Topological Aspects of Random Graphs

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46th Workshop on Graph-Theoretic Concepts in Computer Science (WG20)

Guiding questions/themes

(1) What is a typical genus of Erdős-Rényi random graph?

(2) How does a topological constraint influence the component structure of a random graph?

– planarity

– upper bound on genus

(3) What is the length of the longest and shortest cycles?

– in Erdős-Rényi random graph

– in a random planar graph

Part I.

Erdős-Rényi Random Graphs

Throughout the talk

- let $G(n,m)$ denote a uniform random graph:
  - $\mathcal{G}(n,m) \;=\;$ set of all simple graphs on vertex set $[\,n\,] := \{1,\ldots,n\}$ with $m = m(n)$ edges
  - $G(n,m) \;=\;$ chosen uniformly at random from $\mathcal{G}(n,m)$

- whp $\;=\;$ with probability tending to one as $n \to \infty$

- all asymptotics are taken as $n \to \infty$

Emergence of giant component

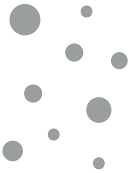
$L_1 \;=\;$ largest component in $G(n, m)$

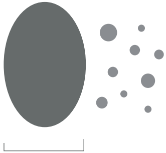
$|L_1| \;=\; \#$ vertices in L_1

$m \;=\; d \cdot \frac{n}{2}$ ($d \;=\;$ average degree)

Theorem [Erdős-Rényi 1959]

- If $d < 1$, whp $|L_1| \;=\; O(\log n)$
- If $d > 1$, whp $|L_1| \;=\; \Theta(n)$





Largest component L_1 in $G(n, m)$

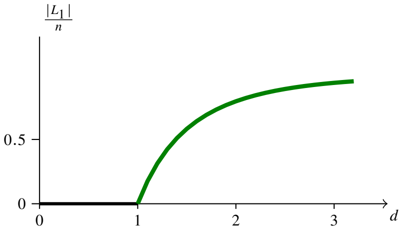
$m \;=\; d \cdot \frac{n}{2}$ for $d > 1$

$1 - \rho \;=\; \exp(-d\,\rho)$

Theorem

whp

$$|L_1| \;=\; (1 + o(1))\,\rho\,n$$



Planarity of $G(n, m)$

$$m = d \cdot \frac{n}{2}$$

Theorem [Erdős-Rényi 1959–60]

- If $d < 1$, whp
  - each component is either a tree or unicyclic component
  - $G(n, m)$ is planar
- If $d > 1$, whp
  - largest component contains $\geq$ two cycles
  - $G(n, m)$ is not planar

Part II.

Random Planar Graphs

Random planar graphs

Let $P(n,m)$ denote a uniform random planar graph:

- $\mathcal{P}(n,m) \quad = \quad$ set of all simple graphs on vertex set $[\,n\,]$
  with $m = m(n)$ edges that are
  embeddable on sphere without crossing edges
- $P(n,m) \quad = \quad$ chosen uniformly at random from $\mathcal{P}(n,m)$

Note

- $\mathcal{P}(n,m) \quad \subset \quad \mathcal{G}(n,m)$
- for $m \; > \; 3n-6$: $\qquad \mathcal{P}(n,m) \quad = \quad \emptyset$
- for $1 \; \leq \; m \; < \; \frac{n}{2}$: $\qquad \dfrac{|\mathcal{P}(n,m)|}{|\mathcal{G}(n,m)|} \quad \xrightarrow{n\to\infty} \quad 1$

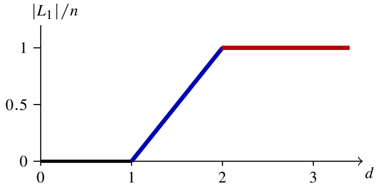
Random planar graph $P(n,m)$

L_1 = largest component in $P(n,m)$

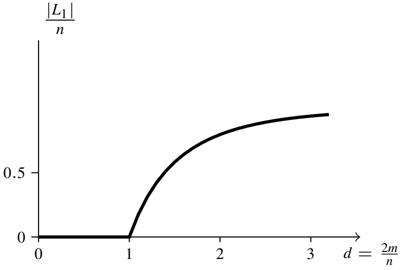
$m = d \cdot \frac{n}{2}$

Theorem [K.–Łuczak 2012]

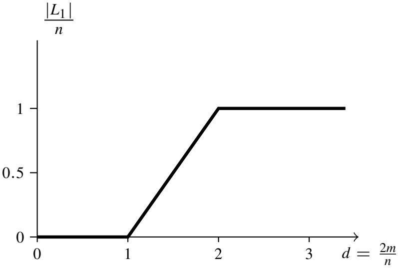
- If $d \in (1,2)$, whp $\quad |L_1| = (1+o(1))\,(d-1)n$
- If $d \in [2,6]$, whp $\quad |L_1| = (1+o(1))\,n$



Phase transitions and critical phases



Uniform random graph $G(n,m)$



Random planar graph $P(n,m)$

Weakly supercritical random graphs

$m \;=\; \frac{n}{2} + s$ for $s > 0$ and $n^{2/3} \ll s \ll n$

Uniform random graph $G(n,m)$ [Bollobás 84; Łuczak 90]

whp $$|L_1| \;=\; (4 + o(1))\, s$$

Random planar graph $P(n,m)$ [K.–Łuczak 2012]

whp $$|L_1| \;=\; (2 + o(1))\, s$$

Part III.

Random Graphs on Surfaces

Random graph on a surface

- Random graph $S_g(n,m)$ on an orientable surface of genus g:
  - $\mathcal{S}_g(n,m) \quad = \quad$ set of all simple graphs on vertex set $[\,n\,]$ with $m = m(n)$ edges and genus $\leq g$
  - $S_g(n,m) \quad = \quad$ chosen uniformly at random from $\mathcal{S}_g(n,m)$

- Note $\quad \mathcal{P}(n,m) \; = \; \mathcal{S}_0(n,m) \quad \subset \quad \mathcal{S}_g(n,m) \quad \subset \quad \mathcal{G}(n,m)$

- For which $g = g(n)$, are $G(n,m)$ and $S_g(n,m)$ indistinguishable under viewpoint of whp–properties?

  In other words, when are $G(n,m)$ and $S_g(n,m)$ contiguous?

Genus of $G(n,m)$

g = genus of $G(n,m)$

$m = \frac{n}{2} + s$ for $s > 0$ and $n^{2/3} \ll s \ll n$

Theorem [Dowden–K.–Krivelevich 2019]

whp $$g = (1+o(1))\,\frac{8s^3}{3n^2}$$

Proof ideas: let $G = G(n,m)$

$$\begin{aligned} \text{genus of } G &= \text{genus of largest component } L_1 \text{ of } G \\ &= \text{genus of 2-core } C \text{ of } L_1 \\ &= \frac{1}{2}\left(e(C) - v(C) - f(C) + 2\right) \end{aligned}$$

$$f(C) \ll e(C) - v(C) = \text{excess}(C) = \text{excess}(L_1)$$

Contiguity threshold

$g \ = \ $ genus of $G(n,m)$

$m \ = \ \frac{n}{2} + s$ for $s > 0$ and $n^{2/3} \ll s \ll n$

$T \ = \ \frac{8s^3}{3n^2} \ = \ $ contiguity threshold

Theorem [Dowden–K.–Krivelevich 2019]

For every $\varepsilon > 0$,

- if $g \geq (1+\varepsilon)T$, $G(n,m)$ and $S_g(n,m)$ are contiguous

  ($\forall$ a property $\mathcal{P}$, whp G satisfies $\mathcal{P}$ iff whp S_g satisfies $\mathcal{P}$)

- if $g \leq (1-\varepsilon)T$, $G(n,m)$ and $S_g(n,m)$ are not contiguous

Largest component in $S_g(n, m)$

L_1 = largest component of $S_g(n, m)$

$m = \frac{n}{2} + s$ for $s > 0$ and $n^{2/3} \ll s \ll n$

$T = \frac{8s^3}{3n^2}$ = contiguity threshold

Theorem [Dowden–K.–Mosshammer–Sprüssel 2019+]

whp

- $|L_1| = (4 + o(1))\, s$ if $g \geq (1 + o(1))T$
- $|L_1| = (2 + o(1))\, s$ if $g = o(T)$

Genus of $G(n,m)$

$g \;=\;$ genus of $G(n,m)$

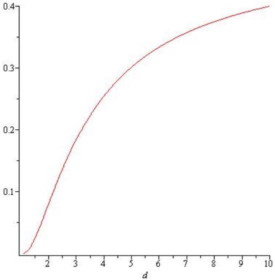
$m \;=\; d \cdot \frac{n}{2}$ for $d > 1$

Theorem [Dowden–K.–Krivelevich 2019]

whp

$$g \;=\; (1+o(1))\;\mu(d)\; d \cdot \frac{n}{2}$$

$g/m \;\sim\; \mu(d)$



Largest component L_1 in $S_g(n, m)$

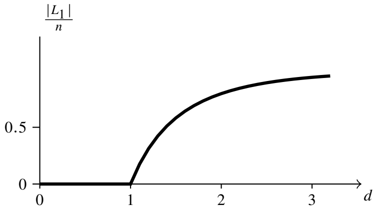
$m = d \cdot \frac{n}{2}$ for $d > 1$

$g \gg n$: thus $S_g(n, m)$ and $G(n, m)$ are contiguous

Theorem [Dowden–K.–Mosshammer–Sprüssel 2019+]

whp

$$|L_1| = (1 + o(1))\, \rho\, n$$



Largest component L_1 in $S_g(n,m)$

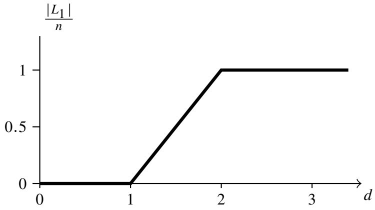
$m \;=\; d \cdot \frac{n}{2}$ for $d > 1$

$g \;\ll\; n$: thus $S_g(n,m)$ and $G(n,m)$ are not contiguous

Theorem [Dowden–K.–Mosshammer–Sprüssel 2019+]

whp

$$|L_1| \quad \sim \quad \begin{cases} (d-1)n & \text{if} \quad d \in (1,2) \\ n & \text{if} \quad d \in [2,6] \end{cases}$$



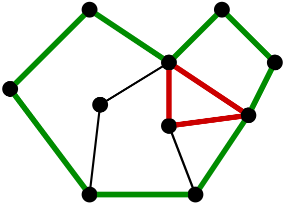
Part IV.

Longest and shortest cycles in random graphs

Cycles in graphs

Given a graph G:

- $c(G) =$ circumference $=$ length of longest cycle
- $g(G) =$ girth $=$ length of shortest cycle
- $L_1 = L_1(G) =$ largest component
- $R = G \setminus L_1 =$ graph outside largest component



Cycles in $G(n, d \cdot \frac{n}{2})$

Theorem [Erdős-Rényi 1960; Ajtai-Komlós-Szemerédi 1981]

- If $d < 1$, whp $\quad c(L_1) = g(L_1) = 0$

  and $\quad c(G) = O_p(1)$

- If $d > 1$, whp $\quad c(L_1) = \Theta(n)$

  and $\quad c(R) = O_p(1), \quad g(L_1) = O_p(1)$

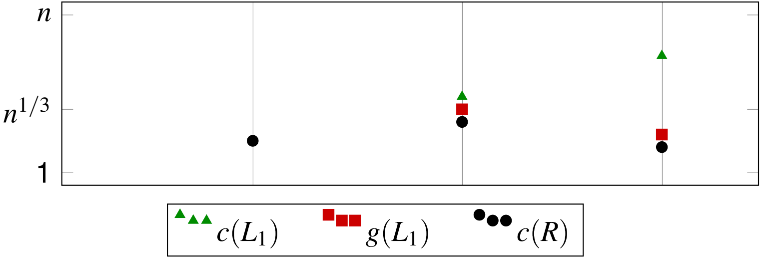
If $d < 1$, whp

- G consists of trees and unicyclic components
- L_1 is a tree

Cycles in $G(n, n/2 + s)$

[Kolchin 1987; Łuczak 1991; Łuczak–Pittel–Wierman 1994]

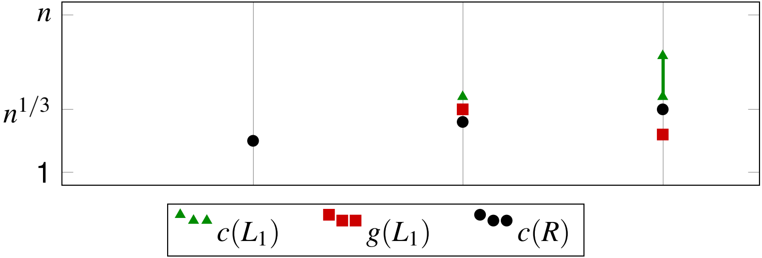
	$s\,n^{-2/3} \to -\infty$	$s\,n^{-2/3} \to c$	$s\,n^{-2/3} \to \infty$
$c(L_1)$	no cycle	$\Theta_p\left(n^{1/3}\right)^*$	$\Theta\left(s^2\,n^{-1}\right)$
$g(L_1)$	no cycle	$\Theta_p\left(n^{1/3}\right)^*$	$\Theta_p\left(n\,s^{-1}\right)$
$c(R)$	$\Theta_p\left(n\,\lvert s\rvert^{-1}\right)$	$\Theta_p\left(n^{1/3}\right)$	$\Theta_p\left(n\,s^{-1}\right)$



Cycles in $P(n, n/2+s)$

[K.–Missethan 2020+]

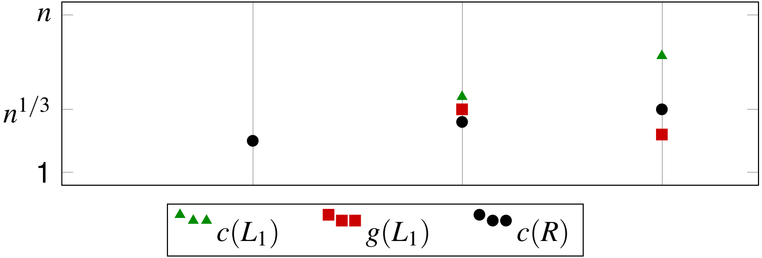
	$s\,n^{-2/3} \to -\infty$	$s\,n^{-2/3} \to c$	$s\,n^{-2/3} \to \infty$
$c(L_1)$	no cycle	$\Theta_p\left(n^{1/3}\right)^*$	$\omega\left(n^{1/3}\right),\ O\left(s\,n^{-1/3}\right)$
$g(L_1)$	no cycle	$\Theta_p\left(n^{1/3}\right)^*$	$\Theta_p\left(n\,s^{-1}\right)$
$c(R)$	$\Theta_p\left(n\,\lvert s\rvert^{-1}\right)$	$\Theta_p\left(n^{1/3}\right)$	$\Theta_p\left(n^{1/3}\right)$



Cycles in $P(n, n/2+s)$

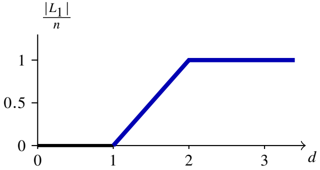
[K.–MISSETHAN 2020+]

	$s\,n^{-2/3} \to -\infty$	$s\,n^{-2/3} \to c$	$s\,n^{-2/3} \to \infty$
$c(L_1)$	no cycle	$\Theta_p\left(n^{1/3}\right)^*$	$\Theta\left(s\,n^{-1/3}\right)$?
$g(L_1)$	no cycle	$\Theta_p\left(n^{1/3}\right)^*$	$\Theta_p\left(n\,s^{-1}\right)$
$c(R)$	$\Theta_p\left(n\,\lvert s\rvert^{-1}\right)$	$\Theta_p\left(n^{1/3}\right)$	$\Theta_p\left(n^{1/3}\right)$

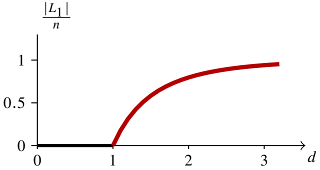


Summary & open problems

(1) Largest component L_1 in $S_g(n, d \cdot n/2)$ for $d > 1$



$S_g(n,m)$ for $g \ll n$



$S_g(n,m)$ for $g \gg n$

$\Longrightarrow$ behaviour of largest component L_1 when $g = \Theta(n)$?

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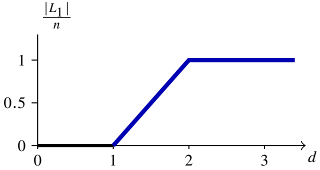
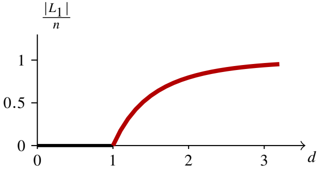


$S_g(n,m)$ for $g \ll n$



$S_g(n,m)$ for $g \gg n$

$\Longrightarrow$ behaviour of largest component L_1 when $g = \Theta(n)$?

(2) Circumference $c(L_1)$ of L_1 in $P(n, n/2+s)$ for $n^{2/3} \ll s \ll n$:

$$n^{1/3} \ll c(L_1) = O\left(s\, n^{-1/3}\right)$$

IF $\exists$ cycle of linear order in a random cubic planar graph, THEN

$$c(L_1) = \Theta\left(s\, n^{-1/3}\right)$$