Topological Aspects of Random Graphs

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1-5 August 2022

Guiding questions/themes

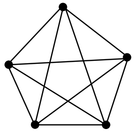
(1) What is a typical genus of the Erdős-Rényi random graph?

* genus of a graph G

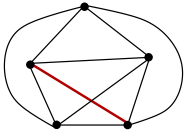
$=$ minimum number of handles that must be attached to sphere

in order to embed G without any crossing edges

* the case when genus is 0 corresponds to planar graphs



K_5



genus of K_5 = 1

Guiding questions/themes

(1) What is a typical genus of the Erdős-Rényi random graph?

(2) How does a topological constraint such as

- being planar
- being embeddable on the orientable surface with given genus

affect the global and local structure of a random graph, e.g.,

- component structures
- local weak limits

Part I.

The Erdős-Rényi random graph

A uniform random graph

$G(n,m) \in_R \mathcal{G}(n,m)$

$$\mathcal{G}(n,m) \quad = \quad \text{set of all vertex-labelled simple graphs}$$

on vertex set $[n] := \{1,\ldots,n\}$ with $m = m(n)$ edges

$$G(n,m) \quad = \quad \text{chosen uniformly at random from } \mathcal{G}(n,m)$$

Throughout the talk

- whp = with high probability
  = with probability tending to one as $n \to \infty$

- all asymptotics are taken as $n \to \infty$

Emergence of the giant component

L_1 = largest component in $G(n, m)$

$|L_1|$ = $\#$ vertices in L_1

$m = d \cdot \frac{n}{2}$

Theorem [Erdős-Rényi 1959]

- If $d < 1$ (subcritical), whp $|L_1| = O(\log n)$
- If $d > 1$ (supercritical), whp $|L_1| = \Theta(n)$

Largest component in supercritical $G(n, m)$

$m \; = \; d \cdot \frac{n}{2}$ for $d > 1$

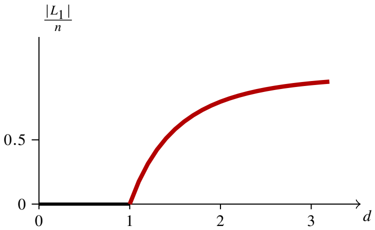
$\rho \; = \; 1 - \exp(-d\,\rho)$

(survival prob. of GW branching process with offspring dist. $\mathrm{Po}\,(d)$)

Theorem

whp

$$|L_1| \; = \; (1 + o(1))\, \rho\, n$$



Planarity of $G(n,m)$

$m = d \cdot \frac{n}{2}$

Theorem [Erdős-Rényi 1959–60]

- If $d < 1$, whp
  - each component is either a tree or unicyclic component
  - $G(n,m)$ is planar
- If $d > 1$, whp
  - largest component contains $\geq$ two cycles
  - $G(n,m)$ is not planar

Genus of supercritical $G(n,m)$

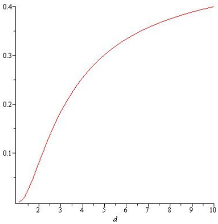
$m = d \cdot \frac{n}{2}$ for $d > 1$

$g =$ genus of $G(n,m)$

Theorem [Dowden–K.–Krivelevich 2019]

whp $$g = (1+o(1))\, \mu(d)\, d \cdot \frac{n}{2}$$

$\frac{g}{m} \sim \mu(d) \nearrow \frac{1}{2}$



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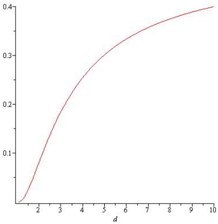

* when $n \ll m \leq \binom{n}{2}$, $\frac{g}{m}$ decreases from $\frac{1}{2}$ to $\frac{1}{6}$ [RÖDL– THOMAS 1995]

Guiding questions/themes

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(2) How does a topological constraint such as

– being planar

– being embeddable on the orientable surface with given genus

affect the global and local structure of a random graph, e.g.,

– component structures

– local weak limits

A uniform random graph on a surface

$g \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$

$S_g(n,m) \in_R \mathcal{S}_g(n,m)$

$\mathcal{S}_g(n,m)$ = set of all vertex-labelled simple graphs on $[n]$
with $m = m(n)$ edges that are
embeddable on the orientable surface of genus g

$S_g(n,m)$ = chosen uniformly at random from $\mathcal{S}_g(n,m)$

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Note

- $\mathcal{S}_0(n,m) \subset \ldots \subset \mathcal{S}_g(n,m) \subset \mathcal{S}_{g+1}(n,m) \subset \ldots \subset \mathcal{G}(n,m)$

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Note

- $\mathcal{S}_0(n,m) \subset \dots \subset \mathcal{S}_g(n,m) \subset \mathcal{S}_{g+1}(n,m) \subset \dots \subset \mathcal{G}(n,m)$
- If $1 \le m < \frac{n}{2}$, then $\frac{|\mathcal{S}_0(n,m)|}{|\mathcal{G}(n,m)|} \xrightarrow{n\to\infty} 1$
- If $m > 3n - 6 + 6g$, then $\mathcal{S}_g(n,m) = \emptyset$

Part II.

Random graphs on surfaces with constant genus

Two critical phases

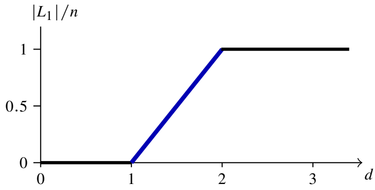
$g \in \mathbb{N}_0$ constant

$L_1 \;=\;$ largest component in $S_g(n,m) \;\in_R\; \mathcal{S}_g(n,m)$

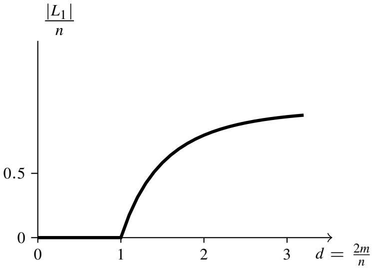
$m \;=\; d \cdot \frac{n}{2}$

Theorem [K.–Łuczak 2012; K.–Mosshammer–Sprüssel 2020]

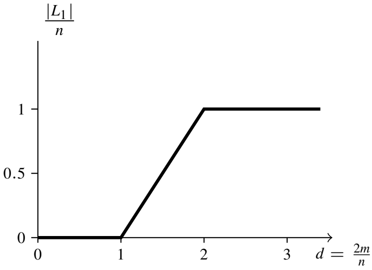
- If $d \in (1,2)$, whp $\quad |L_1| \;=\; (1+o(1))\,(d-1)n$
- If $d \in [2,6]$, whp $\quad |L_1| \;=\; (1+o(1))\,n$



ER random graph vs random graphs on surfaces

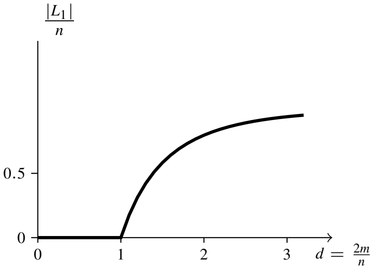


Uniform random graph $G(n, m)$



Random graph on a surface $S_g(n, m)$

ER random graph vs random graphs on surfaces



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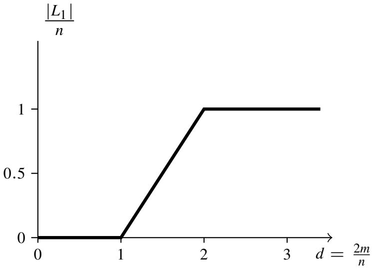
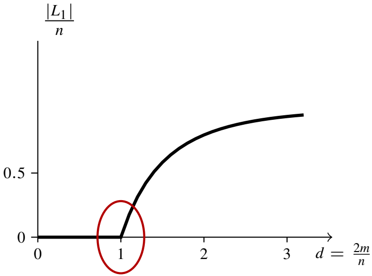
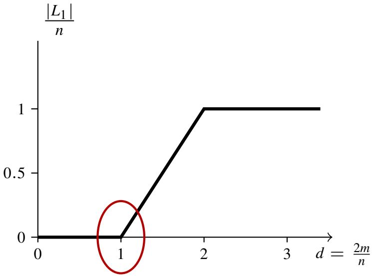


Random graph on a surface $S_g(n,m)$

* fragment $R = G(n,m) \setminus L_1$ is subcritical (i.e., $2m_R/n_R < 1$)
* fragment $R = S_g(n,m) \setminus L_1$ is critical (i.e., $2m_R/n_R \to 1$)

ER random graph vs random graphs on surfaces





Uniform random graph $G(n, m)$

Random graph on a surface $S_g(n, m)$

Weakly supercritical random graphs

$m = \frac{n}{2} + s$ for $s > 0$ and $n^{2/3} \ll s \ll n$

$G(n, m)$ [Bollobás 84; Łuczak 90]

whp $$|L_1| = (4 + o(1))\, s$$

$S_g(n, m)$ [K.–Łuczak 2012; K.–Mosshammer–Sprüssel 2020]

whp $$|L_1| = (2 + o(1))\, s$$

Part III.

Local structure of random graphs

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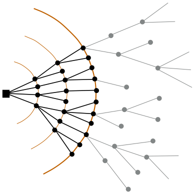
Local structure of random graphs

(1) What does a random graph locally look like?



(2) How does a global structure of a random graph affect its local structure?

Local structure of Erdős-Rényi random graph

$$G = G(n,m) \in_R \mathcal{G}(n,m)$$

$$m = d \cdot \tfrac{n}{2}$$

Local structure of Erdős-Rényi random graph

$G \;=\; G(n,m) \;\in_R\; \mathcal{G}(n,m)$

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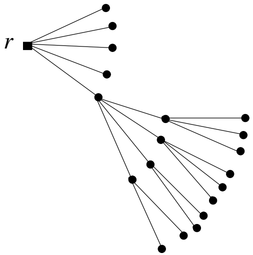
$r \;\in_R\; V(G) \;=\;$ a vertex chosen uniformly at random from $V(G)$



$d^+(r) \;\sim\; \mathrm{Po}\,(d)$

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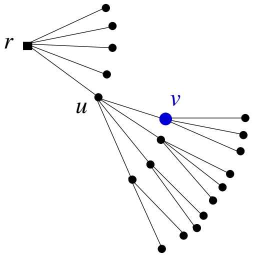

$d^+(r) \;\sim\; \mathrm{Po}\,(d)$

$d^+(u) \;\sim\; \mathrm{Po}\,(d)$

$d^+(v) \;\sim\; \mathrm{Po}\,(d)$

Erdős-Rényi random graph vs Galton–Watson tree

$G = G(n, m) \in_R \mathcal{G}(n, m)$ and $r \in_R V(G)$

If $2m/n \xrightarrow{n\to\infty} d \in [0, \infty)$, then

$$(G, r) \xrightarrow{D} \mathrm{GWT}(d)$$

where $\mathrm{GWT}(d)$ is the Galton–Watson tree with offspring distribution $\mathrm{Po}(d)$

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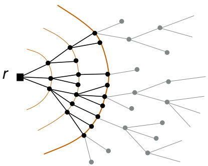
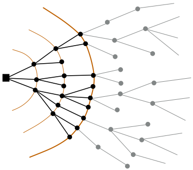


$\xrightarrow{D}$



i.e., for each rooted graph (H, r_H) and $\ell \in \mathbb{N}$, we have

$$\mathbb{P}\Big[B_\ell(G,r) \cong (H,r_H)\Big] \xrightarrow{n\to\infty} \mathbb{P}\Big[B_\ell(\mathrm{GWT}(d)) \cong (H,r_H)\Big]$$

Local weak limit of a random tree

$$T \;=\; T(n) \;\in_R\; \mathcal{T}(n)$$

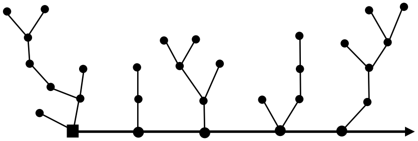
$=$ a tree chosen uniformly at random from the class of all trees on $[n]$

$$r \;\in_R\; V(T)$$

Theorem [Grimmett 1980/1981]

$$(T, r) \quad \xrightarrow{D} \quad T_\infty$$

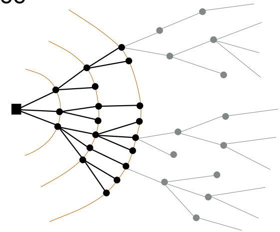
Skeleton tree T_∞



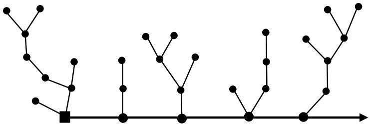
$=$ a rooted tree obtained from an infinite path by replacing each vertex of the path by an independent Galton-Watson tree $\mathrm{GWT}\,(1)$

Benjamini-Schramm local weak limits

$\mathrm{GWT}\,(d)$ Galton–Watson tree



T_∞ Skeleton tree



Local weak limit of a random graph on a surface

$S \ = \ S_g(n,m) \ \in_R \ \mathcal{S}_g(n,m)$

$r \ \in_R \ V(S)$ a vertex chosen uniformly at random from $V(S)$

$g \ \in \ \mathbb{N}_0$ constant and $2m/n \xrightarrow{n\to\infty} d \in [1,2]$

Local weak limit of a random graph on a surface

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Theorem [K.–MISSETHAN 2022+]

$$(S,r) \xrightarrow{D} (2-d)\,\mathrm{GWT}(1) + (d-1)\,T_\infty$$

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$$\mathbb{P}\Big[B_\ell(S,r) \cong (H,r_H)\Big] \xrightarrow{n\to\infty} (2-d)\,\mathbb{P}\Big[B_\ell(\mathrm{GWT}(1)) \cong (H,r_H)\Big] \;+\; (d-1)\,\mathbb{P}\Big[B_\ell(T_\infty) \cong (H,r_H)\Big]$$

Global structure of a random graph on a surface

$S = S_g(n,m) \in_R \mathcal{S}_g(n,m)$ and $2m/n \to d \in (1,2)$

L_1 largest component of S

$R = S \setminus L_1$ fragment of S

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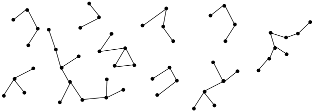
Theorem [K.–ŁUCZAK 2012; K.–MOSSHAMMER–SPRÜSSEL 2020]

- R 'behaves similarly' like a critical ER random graph $G(n_R, m_R)$ with $n_R = (2-d)\,n$ and $2m_R/n_R \to 1$



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C 2-core $=$ max subgraph of L_1 with min deg $\geq$ two

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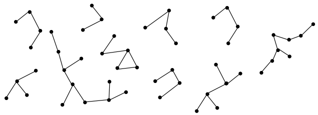
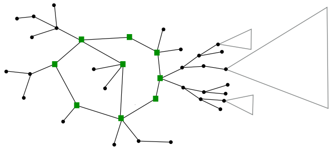
- $L_1 \;=\; C \;+\;$ each vertex in $V(C)$ replaced by a rooted tree





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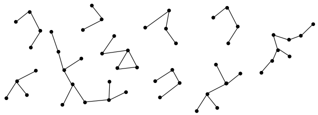

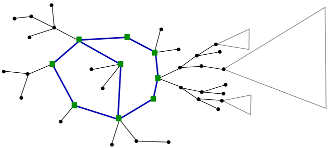

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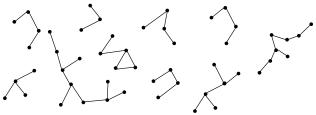
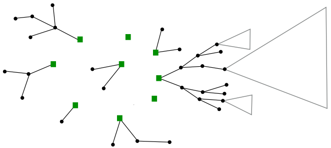




Finer view of local weak limits

$S \;=\; S_g(n,m) \;\in_R\; \mathcal{S}_g(n,m)$ and $2m/n \;\to\; d \in (1,2)$

L_1 largest component of S

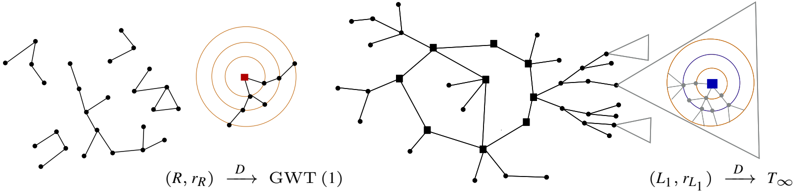
$R = S \setminus L_1 \;\sim\;$ critical ER random graph

$r_R \;\in_R\; V(R), \quad r_{L_1} \;\in_R\; V(L_1)$

Theorem [K.–MISSETHAN 2021+]

$$(R, r_R) \xrightarrow{D} \mathrm{GWT}(1)$$

$$(L_1, r_{L_1}) \xrightarrow{D} T_\infty$$



Finer view of local weak limits

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L_1 largest component of S and $|L_1| \;\sim\; (d-1)\,n$

$R = S \setminus L_1 \;\sim\;$ critical ER random graph and $|R| \;\sim\; (2-d)\,n$

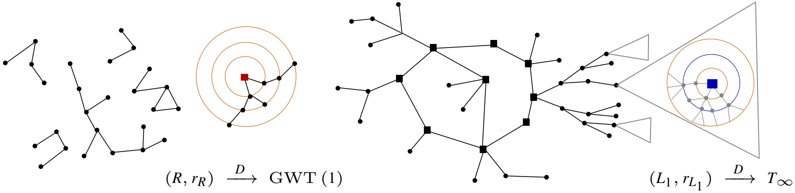
$r_R \;\in_R\; V(R),\;\; r_{L_1} \;\in_R\; V(L_1)$ and $r_S \;\in_R\; V(S)$

Theorem [K.–MISSETHAN 2021+]

$$(R, r_R) \xrightarrow{D} \mathrm{GWT}(1)$$

$$(L_1, r_{L_1}) \xrightarrow{D} T_\infty$$

$$(S, r_S) \xrightarrow{D} (2-d)\,\mathrm{GWT}(1) \;+\; (d-1)\,T_\infty$$



Part IV.

Random graphs on surfaces with non-constant genus

ER random graph vs random graphs on surfaces

IF whp the genus of $G(n,m)$ is $T = T(n,m)$,

THEN $\forall\, g \geq T$

$$\frac{|\mathcal{S}_g(n,m)|}{|\mathcal{G}(n,m)|} \quad \geq \quad \frac{|\mathcal{S}_T(n,m)|}{|\mathcal{G}(n,m)|} \quad \xrightarrow{n\to\infty} \quad 1.$$

ER random graph vs random graphs on surfaces

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In other words, for $\forall\, g \geq T$,

$S_g(n,m)$ is indistinguishable from $G(n,m)$ under viewpoint of whp-properties

If for every property $\mathcal{A}$

whp $G(n,m)$ satisfies $\mathcal{A}$ iff whp $S_g(n,m)$ satisfies $\mathcal{A}$

then we say $G(n,m)$ and $S_g(n,m)$ are contiguous.

Contiguity thresholds

T = contiguity threshold for $S_g(n,m)$ and $G(n,m)$

Theorem [Dowden–K.–Krivelevich 2019]

- If $m = \frac{n}{2} + s$ for $s > 0$ and $n^{2/3} \ll s \ll n$ (weakly supercritical), whp

$$T = \frac{8s^3}{3n^2} \ll n$$

- If $m = d \cdot \frac{n}{2}$ for $d > 1$ (supercritical), whp

$$T = \nu(d) \cdot n$$

Weakly supercritical regime

$m = \frac{n}{2} + s$ for $s = s(n) > 0$ and $n^{2/3} \ll s \ll n$

$T = \frac{8s^3}{3n^2} =$ contiguity threshold

$L_1 =$ largest component

$G(n,m)$ [Bollobás 84; Łuczak 90]

whp $$|L_1| = (4 + o(1))\, s$$

$S_g(n,m)$ [Dowden–K.–Mosshammer–Sprüssel 2022+]

whp

$$|L_1| = (4 + o(1))\, s \qquad \text{if} \quad g \gg T$$

$$|L_1| = (2 + o(1))\, s \qquad \text{if} \quad g \ll T$$

Supercritical regime

$m = d \cdot \frac{n}{2}$ for $1 < d < 2$

$T = \nu(d) \cdot n =$ contiguity threshold

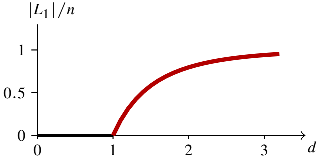
$L_1 =$ largest component

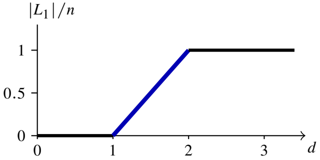
Theorem [Dowden–K.–Mosshammer–Sprüssel 2022+]

whp

$$|L_1| = (1 + o(1))\, \rho\, n \qquad \text{if} \quad g \gg T$$

$$|L_1| = (1 + o(1))\, (d - 1)\, n \qquad \text{if} \quad g \ll T$$

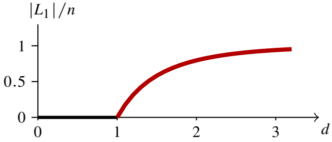




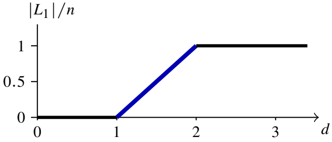
Summary and open problems

Global properties of $S_g(n, m)$ when $m = d \cdot \frac{n}{2}$ for $d > 1$

- contiguity threshold $T = \nu(d) \cdot n$
- largest component L_1



$|L_1| \sim \rho\, n$ if $g \gg T$



$|L_1| \sim (d-1)\, n$ if $g \ll T$

$Q1.$ Order of largest component when $g = \Theta(T)$?

$Q2.$ Length of longest cycle when $g \ll T$ or $g = \Theta(T)$?

* when $g \gg T$, it follows from $G(n, m)$ [AJTAI-KOMLÓS-SZEMERÉDI 1981]