Random subgraphs of the hypercube

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Joint work with

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Confererence celebrating the 100th anniversary of Rényi's birth

Talk outline

I. Erdős–Rényi random graph and random subgraphs

II. Expansion properties and consequences

III. Proof ideas for an expansion property

IV. Mixing time of lazy random walk

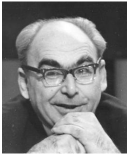
Part I.

Erdős-Rényi random graph and random subgraphs

Erdős-Rényi random graph

$G(n,m) \quad = \quad$ a graph chosen uniformly at random from the set of all graphs on vertex set $[n] := \{1,\ldots,n\}$ with $m = m(n)$ edges

Paul Erdős (1913 – 1996)



Alfréd Rényi (1921 – 1970)

Random subgraphs

Given $p \in (0, 1)$

$$
\begin{aligned}
G(n, p) &= \text{a binomial random graph} \\
&= \text{a graph obtained by retaining each edge of complete graph } K_n \\
&\quad\ \text{independently with probability } p \\
&= \text{bond percolation on complete graph } K_n \text{ with edge probability } p
\end{aligned}
$$

Random subgraphs

Given $p \in (0, 1)$

$G(n, p)$ = a binomial random graph

= a graph obtained by retaining each edge of complete graph K_n independently with probability p

= bond percolation on complete graph K_n with edge probability p

G_p = a graph obtained by retaining each edge of a given base graph G independently with probability p

= bond percolation on G with edge probability p

The hypercube

Given $d \in \mathbb{N}$, the d-dimensional hypercube Q^d is the graph with

- vertex set

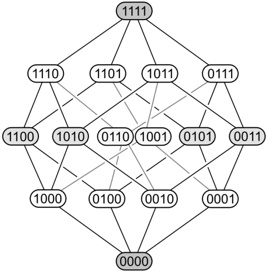
$$V\left(Q^d\right) \;=\; \{0,1\}^d \;=\; \left\{x = (x_1, \ldots, x_d) \;:\; x_i \in \{0,1\},\; 1 \le i \le d\right\}$$

- edge set $E\left(Q^d\right)$: $\quad \forall v = (v_1, \ldots, v_d), w = (w_1, \ldots, w_d) \in V\left(Q^d\right)$,

$$\{v, w\} \in E\left(Q^d\right) \quad \text{iff} \quad v \text{ and } w \text{ differ in exactly one coordinate}$$

Obvious facts:

- $|V\left(Q^d\right)| \;=\; 2^d$
- Q^d is d-regular
- Q^d is bipartite
- diameter of Q^d is d
- $\ldots$



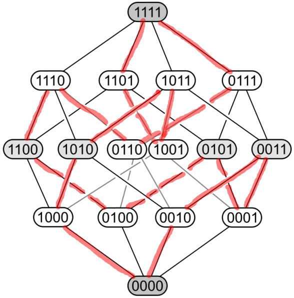
Hasse diagram

A random subgraph of the hypercube

Given $p \in (0,1)$

Q_p^d = a graph obtained by retaining each edge of Q^d independently with probability p

= bond percolation on Q^d with edge probability p



Typical properties of Q_p^d around $p = \frac{1}{2}$

Connectivity [Sapoženko 67; Burtin 67; Erdős–Spencer 79; Bollobás 83]

$p = \frac{1}{2}$ is a sharp threshold for connectedness: $\forall \varepsilon > 0$

$$\mathbb{P}\left[Q_p^d \text{ is connected} \right] \xrightarrow{d \to \infty} \begin{cases} 0 & \text{if} \quad p < \frac{1-\varepsilon}{2} \\ 1 & \text{if} \quad p > \frac{1+\varepsilon}{2} \end{cases}$$

Typical properties of Q_{p}^{d} around $p = \frac{1}{2}$

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Perfect matching [BOLLOBÁS 90]

$p = \frac{1}{2}$ is a sharp threshold for the existence of a perfect matching

Is $p = \frac{1}{2}$ a sharp threshold for Hamiltonicity? [BOLLOBÁS 80'S; FRIEZE 14]

Hamiltonicity [CONDON-ESPUNY-DÍAZ-GIRÃO-KÜHN-OSTHUS 21]

$p = \frac{1}{2}$ is a sharp threshold the existence of a Hamiltonian cycle.

Emergence of the giant component in Q_p^d

Does the component structure of Q_p^d undergo a phase transition at $p = \frac{1}{d}$?

[ERDŐS–SPENCER 79]

Giant component [AJTAI-KOMLÓS- SZEMERÉDI 81]

$p = \frac{1}{d}$ is a sharp threshold: $\forall \varepsilon > 0$

- whp all components are of order $O(d)$ if $p < \frac{1-\varepsilon}{d}$
- whp $\exists$ a unique largest component of order $\Theta(2^d)$ if $p > \frac{1+\varepsilon}{d}$

whp $=$ with high probability $=$ with prob tending to one as $d \to \infty$

Supercritical regime – open questions

$p = \frac{1+\varepsilon}{d}$ for fixed $\varepsilon > 0$

$L_1 =$ the largest component of Q_p^d

- diameter of L_1 ? [Bollobás-Kohayakawa-Łuczak 92]
- circumference of L_1 (= length of the longest cycles) ?
- Hadwiger number of L_1 (= order of the largest complete minor) ?
- mixing time of lazy simple random walk on L_1 ?

[Pete 08; van der Hofstad-Nachmias 17]

Part II.

Expansion properties and consequences

Expanders

[ALON 86; HOORY-LINIAL-WIGDERSON 06; KRIVELEVICH 19; KRIVELEVICH-SUDAKOV 09; SARNAK 04; . . .]

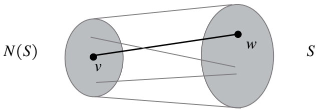
Given a graph G

- $N(S) =$ external neighbourhood of a subset $S \subseteq V(G)$

  $= \{v \in V(G) \setminus S : \exists\, w \in S \text{ with } \{v,w\} \in E(G)\}$

- G is an α-expander if

$$|N(S)| \geq \alpha |S|, \qquad \forall S \subseteq V(G) \quad \text{with} \quad |S| \leq \frac{|V(G)|}{2}$$



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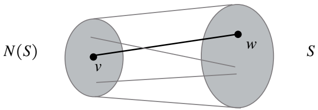


Properties of an expander

- small diameter, long cycles, large complete minor, . . .
- edge-expansion for graphs with bounded max degree

Expansion properties and consequences

L_1 = largest component of Q_p^d when $p = \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

Theorem [Erde-K.-Krivelevich 21+]

whp L_1

- is $c\,d^{-5}-$expander

Expansion properties and consequences

$L_1 \;=\;$ largest component of Q_p^d when $p \;=\; \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

Theorem [ERDE-K.-KRIVELEVICH 21+]

whp L_1

- is $c\,d^{-5}-$expander
- contains a $c'\,d^{-2}\,(\log d)^{-1}-$expander on $\geq 0.99\,|L_1|$ vertices

Expansion properties and consequences

$L_1 \;=\;$ largest component of Q_p^d when $p \;=\; \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

Theorem [ERDE-K.-KRIVELEVICH 21+]

whp L_1

- is $c\, d^{-5}$–expander
- contains a $c'\, d^{-2}\, (\log d)^{-1}$–expander on $\geq 0.99\, |L_1|$ vertices
- has diameter $O\left(d^3\right)$
- contains a cycle of length $\Omega\left(2^d\, d^{-2}\, (\log d)^{-1}\right)$
- contains a complete minor of order $\Omega\left(2^{\frac{d}{2}}\, d^{-2}\, (\log d)^{-1}\right)$
- has Cheeger constant $\Omega\left(d^{-5}\right)$

Part III.

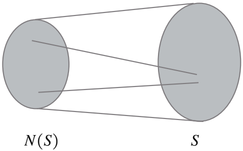
Proof ideas

Theorem

whp L_1 is a $\frac{1}{\text{poly}(d)}$$-$expander

i.e., $\forall S \subseteq V(L_1)$ with $|S| \leq \frac{|V(L_1)|}{2}$,

$$|N(S)| \geq \frac{|S|}{\text{poly}(d)}$$



Sprinkling argument

Sprinkling

$p = \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

$q_1 = \frac{1+\delta_1}{d}$ and $q_2 = \frac{\delta_2}{d}$ s.t. $1 - p = (1-q_1)(1-q_2)$ and $0 < \delta_2 \ll \delta_1$

$Q_p^d \sim Q_{q_1}^d \cup Q_{q_2}^d$

Sprinkling argument

Sprinkling

$$p \;=\; \tfrac{1+\varepsilon}{d} \text{ for } \;\varepsilon > 0$$

$$q_1 = \tfrac{1+\delta_1}{d} \text{ and } q_2 = \tfrac{\delta_2}{d} \;\text{ s.t. }\; 1-p = (1-q_1)(1-q_2) \text{ and } 0 < \delta_2 \ll \delta_1$$

$$Q^d_p \;\sim\; Q^d_{q_1} \,\cup\, Q^d_{q_2}$$

Largest components before and after sprinkling

$$L_1' \;=\; \text{largest component in } Q^d_{q_1} \quad \text{(before sprinkling)}$$

$$L_1 \;=\; \text{largest component in } Q^d_p \quad \text{(after sprinkling)}$$

$$\gamma(x) \;=\; \text{survival probability of } \mathrm{Po}(1+x) \text{ branching process}$$

Lemma [AJTAI-KOMLÓS- SZEMERÉDI 81]

- whp $L_1' \;\sim\; \gamma(\delta_1)\, 2^d$
- whp $L_1 \;\sim\; \gamma(\epsilon)\, 2^d$

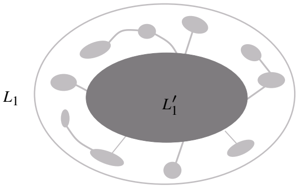
Giant component before and after sprinkling

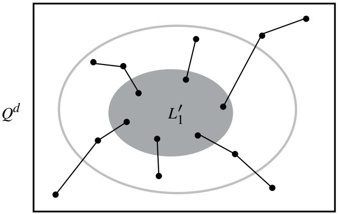
$L_1' \;=\;$ largest component in $Q^d_{q_1}$ (before sprinkling)

$L_1 \;=\;$ largest component in Q^d_p (after sprinkling)

Lemma

- whp $\forall$ connected component in $Q^d_p\,[L_1 - L_1']$ is of order $O(d)$
- whp $\forall$ vertex in $V(Q^d)$ is within distance two from $\geq cd^2$ vertices in L_1'

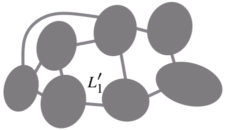




Splitting the largest component into pieces

L_1' = largest component (before sprinkling)

= split into a family $\mathcal{C}$ of vertex-disjoint connected subgraphs ('pieces'), each of order poly(d)



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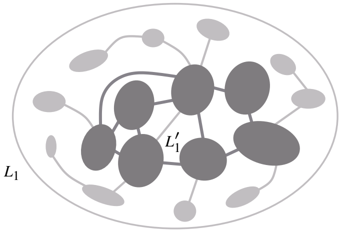


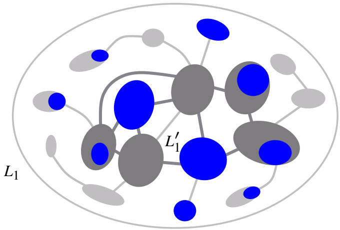
Splitting the largest component into pieces

L_1' = largest component (before sprinkling)

= split into a family $\mathcal{C}$ of vertex-disjoint connected subgraphs ('pieces'), each of order poly(d)

L_1 = largest component (after sprinkling)

S = arbitrary subset of $V(L_1)$ with $|S| \leq \frac{|V(L_1)|}{2}$



Splitting the largest component into pieces

$$
\begin{aligned}
L_1' &= \text{largest component} \quad \text{(before sprinkling)} \\
&= \text{split into a family } \mathcal{C} \text{ of vertex-disjoint connected subgraphs} \\
&\quad\ \text{('pieces'), each of order poly}(d) \\
L_1 &= \text{largest component} \quad \text{(after sprinkling)} \\
S &= \text{arbitrary subset of } V(L_1) \text{ with } |S| \le \tfrac{|V(L_1)|}{2}
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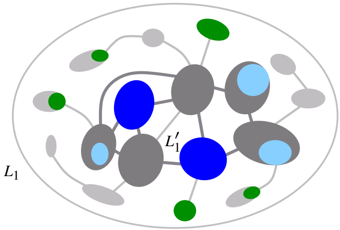
$$



$$
\begin{aligned}
S_1 &= S - L_1' \\
S_2 &= \text{vertices in pieces } C \in \mathcal{C} \text{ with } C \cap S \ne \emptyset \text{ and } S - C \ne \emptyset \\
S_3 &= \text{vertices in pieces } C \in \mathcal{C} \text{ with } C \subseteq S
\end{aligned}
$$

Contribution of S_1 to $N(S)$

$$S_1 = S - L_1'$$

$$S_2 = \text{vertices in pieces } C \in \mathcal{C} \text{ with } C \cap S \neq \emptyset \text{ and } S - C \neq \emptyset$$

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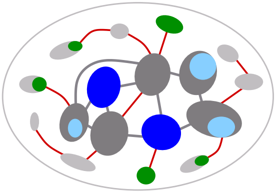


With sprinkling, each component in $Q_p^d[L_1 - L_1']$ which intersects with S_1

- contributes at least one edge to $N(S)$
- or is connected to $S_2 \cup S_3$

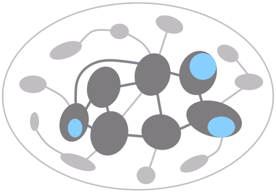
Thus $|N(S)| \geq \frac{c|S_1|}{d}$ or $e(S_1, S_2 \cup S_3) \geq \frac{c|S_1|}{d}$ and thus $|S_2 \cup S_3| \geq \frac{c|S_1|}{d^2}$

Contribution of S_2 to $N(S)$

$L_1' \;=\;$ split into a family $\mathcal{C}$ of pieces, each of order poly(d)

$S_2 \;=\;$ vertices in pieces $C \in \mathcal{C}$ with $C \cap S \neq \emptyset$ and $S - C \neq \emptyset$

$S_3 \;=\;$ vertices in pieces $C \in \mathcal{C}$ with $C \subseteq S$



Each piece $C \in \mathcal{C}$ with $C \cap S \neq \emptyset$ and $S - C \neq \emptyset$

- contributes at least one edge to $N(S)$

and each piece is of order poly(d)

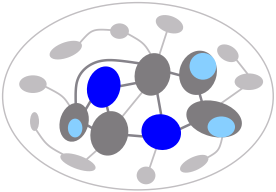
Thus $\quad |N(S)| \geq \frac{|S_2|}{\text{poly}(d)}$

Contribution of S_3 to $N(S)$

$L_1' \;=\;$ split into a family $\mathcal{C}$ of pieces, each of order $\mathrm{poly}(d)$

$S_2 \;=\;$ vertices in pieces $C \in \mathcal{C}$ with $C \cap S \neq \emptyset$ and $S - C \neq \emptyset$

$S_3 \;=\;$ vertices in pieces $C \in \mathcal{C}$ with $C \subseteq S$



(1) Partition the family $\mathcal{C}$ of pieces into two disjoint families $\{\mathcal{A}, \mathcal{B}\}$

$$\mathcal{A} := \{C \in \mathcal{C} \colon C \subseteq S\} \quad \text{and} \quad \mathcal{B} := \mathcal{C} - \mathcal{A}$$

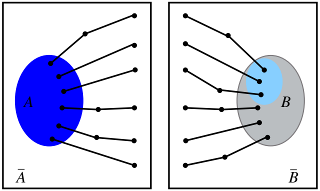
This partitions $V(L_1')$ into two sets A, B where

$$A := V(\mathcal{A}) = S_3 \quad \text{and} \quad B := V(\mathcal{C} - \mathcal{A})$$

Contribution of S_3 to $N(S)$ – extending and connecting

(2) Extending the partition $V(L_1') = A \,\dot\cup\, B$ to a partition $V(Q^d) = \bar{A} \,\dot\cup\, \bar{B}$ s.t.

- every vertex in $\bar{A}$ is within distance 2 of A
- every vertex in $\bar{B}$ is within distance 2 of B



whp every vertex in $V(Q^d)$ is within distance two from vertices in L_1'

Contribution of S_3 to $N(S)$ – extending and connecting

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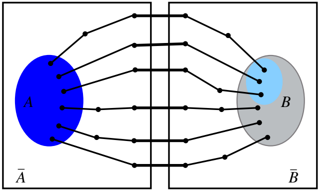

Edge-isoperimetry in Q^d [Harper 64; Lindsey 64; Bernstein 67; Hart 76]

$$|E(X, X^c)| \geq |X| \, (d - \log_2 |X|), \qquad \forall X \subseteq V(Q^d) \quad \text{with} \quad |X| \leq 2^{d-1}$$

Contribution of S_3 to $N(S)$ – extending and connecting

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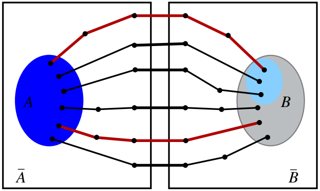


(3) Sprinkle with $q_2 = \frac{\delta_2}{d}$

Lemma

whp $\exists$ at least $\frac{|A|}{\text{poly}(d)}$ vertex-disjoint A-B-paths of length at most 5 in $Q^d_{q_2}$

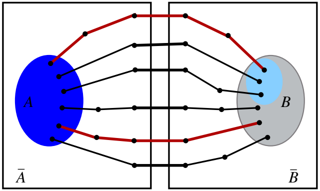
Contribution of S_3 to $N(S)$

L_1' = split into a family $\mathcal{C}$ of 'pieces', each of order poly(d)

S_2 = vertices in pieces $C \in \mathcal{C}$ with $C \cap S \neq \emptyset$ and $S - C \neq \emptyset$

S_3 = vertices in pieces $C \in \mathcal{C}$ with $C \subseteq S$

$\quad = A$



Each A-B-path in $Q_{q_2}^d$ contributes at least one edge to $N(S)$, unless it goes to S_2

Thus $\quad |N(S)| \geq \frac{|S_3|}{\text{poly}(d)} - d|S_2|$

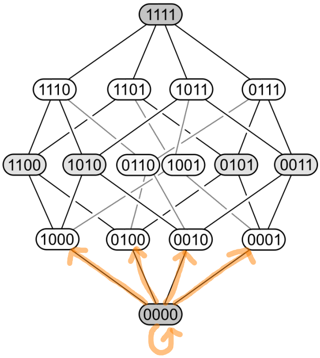
Part IV.

Mixing time of lazy random walk

Mixing time of lazy random walk on Q^d

In each step,

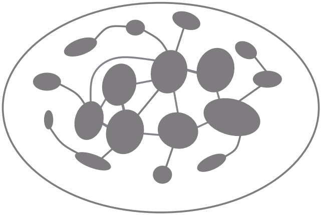
- it remains at the current position with prob $\frac{1}{2}$
- it moves to a uniformly chosen random neighbour with prob $\frac{1}{2}$



Mixing time: $O(d \log d)$

Mixing time of lazy random walk on giant comp of Q_p^d

$L_1 \; = \;$ giant component of Q_p^d when $p \; = \; \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

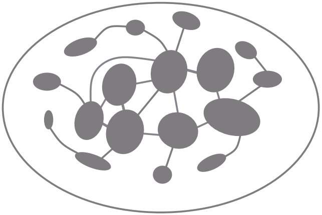


Mixing time of lazy random walk on giant comp of Q_p^d

$L_1 \;=\;$ giant component of Q_p^d when $p \;=\; \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

What is the mixing time of the lazy random walk on L_1 ?

[PETE 08; VAN DER HOFSTAD-NACHMIAS 17]



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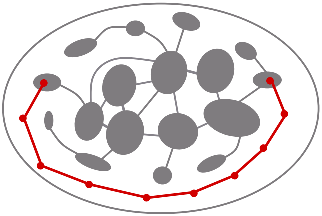

whp L_1 contains bare paths of length $\Omega(d)$

$\Longrightarrow$ mixing time: $\Omega(d^2)$

Mixing time of lazy random walk

Given a graph G,

$$t_{\text{mix}}(G) \;=\; \text{mixing time of a lazy random walk on a graph } G$$

$$\Phi(G) \;=\; \text{Cheeger constant of } G \;\; (= \text{bottleneck ratio})$$

$$\pi_{\min}(G) \;=\; \min\{ \tfrac{d_G(x)}{2|E(G)|} \;:\; x \in V(G)\}$$

[Lawler–Sokal 88; Jerrum–Sinclair 89; Levin–Peres–Wilmer 07]

$$t_{\text{mix}}(G) \;\leq\; \frac{2}{\Phi(G)^2} \log\left(\frac{4}{\pi_{\min}(G)}\right)$$

Mixing time of lazy random walk on giant comp of Q_p^d

Given a graph G,

$$t_{\text{mix}}(G) = \text{mixing time of a lazy random walk on a graph } G$$

$$\Phi(G) = \text{Cheeger constant of } G \ \ (= \text{bottleneck ratio})$$

$$\pi_{\min}(G) = \min\{\tfrac{d_G(x)}{2|E(G)|} \ : \ x \in V(G)\}$$

[LAWLER–SOKAL 88; JERRUM–SINCLAIR 89; LEVIN–PERES–WILMER 07]

$$t_{\text{mix}}(G) \leq \frac{2}{\Phi(G)^2} \log\left(\frac{4}{\pi_{\min}(G)}\right)$$

L_1 = giant component of Q_p^d when $p = \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

[ERDE-K.-KRIVELEVICH 21+]

whp

$$\Phi(L_1) = \Omega\left(d^{-5}\right) \quad \text{and} \quad \pi_{\min}(L_1) = \Omega\left(2^{-d}\right)$$

$$t_{\text{mix}}(L_1) = O\left(d^{11}\right)$$

Summary

$L_1 \;=\;$ largest component of Q_p^d when $p \;=\; \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

Theorem [Erde-K.-Krivelevich 21+]

whp L_1

- is $c\,d^{-5}$–expander
- contains a $c'\,d^{-2}\,(\log d)^{-1}$–expander on $\geq 0.99\,|L_1|$ vertices
- has diameter $O\left(d^3\right)$
- contains a cycle of length $\Omega\left(2^d\,d^{-2}\,(\log d)^{-1}\right)$
- contains a complete minor of order $\Omega\left(2^{\frac{d}{2}}\,d^{-2}\,(\log d)^{-1}\right)$
- has Cheeger constant $\Omega\left(d^{-5}\right)$

whp the mixing time of the lazy simple random walk on L_1 is $O\left(d^{11}\right)$.

Open problems

$L_1 \;=\;$ largest component of Q_p^d when $p \;=\; \frac{1+\varepsilon}{d}$ for $\varepsilon > 0$

Correct order of

- diameter of L_1: $\Theta(d^2)$?
- circumference of L_1: $\Omega(2^d)$?
- Hadwiger number of L_1: $\Omega(2^{\frac{d}{2}})$?
- mixing time of lazy random walk on L_1?