Enumeration methods
for planar graphs and beyond

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Outline of talk

I. Singularity analysis of generating functions

$\triangleright$ the number M of edges is linear in the number n of vertices

II. Probabilistic counting method

$\triangleright$ $M \leq n$

III. Matrix integral method

$\triangleright$ maps vs graphs on surfaces

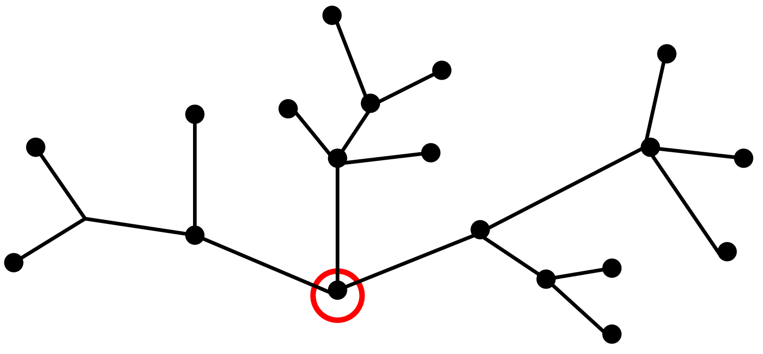
Part I: Singularity analysis of generating functions

- Trees
- Planar graphs

Trees

Let $t(n)$ denote the number of rooted labelled trees on n vertices and $T(z)$ be its exponential generating function defined by

$$T(z) = \sum_n t(n) \frac{z^n}{n!}.$$



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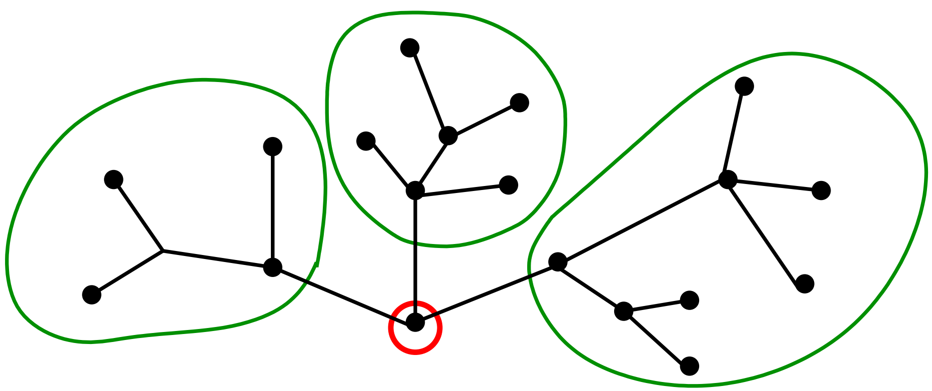

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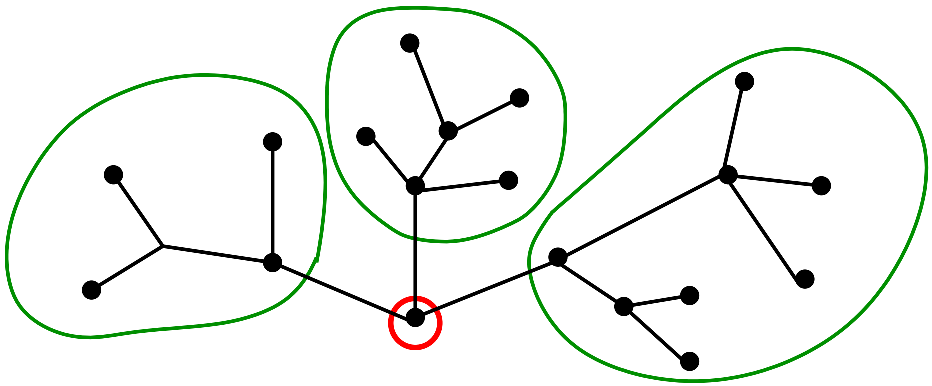

Then

$$T(z) = {\color{red}z} \left(\qquad\qquad\qquad\qquad {\color{green}\frac{T(z)^3}{3!}} \qquad\qquad \right)$$

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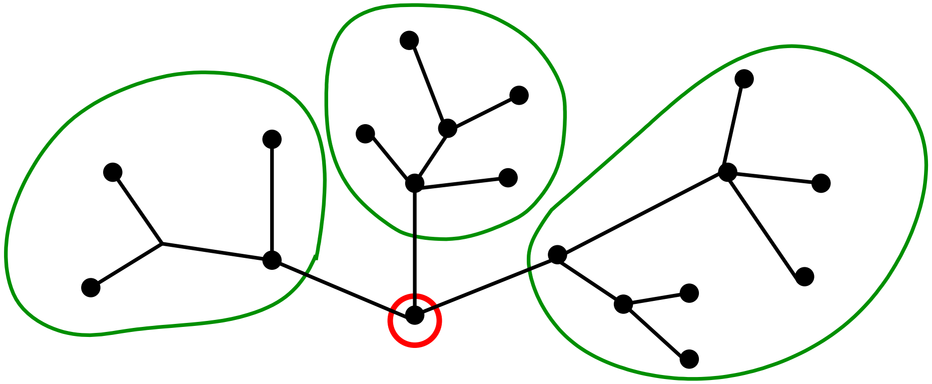

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$$T(z) = z \left(1 + T(z) + \frac{T(z)^2}{2!} + \frac{T(z)^3}{3!} + \cdots \right)$$

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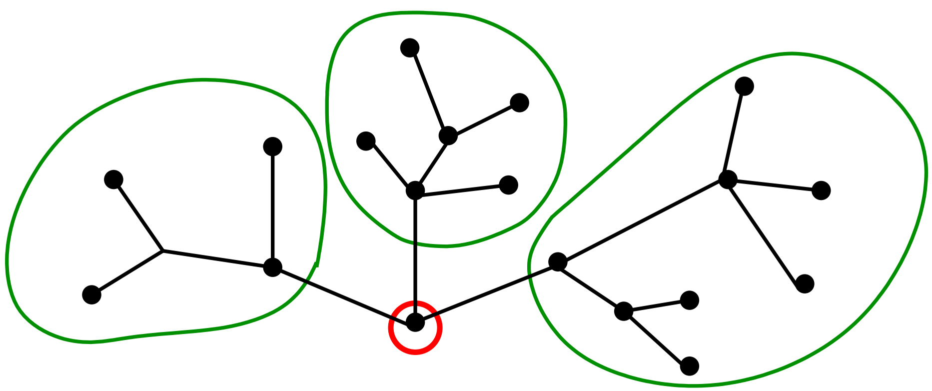

Then

$$T(z) = z\left(1 + T(z) + \frac{T(z)^2}{2!} + \frac{T(z)^3}{3!} + \cdots\right) = ze^{T(z)}.$$

Asymptotic number

View the generating function $T(z) = \sum_n t(n) \frac{z^n}{n!}$ as a complex-valued function that is analytic at the origin. Let $[z^n]T(z) = t(n)/n!$.

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Let R be the radius of convergence of $T(z)$. Then

$$[z^n]T(z) = \theta(n) R^{-n}, \quad \textit{where} \quad \limsup_{n \to \infty} |\theta(n)|^{1/n} = 1.$$

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How to determine

- the dominant singularity R and
- the subexponential factor $\theta(n)$?

Singularity analysis

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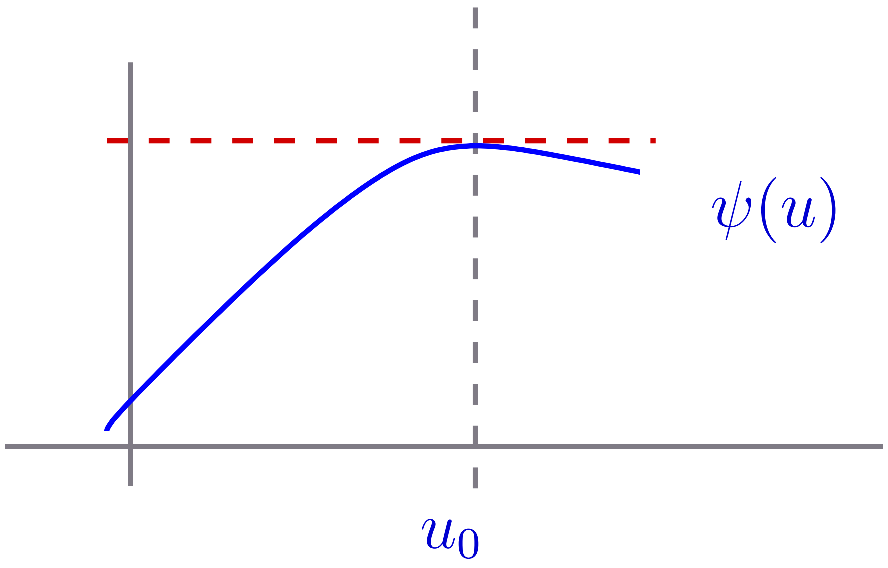

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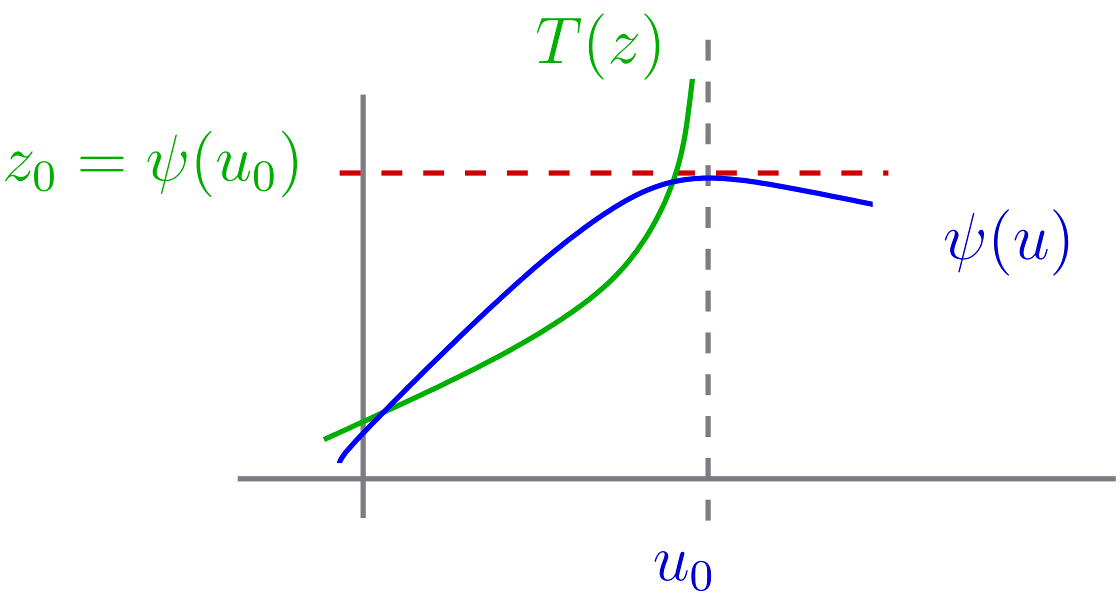

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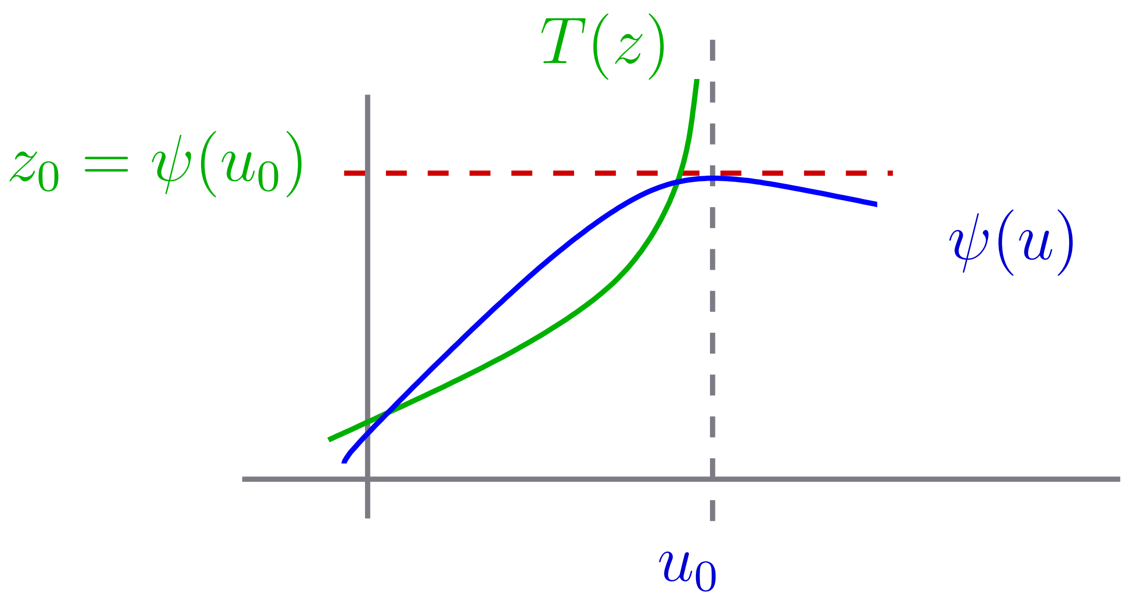

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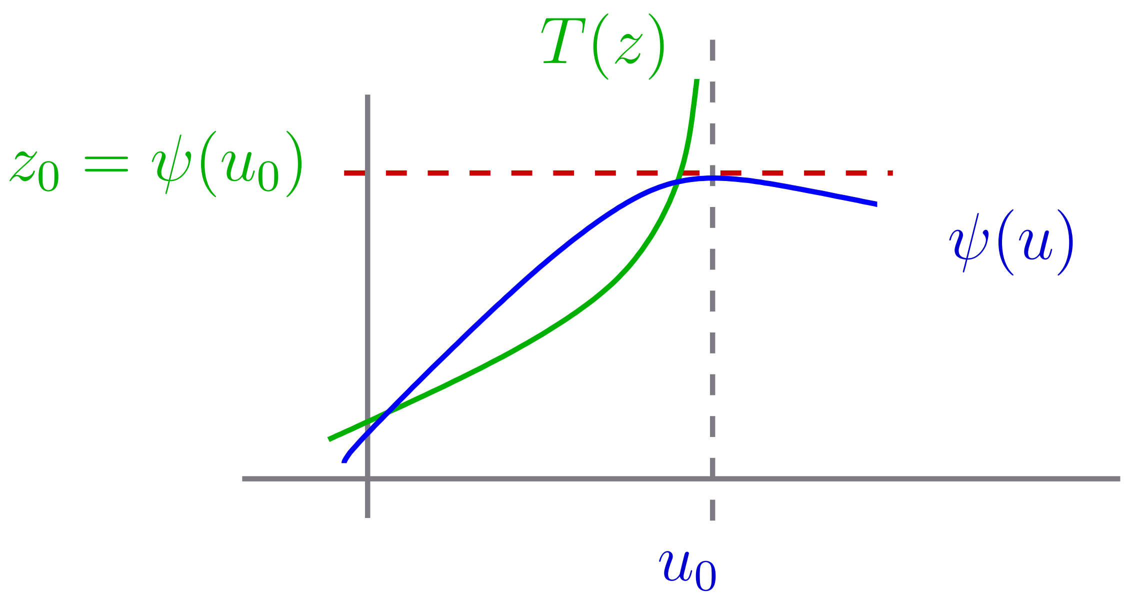

Indeed, $z_0 = e^{-1}$ and thus $\frac{t(n)}{n!} = \theta(n)e^n$, where $\limsup |\theta(n)|^{1/n} = 1$.

Local dependency

Taylor expansion of $z = \psi(u)$ at u_0 is of the form

$$\psi(u) = \psi(u_0) + \frac{1}{2}\psi''(u_0)(u - u_0)^2 + \cdots .$$

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It implies a locally quadratic dependency between z and $u = T(z)$:

$$(T(z) - T(z_0))^2 = (u - u_0)^2 \sim \frac{2}{\psi''(u_0)}(z - z_0) = -\frac{2\psi(u_0)}{\psi''(u_0)}(1 - z/z_0)$$

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Using Δ-analycity of $T(z)$ and transfer theorem, we have

$$[z^n]T(z) \sim -\sqrt{-2\psi(u_0)/\psi''(u_0)}[z^n](1 - z/z_0)^{1/2}$$

Basic scale

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RESCALING RULE/ GENERALISED BINOMIAL THEOREM

$$[z^n]\,(1 - e{\cdot}z)^{1/2} = \binom{n - 3/2}{n}\,e^n \sim \frac{n^{-3/2}}{-2\sqrt{\pi}}\,e^n.$$

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Block structure of a graph

[Harary–Palmer 78]

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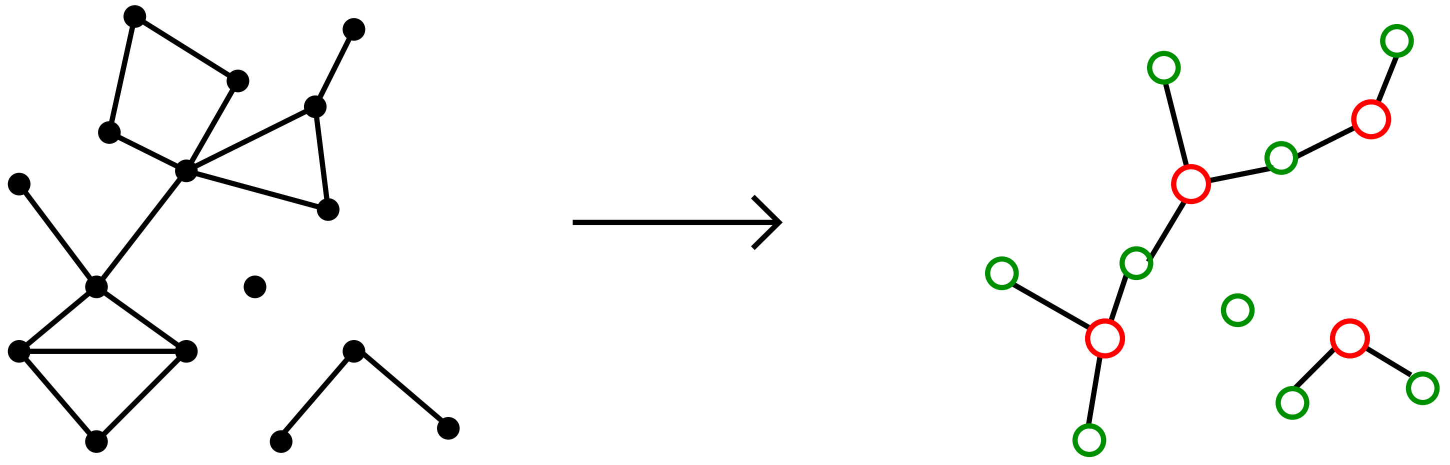
The block structure of a graph is a forest with two types of vertices: the blocks and the cut-vertices of the graph.



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Let $P(x)$ (resp. $C(x)$, $B(x)$, $T(x)$) be the EGF for labelled (resp. connected, biconnected, triconnected) planar graphs:

$$\begin{aligned} P(x) &= \exp(C(x)) \\ xC'(x) &= x\exp(B'(xC'(x))) \end{aligned}$$

Tutte’s structure theorem

2-connected graphs with one edge distinguished and oriented

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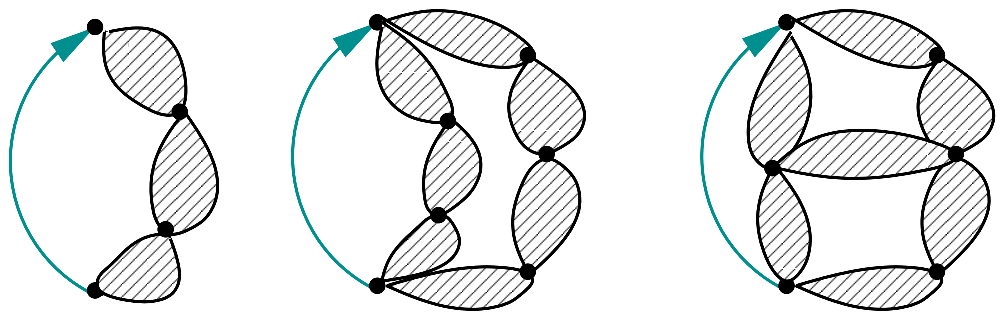


s-network p-network h-network

$$N = S + P + H$$

$$S = x(N - S)N \quad \Rightarrow \quad S = \frac{xN^2}{1 + xN}$$

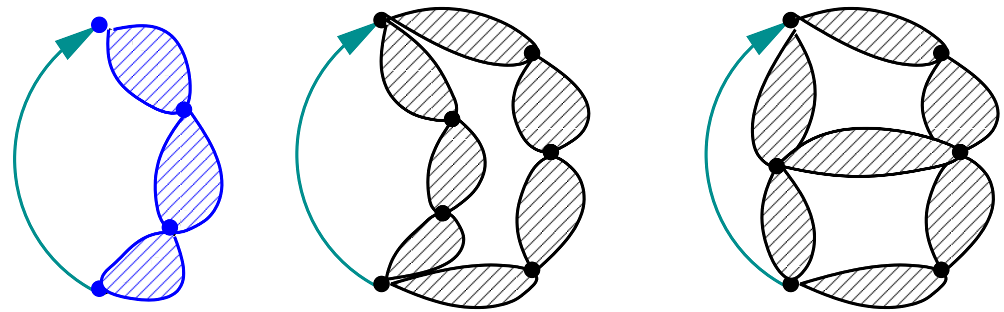
$$P = (1 + y)\exp(N - P) - (N - P) - 1 \quad \Rightarrow \quad 1 + N = (1 + y)\exp(S + H)$$

$$H = \frac{M(x, N)}{2x^2 N}$$

Tutte's structure theorem

2-connected graphs with one edge distinguished and oriented

$\Longleftrightarrow$ networks [Trakhtenbrot 58; Tutte 63; Walsh 82]



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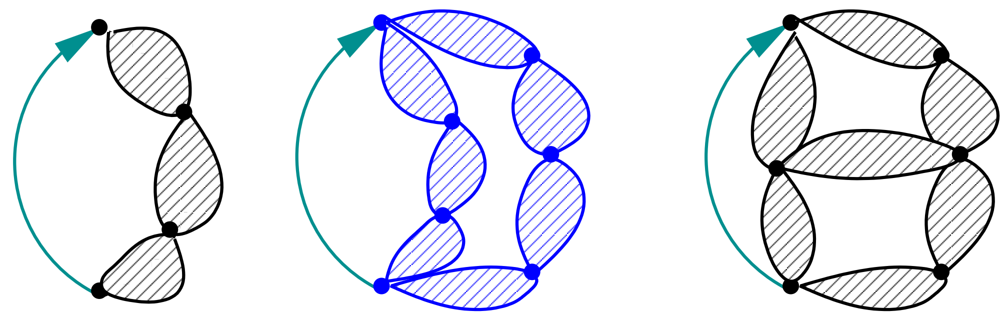

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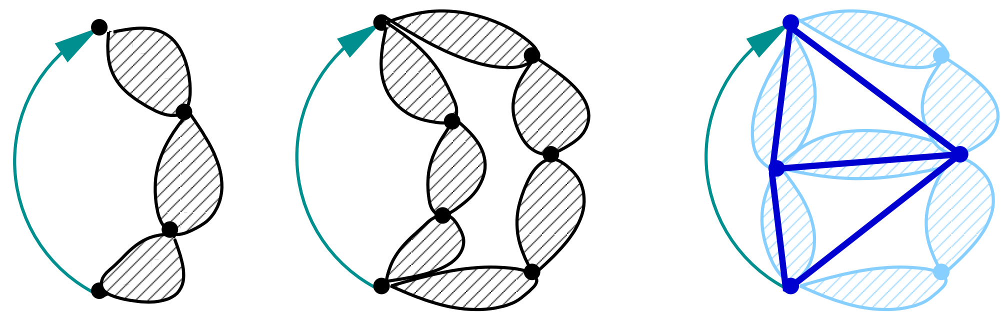

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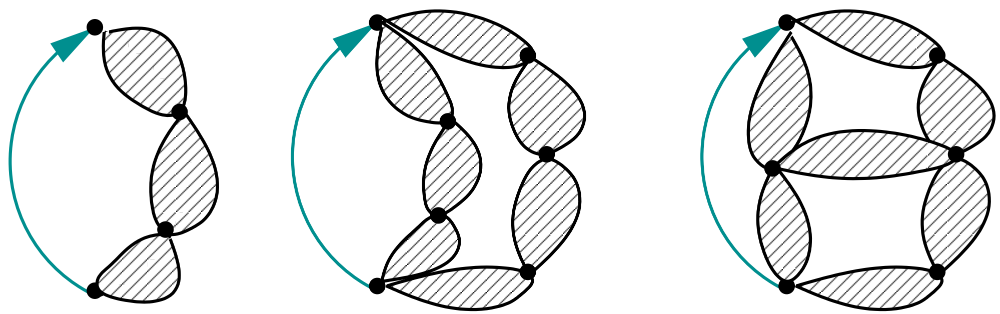

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$$\Rightarrow \qquad \log\frac{1+N}{1+y} = S + H = \frac{xN^2}{1+xN} + \frac{M(x,N)}{2x^2N}$$

Unique embedding

Unique embedding of 3-conn. planar graphs on the sphere [WHITNEY 32]

3-conn. planar graphs $\Longleftrightarrow$ 3-conn. planar maps

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Tutte rooting: given a 3-conn. planar graph,

select one edge, a direction on the edge and a side of the edge

$\Longleftrightarrow$ c-net [TUTTE 63; MULLIN–SCHELLENBERG 68]

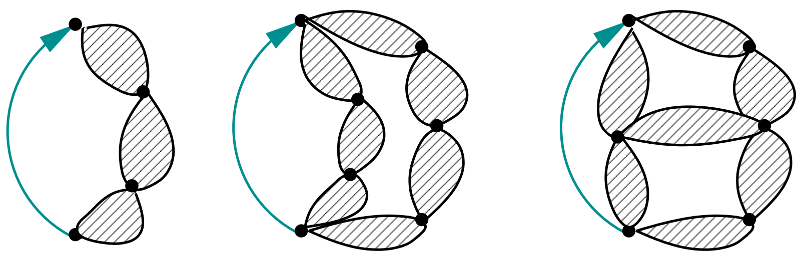
$$\begin{aligned}
\frac{\partial T(x,y)}{\partial y} &= \frac{M(x,y)}{4y} \\
M(x,y) &= x^2y^2\left(\frac{1}{1+xy}+\frac{1}{1+y}-1-\frac{(1+u)^2(1+v)^2}{(1+u+v)^3}\right) \\
u &= xy(1+v)^2 \\
v &= y(1+u)^2
\end{aligned}$$

Generating functions

Connected graphs $\Longleftrightarrow$ block structure [Harary–Palmer 78]

$$C'(x) = \exp(B'(xC'(x)))$$

2-connected graphs $\Longleftrightarrow$ networks [Trakhtenbrot 58; Tutte 63; Walsh 82]



$$\frac{\partial B(x,y)}{\partial y} = \frac{x^2(1+N(x,y))}{2(1+y)}$$

$$\frac{xN^2}{1+xN} - \log\frac{1+N}{1+y} + \frac{M(x,N)}{2xN^2} = 0$$

3-conn. planar graphs $\Longleftrightarrow$ c-nets [Tutte 63; Mullin–Schellenberg 68]

$$\begin{aligned} M(x,y) &= x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+u)^2(1+v)^2}{(1+u+v)^3}\right) \\ u &= xy(1+v)^2, \quad v = y(1+u)^2 \end{aligned}$$

Singularity analysis

c-nets $\iff$ 3-conn. planar graphs [BENDER–RICHMOND 84; BGJK 05]

- $T(n) \sim a\, n^{-7/2}\, \eta^n\, n!,$ $\eta \doteq 21.05$
- $T(n,m) \sim a_d\, n^{-4}\, \eta_d^n\, n!,$ $m = dn, \quad d \in (1,3)$

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Networks $\iff$ 2-conn. planar graphs [BENDER–GAO–WORMALD 02]

- $N(x,y) =$ *analytic part* $+\, g(y)(1 - x/R(y))^{3/2}$, *for* $y \sim 1$
- $N(n) \ \sim \ \beta\, n^{-5/2}\, \lambda^n\, n!, \qquad \lambda \doteq 26.1$
- $N(n,m) \ \sim \ \beta_d\, n^{-3}\, \lambda_d^n\, n!, \qquad m = dn, \quad d \in (1,3)$

Inserting an edge between poles and unrooting networks, we have

- $B(n) \ \sim \ b\, n^{-7/2}\, \lambda^n\, n!, \qquad \lambda \doteq 26.1$
- $B(n,m) \ \sim \ b_d\, n^{-4}\, \lambda_d^n\, n!, \qquad m = dn, \quad d \in (1,3)$

Singularity analysis

2-conn. planar graphs $\Rightarrow$ conn. planar graphs: $C'(x) = \exp(B'(xC'(x)))$

- Difficulty: integration of implicitly defined function

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\begin{aligned}
B(x,y) &= \frac{x^2}{2}\int_0^y \frac{1+N(x,t)}{1+t}dt, \qquad \frac{xN^2}{1+xN} - \log\frac{1+N}{1+y} + \frac{M(x,N)}{2x^2N} = 0 \\
N(x,y) &= \textit{analytic part} \; + \; g(y)(1-x/R(y))^{3/2}
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(Connected) planar graphs [GIMÉNEZ–NOY 09]

- $C(n) \sim c\, n^{-7/2}\, \gamma^n\, n!, \qquad \gamma \doteq 27.2$
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- Difficulty: integration of implicitly defined function

$$B(x,y) \;=\; \frac{x^2}{2}\int_0^y \frac{1+N(x,t)}{1+t}dt, \qquad \frac{xN^2}{1+xN} - \log\frac{1+N}{1+y} + \frac{M(x,N)}{2x^2N} = 0$$

$$N(x,y) \;=\; \textit{analytic part} \;+\; g(y)(1-x/R(y))^{3/2}$$

(Connected) planar graphs [Giménez–Noy 09]

- $C(n) \;\sim\; c\, n^{-7/2}\, \gamma^n\, n!, \qquad \gamma \doteq 27.2$
- $C(n,m) \;\sim\; c_d\, n^{-4}\, \gamma_d^n\, n!, \qquad m = dn, \quad d \in (1,3)$

With probability tending to one as $n \to \infty$ (for short w.h.p.) a random planar graph $\mathcal{P}(n,m)$ contains the giant component of size $n - O(1)$.

Part II: Probabilistic counting method

- Erdős–Rényi random graph $\mathcal{G}(n,m)$
- Random planar graph $\mathcal{P}(n,m)$

Planarity

Erdős–Rényi random graph [Janson–Knuth–Łuczak–Pittel 93]

Let $m = n/2 + s$ with $s = o(n)$.

- If $s\, n^{-2/3} \to -\infty$, w.h.p. $\mathcal{G}(n, m)$ is planar.
- If $s\, n^{-2/3} \to \lambda$, with positive probability $\mathcal{G}(n, m)$ is non-planar.
- If $s\, n^{-2/3} \to +\infty$, w.h.p. $\mathcal{G}(n, m)$ is non-planar.

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ERDŐS–RÉNYI RANDOM GRAPH [JANSON–KNUTH–ŁUCZAK–PITTEL 93]

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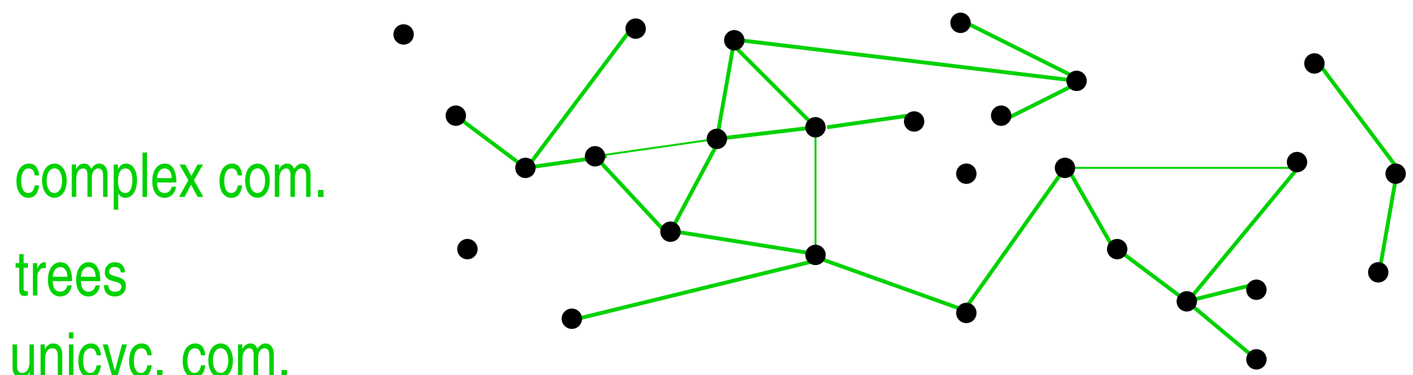
Internal structure $\Rightarrow$ core and kernel of complex graphs



Core and kernel

- Core: maximal subgraph with minimum degree two.
- Kernel: obtained from the core by replacing each path whose internal vertices are all of degree two by a single edge.

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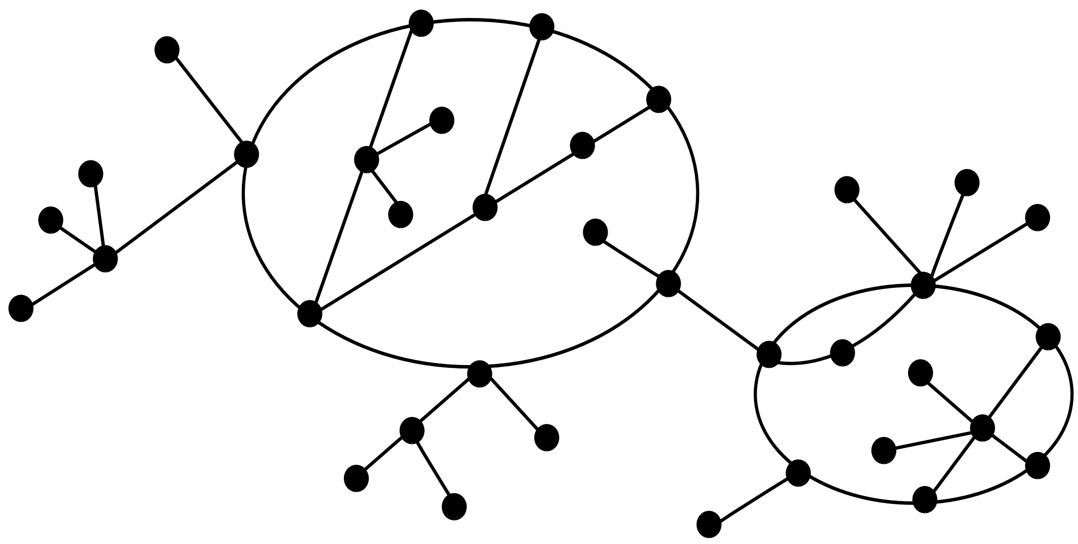

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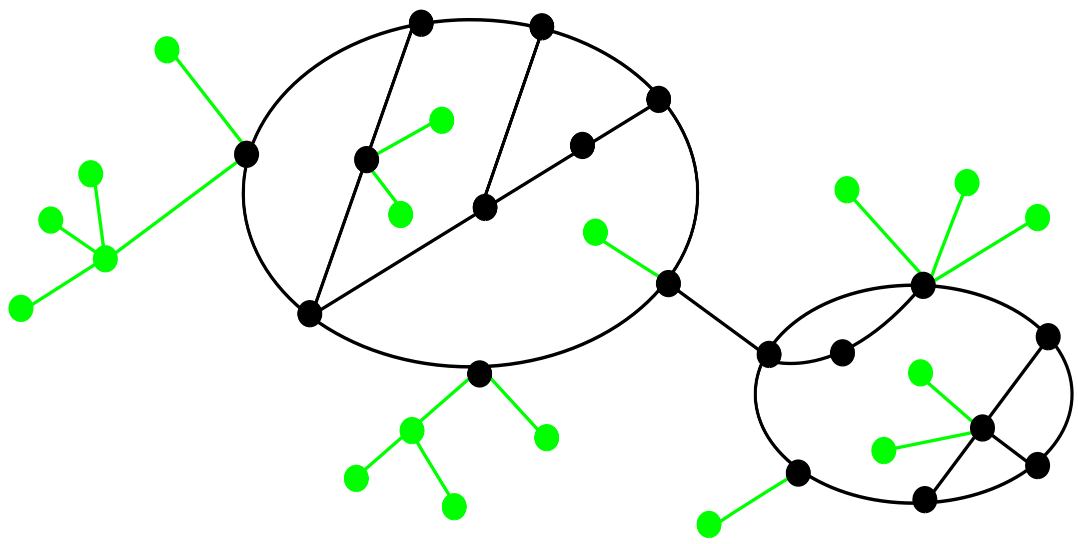

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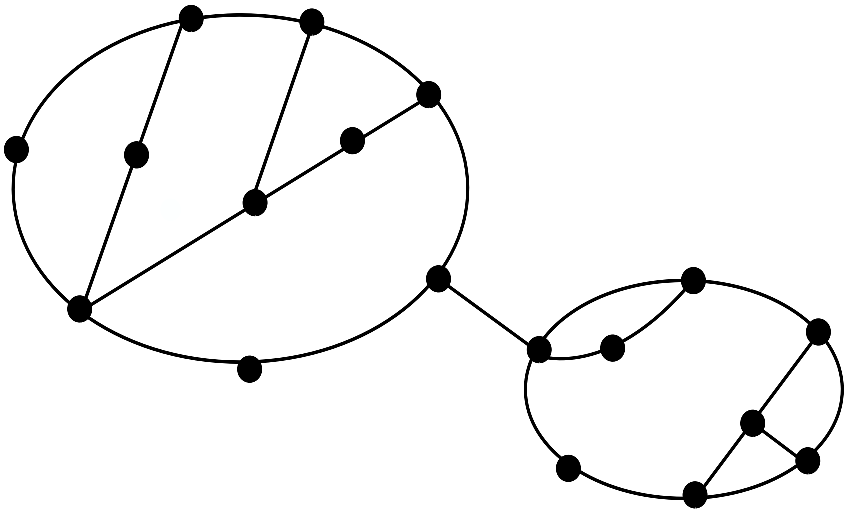

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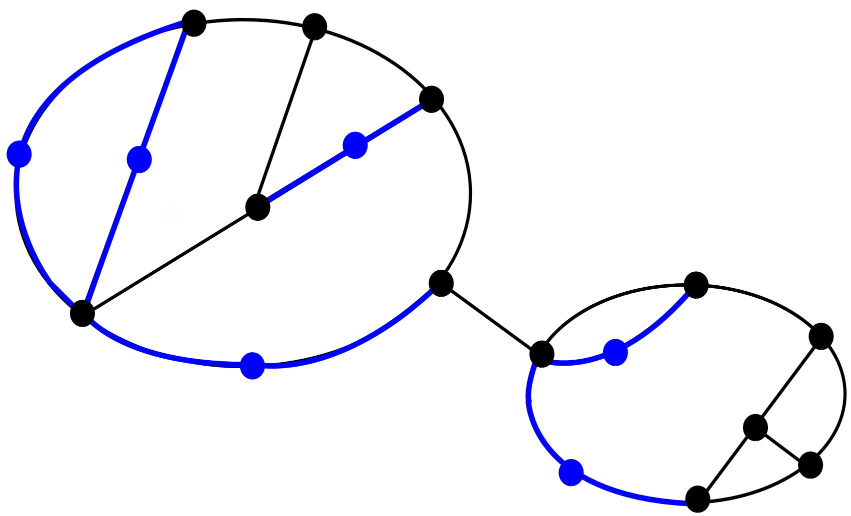

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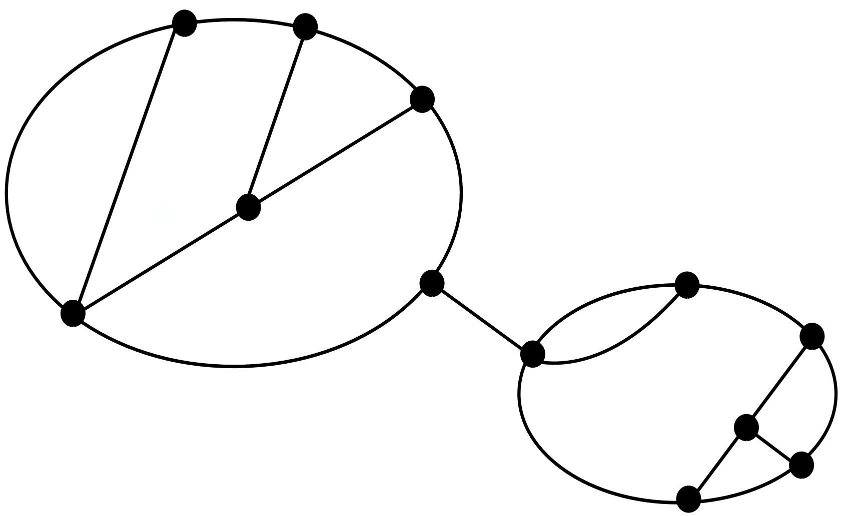

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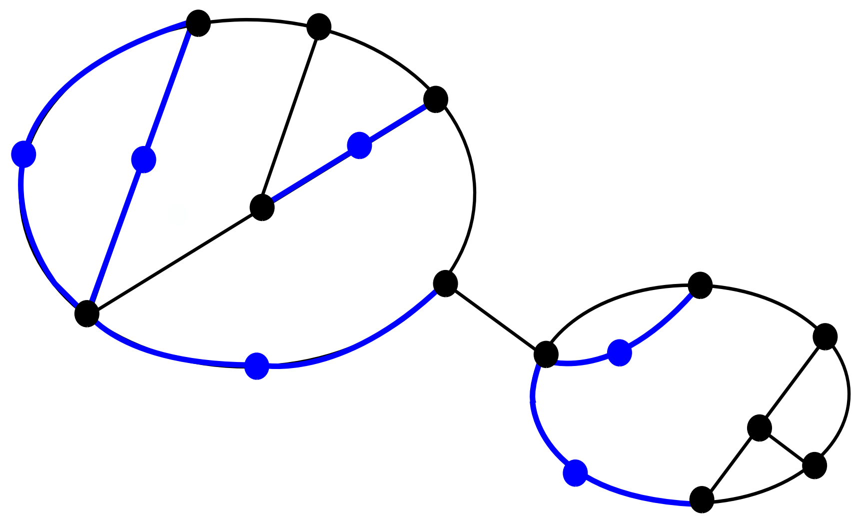

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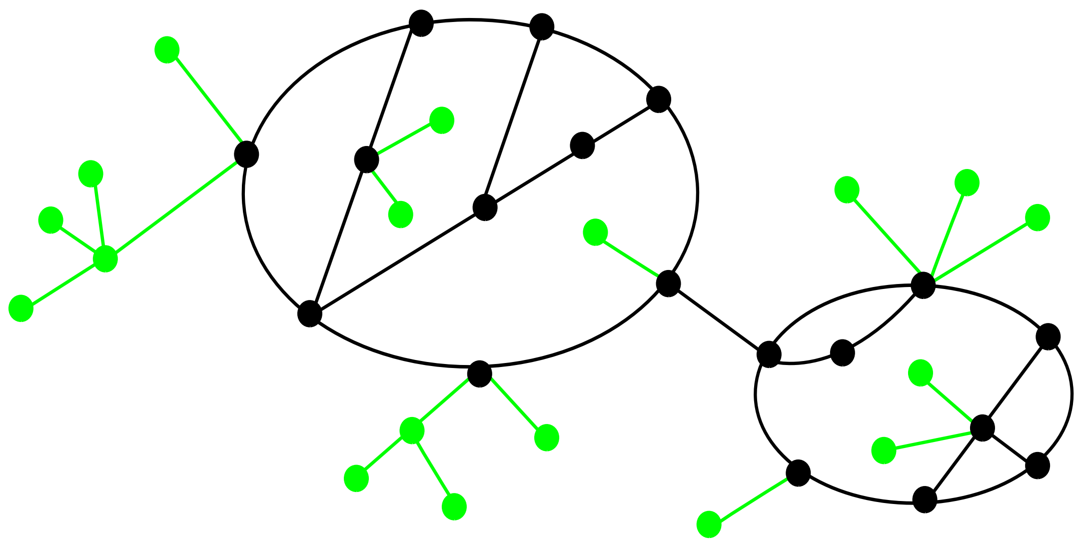

Complex planar graphs

CUBIC PLANAR GRAPHS [K.– ŁUCZACK 09+]

Let $K(n)$ denote the number of all cubic planar weighted multigraphs on n vertices. Then

$$K(n) \sim g\, n^{-7/2}\, \gamma^n\, n!$$

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COMPLEX PLANAR GRAPHS [K.– ŁUCZACK 09+]

Let $C(n, n+\ell)$ denote the number of all connected planar graphs on n vertices with $n+\ell$ edges where $\ell > 0$. Then for $\ell = o(n^{1/3})$

$$C(n, n+\ell) \sim \alpha\, \beta^{\ell}\, n^{n+3\ell/2-1/2}\, \ell^{-3\ell/2-3}$$

where α, β are analytic constants.

Number of planar graphs

[K.– ŁUCZACK 09+]

Let $m = n/2 + s$, $s = o(n)$.

- $s\, n^{-2/3} \to -\infty$:

$$P(n,m) \ \sim \ \alpha\, n^{n+2s}\,(n+2s)^{-n/2-s-1/2}\, e^{n/2+s-1/2}$$

- $s\, n^{-2/3} \to \lambda$, $\lambda \in (-\infty, \infty)$:

$$P(n,m) \ \sim \ \beta_\lambda\, n^{n-1/2}\,(n-2s)^{-n/2+s}\, e^{n/2-s+a\,\lambda\,(n-2s)^{-2/3}}$$

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$\cdots\cdots\cdots$ $P(n,m)$ for $m = an$, $a \in (1/2, 1)$, $m = n + o(n)$

Critical phase

[K.– Łuczack 09+]

Let $m = n/2 + s$, $s = o(n)$ and $L(n)$ be the number of vertices in the largest component in a random planar graph $\mathcal{P}(n, m)$. Then w.h.p.

- $s\, n^{-2/3} \to -\infty$: $L(n) = o(n^{2/3})$
- $s\, n^{-2/3} \to \lambda$, $\lambda \in (-\infty, \infty)$: $L(n) = \Theta(n^{2/3})$
- $s\, n^{-2/3} \to +\infty$: $L(n) \sim 2s$

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▷ In a random graph $\mathcal{G}(n,m)$: $L(n) \sim 4s$

▷ In a random forest $\mathcal{F}(n,m)$: $L(n) \sim 2s$

Critical phase

[K.– Łuczack 09+]

Let $m = n + t$, $t = o(n)$ and $R(n)$ be the number of vertices outside the giant component in a random planar graph $\mathcal{P}(n, m)$. Then w.h.p.

- $t\, n^{-3/5} \to -\infty$:

$$R(n) \ \sim \ \alpha(n + 2t)|t|^{-2/3} - t/2$$

- $t\, n^{-3/5} \to \lambda, \ \lambda \in (-\infty, \infty)$:

$$R(n) \ \sim \ \alpha_\lambda \, n^{3/5}$$

- $t\, n^{-3/5} \to +\infty$:

$$R(n) \ \sim \ \beta \, n^{3/2} t^{-3/2}$$

Part III: Matrix integral method

- Feynman diagram
- Fat graphs = maps
- Graphs embeddable on surfaces

Gaussian integral

The Gaussian integral is defined by

$$\langle f \rangle \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-\frac{x^2}{2}}\, dx.$$

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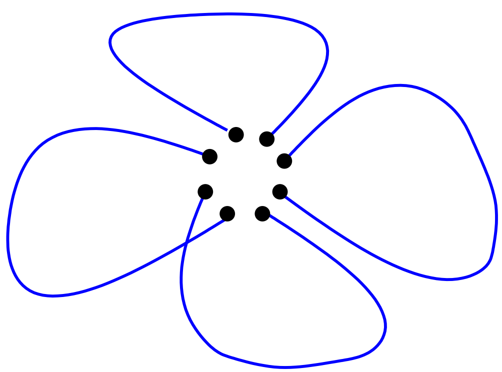
$$\langle x^n \rangle \;=\; \begin{cases} (n-1)!! & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$



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Using the source integral $\langle e^{xs} \rangle \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{2} + \frac{s^2}{2}}\, dx \;= e^{\frac{s^2}{2}}$

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$$\langle x^n \rangle = \langle \frac{d^n}{ds^n} e^{xs} \Big|_{s=0} \rangle = \frac{d^n}{ds^n} \langle e^{xs} \rangle \Big|_{s=0} = \frac{d^n}{ds^n} e^{\frac{s^2}{2}} \Big|_{s=0}$$

Gaussian matrix integral

Let $\mathcal{H}_N$ denote the set of all $N \times N$ Hermitian matrices $M = (M_{ij})$

Let $M_{ij} = x_{ij} + \vec{i}\, y_{ij}$ for $x_{ij}, y_{ij} \in \mathbb{R}$. Then $M_{ji} = x_{ij} - \vec{i}\, y_{ij}$.

$$M = \begin{pmatrix} x_{11} & & & & \\ & x_{22} & & & \\ & & & x_{ij} + \vec{i}\, y_{ij} & \\ & & \cdots & & \\ & x_{ij} - \vec{i}\, y_{ij} & & & \\ & & & & x_{NN} \end{pmatrix}$$

and $\mathcal{H}_N$ forms an N^2-dimensional vector space over the real numbers.

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Let $\mathcal{H}_N$ denote the set of all $N \times N$ Hermitian matrices $M = (M_{ij})$ and $dM = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}(M_{ij}) d\,\mathrm{Im}(M_{ij})$ the Haar measure on $\mathcal{H}_N$.

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$$\langle f \rangle = \frac{\int_{\mathcal{H}_N} f(M) e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} dM}{\int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} dM}.$$

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Using the source integral $\langle e^{\mathrm{Tr}(MS)} \rangle \;=\; e^{\frac{\mathrm{Tr}(S^2)}{2N}}$, we obtain

$$\langle M_{ij} M_{kl} \rangle = \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} \langle e^{\mathrm{Tr}(MS)} \rangle \Big|_{S=0} = \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} e^{\frac{\mathrm{Tr}(S^2)}{2N}} \Big|_{S=0} = \frac{\delta_{il}\delta_{jk}}{N}$$

Wick's Theorem

and

$$
\begin{aligned}
\langle M_{ij} M_{kl} M_{mn} \cdots \rangle \qquad &= \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} \frac{\partial}{\partial S_{nm}} \cdots \langle e^{\mathrm{Tr}(MS)} \rangle \Big|_{S=0} \\
&= \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} \frac{\partial}{\partial S_{nm}} \cdots e^{\frac{\mathrm{Tr}(S^2)}{2N}} \Big|_{S=0} \\
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The derivatives must be taken in pairs to get a non-zero contribution.

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[WICK 50]

Let $M \in \mathcal{H}_N$ and I be a multiset of elements of $N \times N$. Then

$$\begin{aligned}\langle \prod_{ij\in I} M_{ij}\rangle &= \sum_{\text{pairing } P\subset I^2}\ \prod_{(ij,kl)\in P}\langle M_{ij}M_{kl}\rangle\\ &= \sum_{\text{pairing } P\subset I^2}\ \prod_{(ij,kl)\in P}\frac{\delta_{il}\delta_{jk}}{N}\end{aligned}$$

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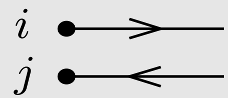
&= \sum_I c_I \sum_{\text{pairing } P \subset I^2} \prod_{(ij,kl) \in P} \frac{\delta_{il}\delta_{jk}}{N}
\end{aligned}$$

Pictorial interpretation

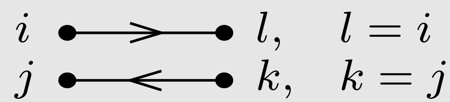
[Brézin–Itzykson–Parisi–Zuber 78; Zvonkin 97; Di Francesco 04]

Pictorial interpretation from $\langle M_{ij} M_{kl} \rangle = \frac{\delta_{il}\delta_{jk}}{N}$:

$M_{ij} \quad \longleftrightarrow$



$< M_{ij} M_{kl} >= \frac{1}{N} \quad \longleftrightarrow$



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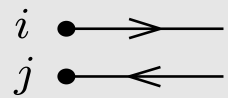

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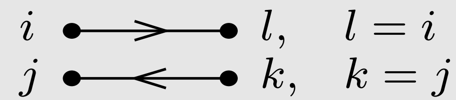

$$\mathrm{Tr}(M^n) \quad = \sum_{1 \leq i_1, i_2, \cdots, i_n \leq N} M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1}$$

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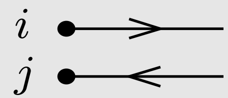

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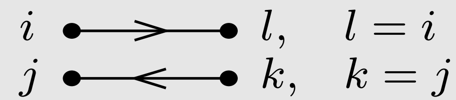

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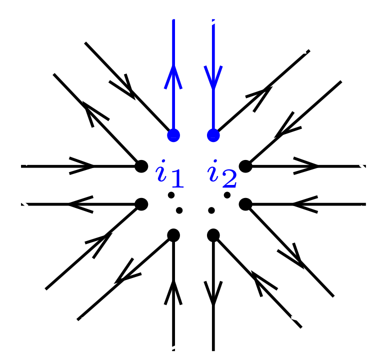

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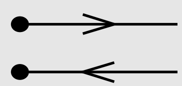

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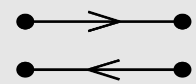


$l, \quad l = i$

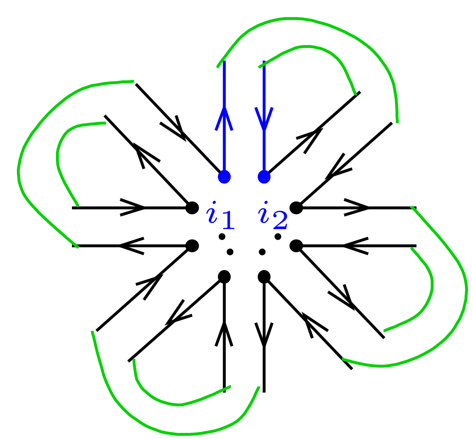
$k, \quad k = j$

$$\langle\, \mathrm{Tr}(M^n)\, \rangle \;=\; \langle \sum_{1 \le i_1, i_2, \cdots, i_n \le N} M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1} \rangle$$

$$= \sum_{1 \le i_1, i_2, \cdots, i_n \le N} \; \sum_{P} \; \prod_{(i_k i_{k+1}, i_l i_{l+1}) \in P} \frac{\delta_{i_k i_{l+1}} \delta_{i_l i_{k+1}}}{N}$$

where P is a partition of $\{i_1 i_2, i_2 i_3, \cdots, i_n i_1\}$ into pairs.

$< M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_n i_1} > \qquad \longleftrightarrow$



Fat graphs

[Brézin–Itzykson–Parisi–Zuber 78; Zvonkin 97; Di Francesco 04]

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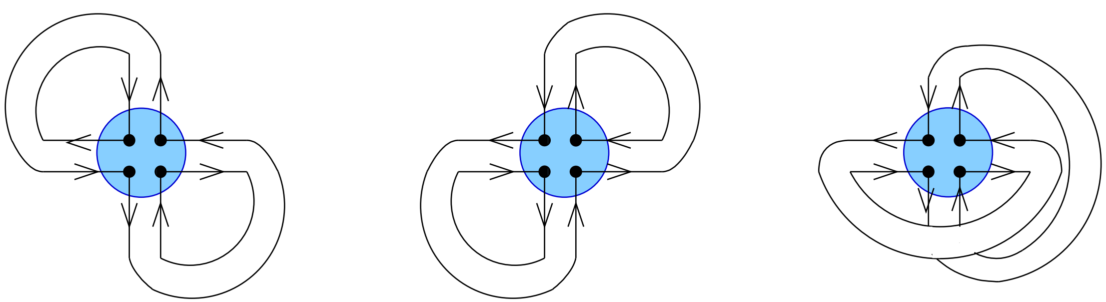
A pairing P with non-zero contribution to $\langle \operatorname{Tr}(M^n) \rangle$

$\Longleftrightarrow$ a fat graph with one island and $n/2$ fat edges .



Fat graphs

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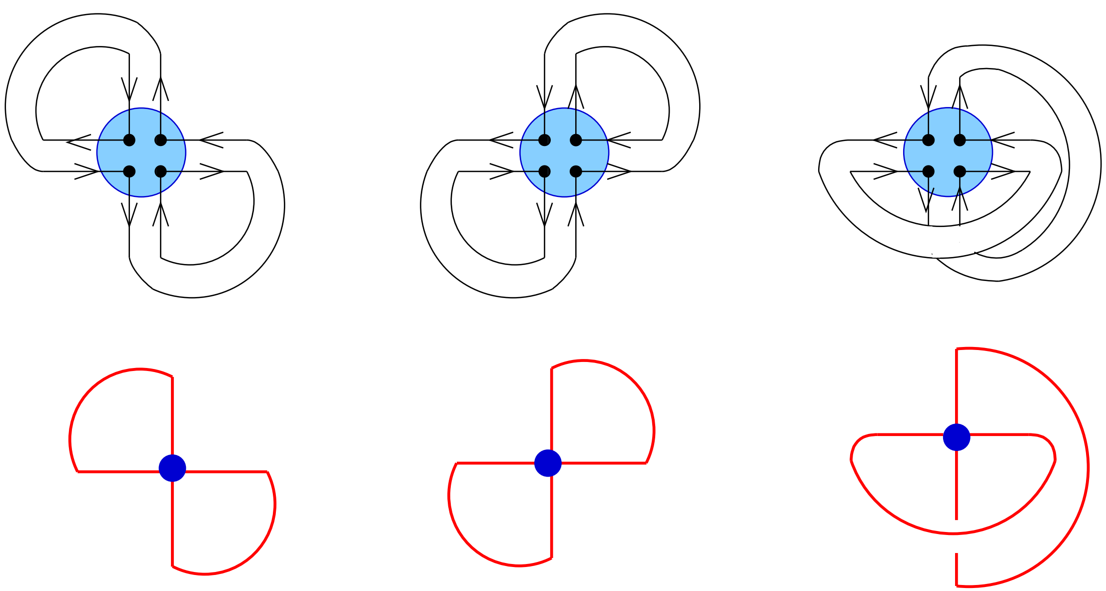
A pairing P with non-zero contribution to $\langle \operatorname{Tr}(M^n) \rangle$

$\iff$ a fat graph with one island and $n/2$ fat edges ordered cyclically.

(It defines uniquely an embedding on a surface: a map!)



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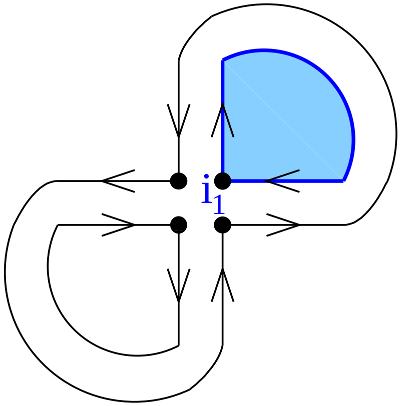
Let F be a fat graph with one island, $e(F)$ edges and $f(F)$ faces.

- The edges contribute $N^{-e(F)}$, since each edge contributes N^{-1}.
- The faces contribute $N^{f(F)}$, since each face attains independently any index from 1 to N.



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Thus

$$\langle \operatorname{Tr}(M^n) \rangle = \sum_F N^{-e(F)+f(F)}$$

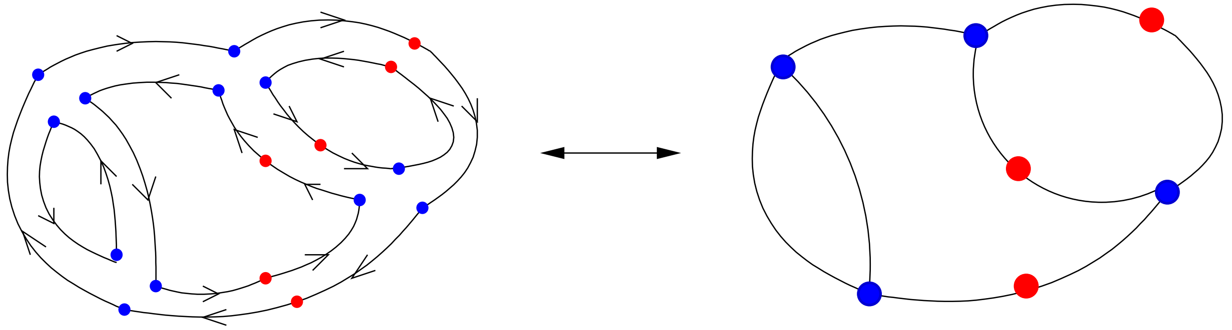
where the sum is over all fat graphs F with one island.

Maps on a surface

For example,

$$\left\langle \left[\mathrm{Tr}(M^3)\quad\right]^4 \left[\mathrm{Tr}(M^2)\quad\right]^3 \right\rangle = \sum_F N^{f(F)-e(F)} \qquad ,$$

where the sum is over all fat graphs (i.e. maps) F with four vertices of degree 3, and three vertices of degree 2.



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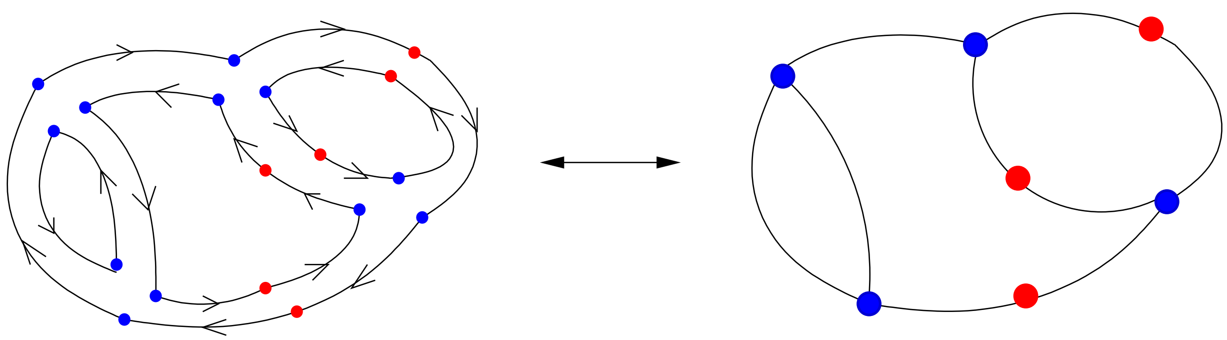

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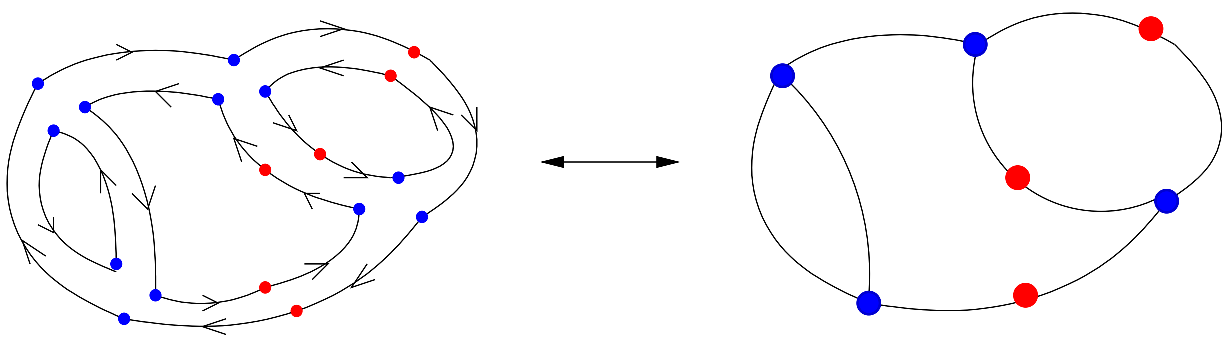

[Brézin–Itzykson–Parisi–Zuber 78; Zvonkin 97; di Francesco 04]

The generating function of maps on a surface can be formulated as a matrix integral of functions of traces.

Maps on a surface

[Brézin–Itzykson–Parisi–Zuber 78]

Consider a function ψ which maps $M \in \mathcal{H}_N$ to

$$\psi(M; z_i, i \in \mathbb{N}) := \exp(-N \sum_{i\in\mathbb{N}} \mathrm{Tr}\left(M^i\right) z_i/i),$$

where $\mathbb{N}$ is the set of all positive integers and z_i's are formal variables.

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$$\log \langle\psi\rangle \ '=' \ \sum_{g\geq 0} N^{2-2g} \sum_{(n_1,\cdots,n_k)\in\mathbb{N}^k} \prod_{i=1}^{k} M_g(n_1,\cdots,n_k) \frac{(-z_i)^{n_i}}{n_i!}$$

where $M_g(n_1,\cdots,n_k)$ denotes the number of connected maps with genus g and n_i vertices of degree i for $1 \leq i \leq k$.

The expression $f(z) \ '=' \ g(z)$ means that all the derivatives of f, g are equal when the variable z is set to zero.

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Partial results on the convergence of the above generating function

[Ercolani–McLaughlin 03; Guionnet 04]

Maps on a surface

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$$[z^n] \lim_{N\to\infty} \frac{1}{N^2} \log \langle\psi\rangle \Big|_{z_i=z}$$ is the number of connected planar maps on n vertices

Graphs on a surface

Matrix integral for graphs embeddable on a surface?

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Graphs on a surface

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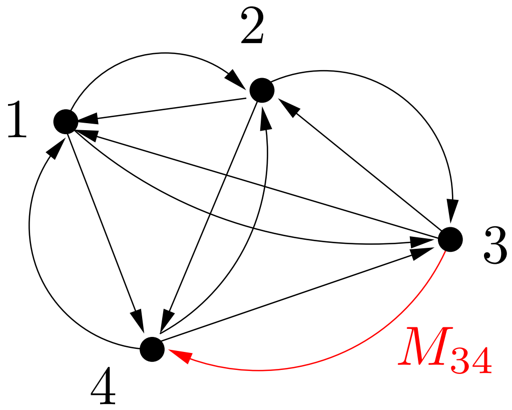
$M \in \mathcal{H}_4$ $\longrightarrow$



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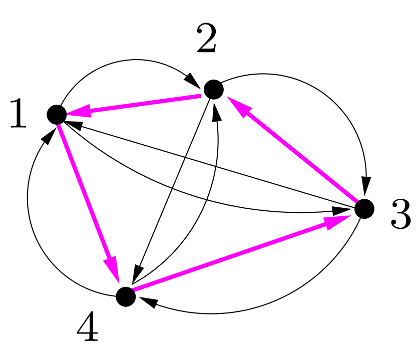

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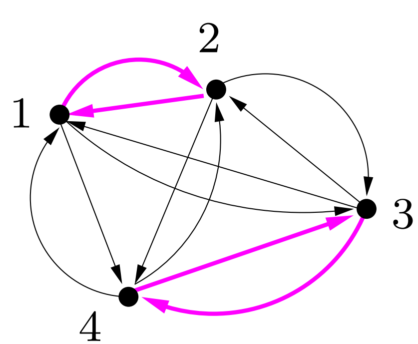

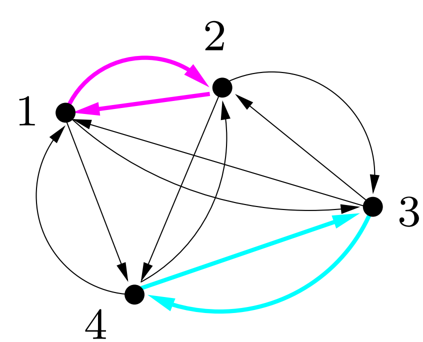

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$$\rho(M; y, z) := \sum_{A \subset N \times N \text{ eulerian}} \left(\prod_{e \in A} M_e\right) \prod_{C\text{: component of } A} \frac{(Nz)^{|C|/2+1} - Nzy^{|C|/2}}{Nz - y}$$

Graphs on a surface

[K.–Loebl 09]

Let $s(N) \leq \frac{N}{2}$ be a function in N tending to ∞ slower than N when $N \to \infty$.

$$\exp\left(\sum_{n\geq 1} \sum_{r \geq 0} [\, p(n,r) + e_1(N,n,r)\,] \frac{z^r y^{n-2}}{n!} \right) \;{}'\leq_{s(N)}{}'$$

$$e^{-N^2} \langle \rho(M; y, z) \rangle \quad {}'\leq_{\frac{N}{2}}{}' \; \exp\left(\sum_{n\geq 1} \sum_{r \geq 0} [\, p(n,r) + e_2(N,n,r)\,] \frac{z^r y^{n-2}}{n!} \right),$$

where $p(n,r)$ denotes the number of labelled connected graphs on $n \leq N$ vertices which have planar embeddings with r faces, and $e_1(N,n,r), e_2(N,n,r)$ are functions of type $O(N^{-1})$.

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$$\sum_{n\geq 1}\sum_{r\geq 0} p(n,r)\frac{y^{n-2}}{n!} = \qquad \lim_{N\to\infty}\frac{1}{N^2}\log\left\langle\rho\right\rangle\Big|_{z=1}$$

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$$[y^{n-2}]\sum_{n\geq 1}\sum_{r\geq 0} p(n,r)\frac{y^{n-2}}{n!} = [y^{n-2}]\lim_{N\to\infty}\frac{1}{N^2}\log\langle\rho\rangle\Big|_{z=1} \quad ??? \ \sim c\, n^{-7/2}\,\gamma^n$$

Open questions

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[K.– Łuczack 09+]

Let $\mathcal{P}(n,m)$ be a random planar graph with n vertices and m edges and $\epsilon > 0$.

- If $m \leq (1-\epsilon)n$, w.h.p. $\chi(\mathcal{P}(n,m)) = 3$.
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- What is the threshold for the appearance of K_4 in $\mathcal{P}(n,m)$?

$\triangleright$ heuristic: $m = n + O(n^{7/9})$