Concrete Mathematics

Lecturer: Dijana Kreso
E-mail: kreso@math.tugraz.at
Due: June 02, 2016

Let φ be the Euler function.

1. (2 points) Find the sequence $(a_n)_{n\geq 0}$ whose ordinary generating function is $A(x) = 3/(1+5x)+(x/(1+7x))^4$.

2. (2 points) Let D_n be the number of permutations σ of the set $\{1,2,\ldots,n\}$ without fixed points, i.e. such that $\sigma(i) \neq i$ for all i. We showed that the exponential generating function for the sequence $(D_n)_{n\geq 0}$ is $e^{-x}/(1-x)$. Show that $\lim_{n\to\infty} D_n/n! = 1/e$.

3. (a) (***2 extra points)** Let P_n be the number of permutations σ of the set $\{1,2,\ldots,n\}$ such that $\sigma(2i-1) > \sigma(2i)$, $\sigma(2i) < \sigma(2i+1)$ for all $1 \leq i \leq (n-1)/2$. Prove or disprove that that the sequence $(P_n)_{n\geq 0}$ satisfies
$$P_{n+1} = \sum_{0\leq k\leq n,\ k \text{ even}} \binom{n}{k} P_k P_{n-k}. \tag{1}$$
   (b) (*2 points) Show that the exponential generating function $P(x)$ of the sequence $(P_n)_{n\geq 0}$ given by (1) satisfies $2P'(x) = P(x)(P(x)+P(-x))$.
   (c) (*1 points) Show that $P(x) = \tan x + 1/\cos x$.

4. (2 points) Let a_n be the number of positive integers with n digits whose digits are in the set $\{2,3,7,9\}$ and which are divisible by 3. Find a_n.

5. (1 point) Find the Dirichlet generating function of the sequence $(\varphi(n))_{n\geq 1}$.

6. (2 points) Let n be a positive integer. Let P_k be the number of pairs (a,b) of nonnegative integers such that $ka+(k+1)b = n+1-k$. Find $P_1+P_2+\cdots+P_{n+1}$.

7. Let $N(m,n)$ be the number of ways to string beads of n different colors into a necklace of length m, where we consider two necklaces to be the same if one can be obtained from another by rotation, and $N_r(m,n)$ be that number, where we consider two necklaces to be the same if one can be obtained from another by rotation or vertical axis reflection. We showed that $mN(m,n) = \sum_{d|m} \varphi(d) n^{m/d}$.
   (a) (*2 points) Show that $\sum_{d|m} \varphi(d) n^{m/d} \equiv 0 \pmod{m}$, $m,n \in \mathbb{N}$, using number-theoretic arguments.
   (b) (*1 points) Find $N_r(10,2)$.

8. (***1 extra points)** For an integer n and a prime p, show that $n^{p^k} \equiv n^{p^{k-1}} \pmod{p^k}$, for $k \in \mathbb{N}$.