Benjamini-Schramm local limits of sparse random planar graphs

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Joint work with Michael Missethan

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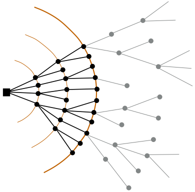
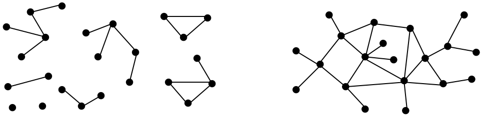
Guiding questions/themes

(1) What does a random graph locally look like?



(2) How does a global structure of a random graph affect its local structure?



(3) What about a local structure of a random graph if a global constraint (e.g., planarity) is imposed on a random graph?

Part I.

Erdős-Rényi random graph

- $G(n,m) \in_R \mathcal{G}(n,m)$

  a graph chosen uniformly at random from the class $\mathcal{G}(n,m)$ of all vertex-labelled simple graphs on vertex set $[n] := \{1,\ldots,n\}$ with $m = m(n)$ edges

- all asymptotics are taken as $n \to \infty$

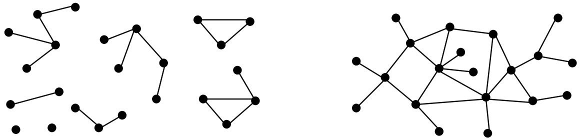
- whp = with high probability
  = with probability tending to one as $n \to \infty$

Phase transition in Erdős-Rényi random graph

$G(n,m) \in_R \mathcal{G}(n,m)$ and $2m/n \xrightarrow{n\to\infty} c \in [0,\infty)$

Theorem [Erdős-Rényi 1959–60]

- If $c < 1$ ('subcritical'), then whp
  - each component in $G(n,m)$ is of order $O(\log n)$
  - $G(n,m)$ consists of tree or unicyclic components
- If $c > 1$ ('supercritical'), then whp
  - $G(n,m)$ contains a unique largest ('giant') component of order $\Theta(n)$
  - giant comp contains $\geq$ two cycles ('complex') and is not planar



Largest component in Erdős-Rényi random graph

$G(n,m) \in_R \mathcal{G}(n,m)$ and $2m/n \xrightarrow{n\to\infty} c \in [0,\infty)$

L largest component in $G(n,m)$

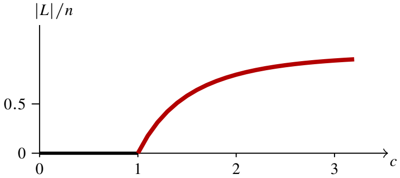
ρ unique positive solution of $1-\rho = \exp(-c\,\rho)$

(= survival prob of $\mathrm{Po}\,(c)$ Galton-Watson branching process*)

Theorem

whp

$$|L| = (1+o(1))\,\rho\,n$$



* coupling a BFS spanning tree of a component with a Galton-Watson tree

Part II.

Local structure of Erdős-Rényi random graph

$G(n,m) \in_R \mathcal{G}(n,m)$ and $2m/n \xrightarrow{n\to\infty} c \in [0,\infty)$

$r \in_R V\left(G(n,m)\right)$ a vertex chosen uniformly at random from $V(G(n,m))$

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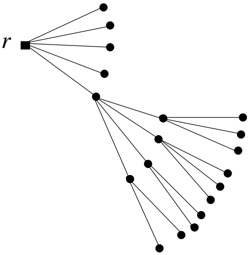

$$d^{+}(r) \sim \mathrm{Po}(c)$$

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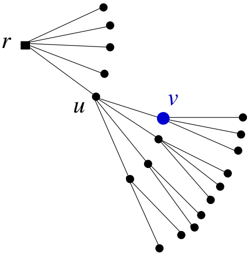


$d^+(r) \sim \mathrm{Po}(c)$

$d^+(u) \sim \mathrm{Po}(c)$

$d^+(v) \sim \mathrm{Po}(c)$

Benjamini–Schramm local weak limit

[Benjamini–Schramm 2001; Aldous–Steele 2004]

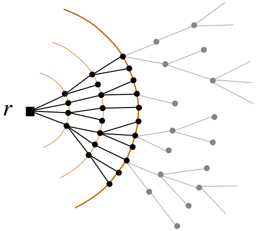
- a rooted graph is a pair (H, r) of a graph H and a vertex $r \in V(H)$
- two rooted graphs (H_1, r_1) and (H_2, r_2) are isomorphic,

$$(H_1, r_1) \;\cong\; (H_2, r_2)$$

  if $\exists$ isomorphism ϕ from H_1 onto H_2 with $\phi(r_1) = r_2$

- given a rooted graph (H, r) and $\ell \in \mathbb{N} := \{1, 2, \ldots\}$, let

$$B_\ell\,(H, r) \;:=\; H\left[\{v \in V(H) : d_H(v, r) \le \ell\}\right]$$



Benjamini–Schramm local weak limit — cont'd

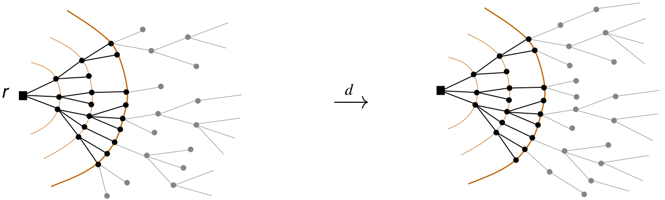
Definition [Benjamini–Schramm 2001; Aldous–Steele 2004]

Given two random rooted graphs (G_1, r_1) and (G_2, r_2) with $G_1 = G_1(n)$, the local weak limit of (G_1, r_1) is (G_2, r_2), denoted by

$$(G_1, r_1) \quad \xrightarrow{d} \quad (G_2, r_2)$$

if for each fixed rooted graph (H, r_H) and $\ell \in \mathbb{N}$

$$\mathbb{P}\Big[B_\ell\,(G_1, r_1) \;\cong\; (H, r_H)\Big] \quad \xrightarrow{n\to\infty} \quad \mathbb{P}\Big[B_\ell\,(G_2, r_2) \;\cong\; (H, r_H)\Big]$$



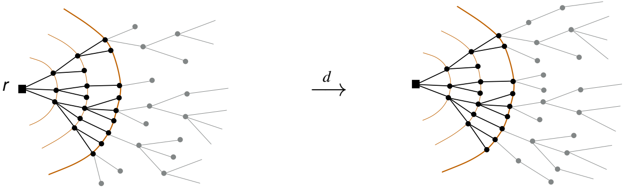
Erdős-Rényi random graph vs Galton–Watson tree

- $G = G(n,m) \in_R \mathcal{G}(n,m)$
- $r \in_R V(G)$

If $2m/n \xrightarrow{n\to\infty} c \in [0,\infty)$, then

$$(G,r) \xrightarrow{d} \mathrm{GWT}(c)$$

where $\mathrm{GWT}(c)$ is the Galton–Watson tree with offspring distribution $\mathrm{Po}(c)$



i.e., $\mathbb{P}\left[B_\ell(G,r) \cong (H,r_H)\right] \xrightarrow{n\to\infty} \mathbb{P}\left[B_\ell(\mathrm{GWT}(c)) \cong (H,r_H)\right]$

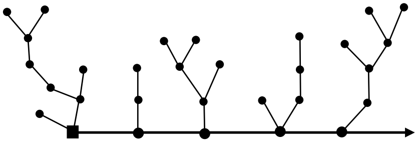
Local weak limit of a random tree

- $T = T(n) \in_R \mathcal{T}(n)$
  a tree chosen uniformly at random from the class of all trees
  (i.e., acyclic connected graphs) on vertex set $[n]$
- $r \in_R V(T)$

Theorem [GRIMMETT 1980/1981]

$$(T, r) \quad \xrightarrow{d} \quad T_\infty$$

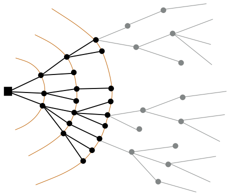
Skeleton tree T_∞



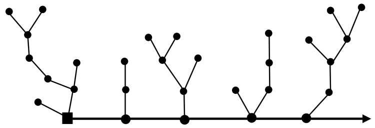
a rooted tree obtained from an infinite path by replacing each vertex of the path by an independent Galton-Watson tree $\mathrm{GWT}(1)$

Local weak limits

$\mathrm{GWT}\,(c)$ Galton–Watson tree: local weak limit of ER random graph



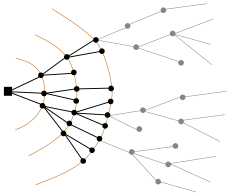
T_∞ Skeleton tree: local weak limit of a uniform random tree



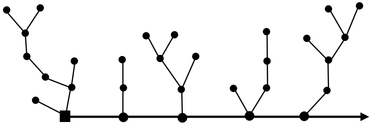
Part III.

Local weak limit of a random planar graph

$\mathrm{GWT}\,(c)$ Galton–Watson tree



T_∞ Skeleton tree



Local weak limit of a random planar graph

- $P = P(n,m) \in_R \mathcal{P}(n,m)$ a uniform random planar graph
- $r \in_R V(P)$ a vertex chosen uniformly at random from $V(P)$

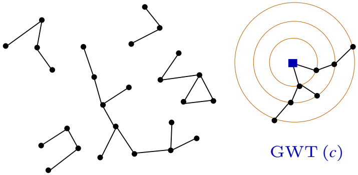
Local weak limit of a random planar graph

- $P = P(n,m) \in_R \mathcal{P}(n,m)$ a uniform random planar graph
- $r \in_R V(P)$ a vertex chosen uniformly at random from $V(P)$

Theorem [K.–MISSETHAN 2021+]

- If $2m/n \xrightarrow{n\to\infty} c \in [0,1]$, then

$$(P, r) \quad \xrightarrow{d} \quad \mathrm{GWT}(c)$$

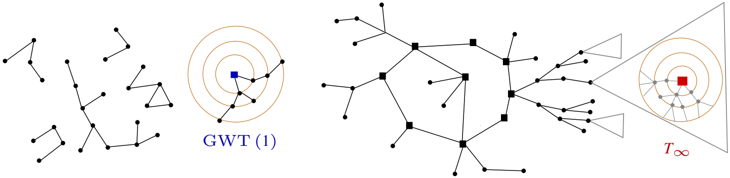


Local weak limit of a random planar graph

- $P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \xrightarrow{n\to\infty} c \in (1,2)$
- $r \in_R V(P)$

Theorem [K.–MISSETHAN 2021+]

$$(P,r) \xrightarrow{d} (2-c)\,\mathrm{GWT}(1) + (c-1)\,T_\infty$$



i.e., for each rooted graph (H, r_H) and $\ell \in \mathbb{N}$, we have

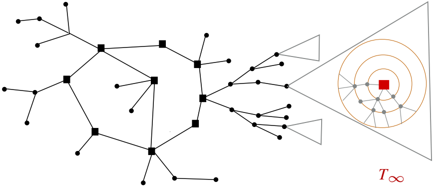
$$\mathbb{P}\left[B_\ell(P,r) \cong (H,r_H)\right] \xrightarrow{n\to\infty} (2-c)\,\mathbb{P}\left[B_\ell(\mathrm{GWT}(1)) \cong (H,r_H)\right] + (c-1)\,\mathbb{P}\left[B_\ell(T_\infty) \cong (H,r_H)\right]$$

Local weak limit of a random planar graph

- $P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \xrightarrow{n\to\infty} 2$
- $r \in_R V(P)$

Theorem [K.–MISSETHAN 2021+]

$$(P, r) \xrightarrow{d} T_\infty$$



Part IV.

Main proof ideas

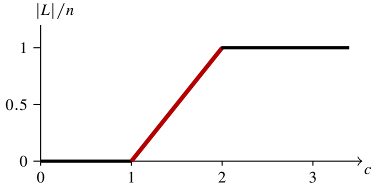
Phase transition in a random planar graph

$P \;=\; P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \;\to\; c \in (1,6]$

L largest component of P

Theorem [K.–ŁUCZAK 2012]

- If $c \;\in\; (1,2)$, whp $|L| \;=\; (1+o(1))\,(c-1)\,n$
- If $c \;\in\; [2,6]$, whp $|L| \;=\; (1+o(1))\,n$

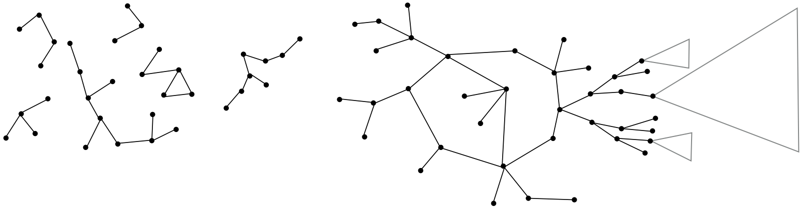


Global structure of a random planar graph

$P \ = \ P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \ \to \ c \in (1,2)$

L largest component of P

$S \ = \ P \setminus L$ 'small' part of P



Global structure of the small part

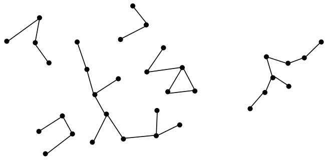
$P \;=\; P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \;\to\; c \in (1,2)$

L largest component of P

$S \;=\; P \setminus L$ 'small' part of P

The small part S 'behaves similarly' like

a critical ER random graph $G(\bar{n},\bar{m})$ with $\bar{n} = (2-c)\,n$ and $2\bar{m}/\bar{n} \;\to\; 1$



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$r_S \in_R V(S)$

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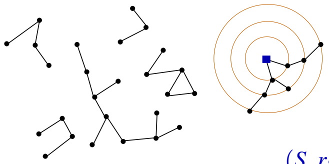


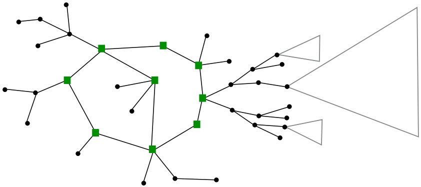
$(S, r_S) \xrightarrow{d} \mathrm{GWT}(1)$

Internal structure of the largest component

$P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \to c \in (1,2)$

L largest component of P

C 2-core $=$ max subgraph of L with min deg $\geq$ two



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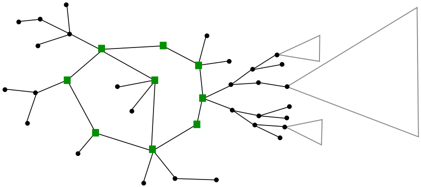
- $|L| \; = \; (1+o(1))\,(c-1)\,n$ and $|C| \; = \; o(n)$
- $L \; = \; C \; +$ each vertex in $V(C)$ replaced by a rooted tree



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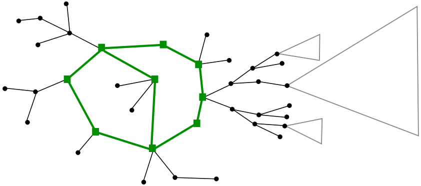

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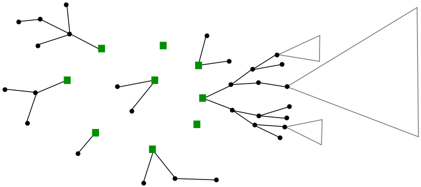
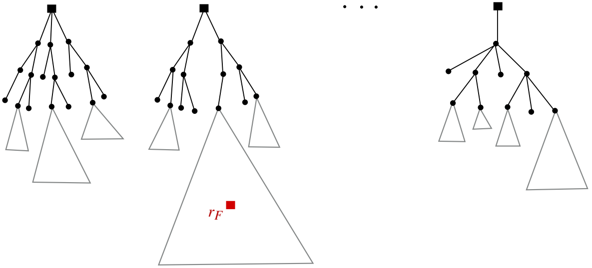


Local weak limit of a random forest

- $F = F(n,t) \in_R \mathcal{F}(n,t)$ a forest on vertex set $[n]$ with t tree components
- $r_F \in_R V(F)$ a vertex chosen uniformly at random from $V(F)$



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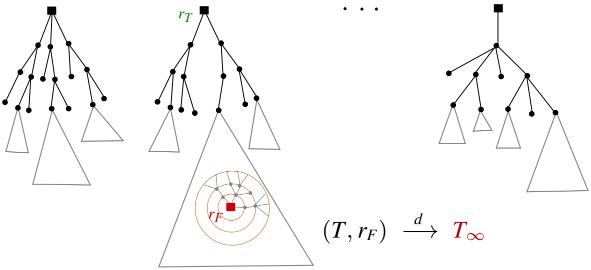
- $r_F \in_R V(F)$ a vertex chosen uniformly at random from $V(F)$
- r_T the root of the tree component T in F that contains r_F

Lemma [K.–MISSETHAN 2021+]

If $t = t(n) = o(n)$, then whp $d_F(r_F, r_T) = \omega(1)$ and

$$(F, r_F) \xrightarrow{d} T_\infty$$



Internal structure of the largest component

$P \ = \ P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \ \to \ c \in (1,2)$

L largest component of P

C 2-core $=$ max subgraph of L with min deg $\geq$ two

- $|L| \ = \ (1+o(1))\,(c-1)\,n$ and $|C| \ = \ o(n)$
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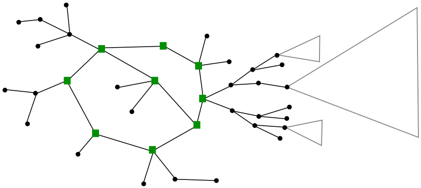

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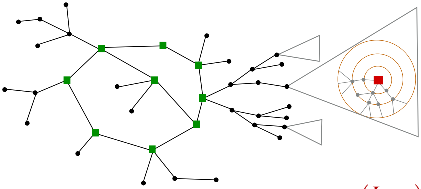


$(L, r_L) \xrightarrow{d} T_\infty$

Finer view of local weak limits

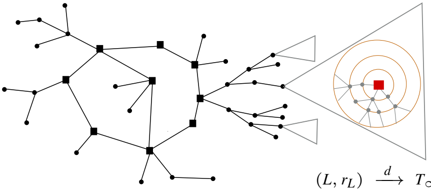
$P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \to c \in (1,2)$

L largest component of P and $|L| \sim (c-1)\,n$

$r_L \in_R V(L)$

Theorem [K.–MISSETHAN 2021+]

$$(L, r_L) \xrightarrow{d} T_\infty$$



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Finer view of local weak limits

$P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \to c \in (1,2)$

L largest component of P and $|L| \sim (c-1)\,n$

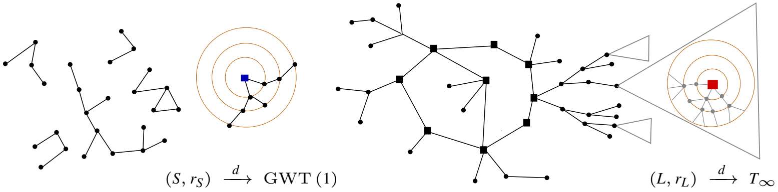
$S = P \setminus L \sim$ crtitical ER random graph and $|S| \sim (2-c)\,n$

$r_S \in_R V(S), \quad r_L \in_R V(L)$

Theorem [K.–MISSETHAN 2021+]

$$(S, r_S) \xrightarrow{d} \mathrm{GWT}(1)$$

$$(L, r_L) \xrightarrow{d} T_\infty$$



Finer view of local weak limits

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L largest component of P and $|L| \sim (c-1)\,n$

$S = P \setminus L \sim$ crtitical ER random graph and $|S| \sim (2-c)\,n$

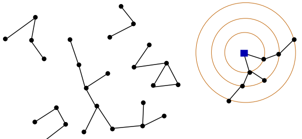
$r_S \in_R V(S)$, $r_L \in_R V(L)$, and $r_P \in_R V(P)$

Theorem [K.–MISSETHAN 2021+]

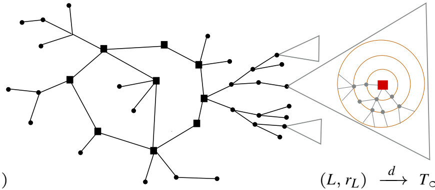
$$(S, r_S) \xrightarrow{d} \mathrm{GWT}(1)$$

$$(L, r_L) \xrightarrow{d} T_\infty$$

$$(P, r_P) \xrightarrow{d} (2-c)\,\mathrm{GWT}(1) + (c-1)\,T_\infty$$



$(S, r_S) \xrightarrow{d} \mathrm{GWT}(1)$

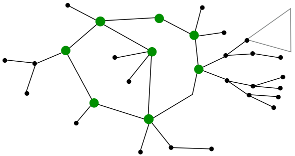


$(L, r_L) \xrightarrow{d} T_\infty$

Local limit of a random planar graph with root in 2-core

$P \; = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \; \to \; c \in (1,2)$

C 2-core = maximal subgraph of largest comp L of P with min deg ≥ 2



Local limit of a random planar graph with root in 2-core

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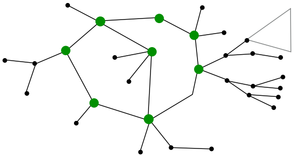

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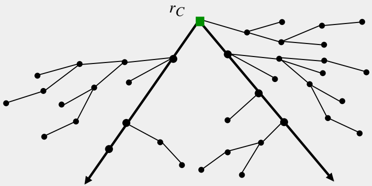

$T_\infty^{(\ell)}$ a rooted tree obtained by replacing each vertex of ℓ infinite paths rooted at a common vertex by an independent $\mathrm{GWT}(1)$

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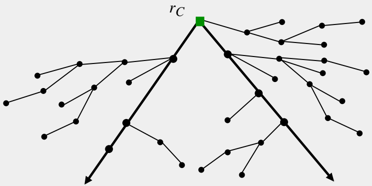

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cf. local limit of small part S and largest compopnent L of P

$$(S, r_S) \xrightarrow{d} \mathrm{GWT}(1) = T_\infty^{(0)}$$

$$(L, r_L) \xrightarrow{d} T_\infty = T_\infty^{(1)}$$

Local limit of a random planar graph with root in kernel

$P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \to c \in (1,2)$

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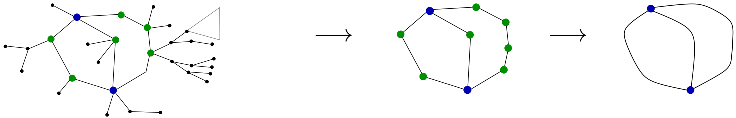
K kernel = multigraph obtained from C by replacing each bare path (i.e., maximal path with only internal vertices of degree two) by an edge



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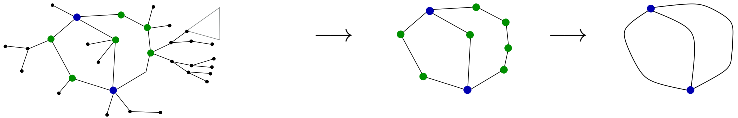
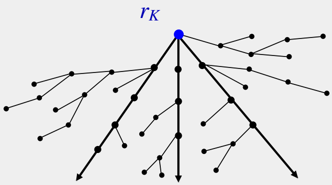


$$(P, r_K) \xrightarrow{d} T_{\infty}^{(3)}$$



Local weak limits of a random planar graph

$P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \to c \in (1,2)$

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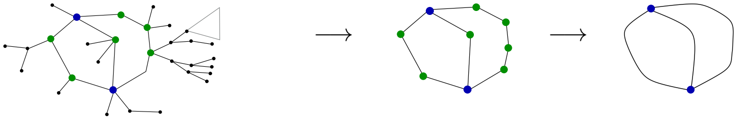
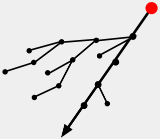
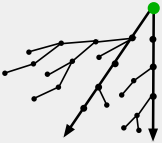
K kernel = multigraph obtained from C by replacing bare path by edge

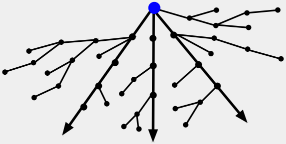
$r_K \in_R V(K)$, $r_C \in_R V(C)$, $r_L \in_R V(L)$



$(P, r_L) \xrightarrow{d} T_\infty^{(1)}$ $(P, r_C) \xrightarrow{d} T_\infty^{(2)}$ $(P, r_K) \xrightarrow{d} T_\infty^{(3)}$



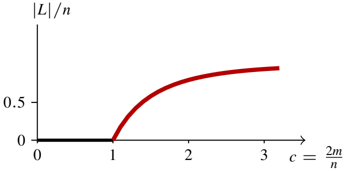




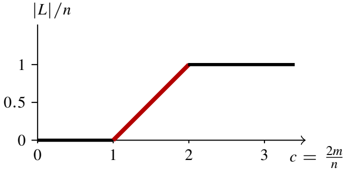
Summary and open problems

(1) Phase transitions and critical phenomena

Uniform random graph $G(n,m)$



Random planar graph $P(n,m)$



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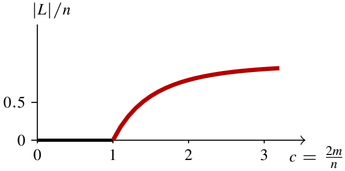

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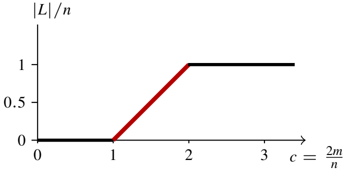


$2m/n \;\to\; c > 1$ (supercritical)

- $S \;=\; G(n,m) \setminus L$ 'behaves similarly' like a subcritical ER random graph $G(\bar{n},\bar{m})$ with $\bar{n} = (1-\rho)\,n$ and $2\bar{m}/\bar{n} \;<\; 1$

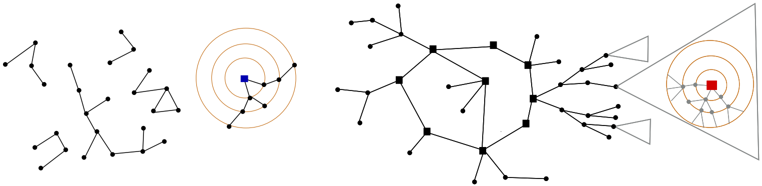
- $S \;=\; P(n,m) \setminus L$ 'behaves similarly' like a critical ER random graph $G(\bar{n},\bar{m})$ with $\bar{n} = (2-c)\,n$ and $2\bar{m}/\bar{n} \;\to\; 1$

Summary and open problems — cont'd

(2) Local weak limit of a random planar graph

If $P = P(n,m) \in_R \mathcal{P}(n,m)$, $2m/n \to c \in (1,2)$, and $r_P \in_R V(P)$, then

$$(P, r_P) \quad \xrightarrow{d} \quad {\color{blue}(2-c)\,\mathrm{GWT}(1)} \;+\; {\color{red}(c-1)\,T_\infty}$$



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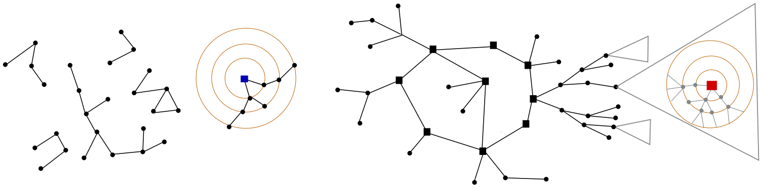


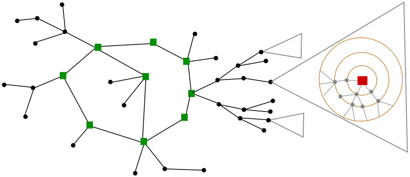
$\longrightarrow$ extended to a random graph on a surface with constant genus

Summary and open problems — cont'd

Let $P = P(n,m) \in_R \mathcal{P}(n,m)$ and $r_P \in_R V(P)$

(3) In 2nd critical regime when $2m/n \to 2$

$$|L| \sim n, \quad |2\text{-core}| = o(n), \quad \text{and} \quad (P, r_P) \xrightarrow{d} T_\infty = T_\infty^{(1)}$$



Summary and open problems — cont'd

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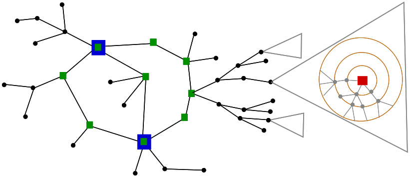

(4) Conjecture: $\exists\, 2 < t_c < 4.42$ and $0 < a, b < 1$ s.t. $2m/n \to \beta \in (2, t_c)$,

- $|\text{2-core}| = (a + b + o(1))\, n$, $|\text{kernel}| = (b + o(1))\, n$, and
- $(P, r_P) \xrightarrow{d} (1 - a - b)\, T_\infty^{(1)} + a\, T_\infty^{(2)} + b\, T_\infty^{(\geq 3)}$