coagulation equation:

$$x_i'(t) = -2(1 - x_1^2(t))\, i\, x_i(t) + (1 - x_1^2(t))\, i \sum_{1\leq j<i} x_j(t)x_{i-j}(t)$$

Bohman-Frieze Process

Phase transition

[Spencer–Wormald 07; Janson–Spencer 12]

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Small components

[K.–Perkins–Spencer 13]

$$x_i(t_c \pm \epsilon) \;\sim\; a\, i^{-3/2} \exp\left(-\,\epsilon^2 i\, b\right)$$

Achlioptas Processes

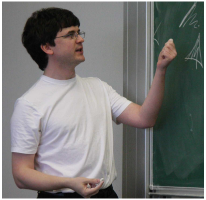
Merging ℓ-vertex rule that is well-behaved [Riordan–Warnke 13]

$L(t) = \#$ vertices in the largest component after $t\,n$ steps

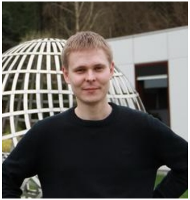
Provided that a rule-dependent system of ODEs has a unique solution, whp

$$n^{-1} L(t) = 1 - \sum_{i \geq 1} x_i(t) + o(1)$$

$$n^{-1} X_i(t,n) = x_i(t) + o(1)$$



Riordan



Warnke

Achlioptas Processes

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$$n^{-1}\, X_i(t,n) = x_i(t) + o(1)$$

ℓ-vertex bounded-size rule [Drmota–K.–Panagiotou 13+]

$$n^{-1} L(t_R + \epsilon) = c_R\, \epsilon + O(\epsilon^2)$$

$$\limsup_{i \to \infty} i^{-1} \log x_i(t_R + \epsilon) = -d_R\, \epsilon^2 + O(\epsilon^3)$$

Proof: Analytic Framework

Variant of Smoluchowski's coagulation equation:

$$x_i' \;=\; f_i(x_1, \ldots, x_K) \;+\; g(x_1, \ldots, x_K)\,\big(-2ix_i + i\sum\nolimits_{j+j'=i} x_j x_{j'}\big)$$

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Moment generating function $D(t,z) = \sum_{i \geq 1} x_i(t)\, z^i$ satisfies

$$D_t + 2\,z\,g(t)\,(1-D)\,D_z = h(t,z), \quad D(0,z) = z$$

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E.g. Erdős-Rényi process:

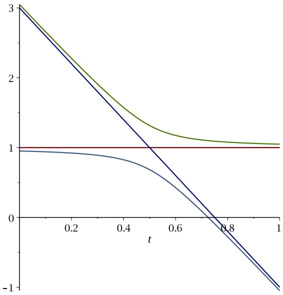
$g(t) = 1, \quad h(t,z) = 0$

Critical point: $t = 1/2,\; z = 1$

Solutions to $D \;=\; z\,e^{2t(D-1)}$:

double point if $z = 1$

hyperbola-like if $z \neq 1$



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Method of characteristics for general case