Concrete Mathematics

Lecturer: Dijana Kreso
E-mail: kreso@math.tugraz.at
Due: May 03, 2016

Let μ be the Möbius function and φ be the Euler function.

1. (2 points) Let A_n be the number of positive integers up to n which are square-free. Show that

$$\lim_{n\to\infty}\frac{A_n}{n}=\lim_{n\to\infty}\left(\sum_{k\le\lfloor\sqrt{n}\rfloor}\frac{\mu(k)}{k^2}-\frac{1}{n}\sum_{k\le\lfloor\sqrt{n}\rfloor}\left\{\frac{n}{k^2}\right\}\mu(k)\right)=\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2}=\frac{6}{\pi^2}.$$

   where $\{x\}$ denotes the fractional part of x. i.e. $\{x\}=x-\lfloor x\rfloor$. Recall that we showed in lectures that the last equality holds. **Hint:** Show that $\mu^2(n)=\sum_{k^2|n}\mu(k)$.

2. (2 points) Let $(a_n)_{n\ge1}\subseteq\mathbb{N}$ be a sequence of positive integers such that $a_{m+n}\ge a_m\cdot a_n$ for all $m,n\in\mathbb{N}$. Show that there exists $\lim_{n\to\infty}a_n^{1/n}$ (which can be ∞).

3. (2 points) Suppose we have beads of n different colors. Let $N(m,n)$ be the number of different ways to string them into circular necklaces of length m. Show that

$$mN(m,n)=\sum_{k=0}^{n}n^{\gcd(k,m)}.$$

   Recall that in lectures we showed that $\sum_{k=0}^{n}n^{\gcd(k,m)}=\sum_{d|m}\varphi(d)n^{m/d}$.

4. (2 points) Find a closed formula for the number a_n of sequences of 0's and 1's of length n with no two consecutive 1's and for which a run of 0's always has length 2 or 3 (including possibly at the beginning or end of the sequence).

5. (1 point) Find a closed formula for a_n if $a_0=0$, $a_1=3$, $a_2=-1$ and $a_{n+3}=9a_{n+2}-15a_{n+1}-25a_n$, $n\ge0$.

6. (2 points) Let $(F_n)_n$ denote the sequence of Fibonacci numbers. Show that if $m\mid n$, then $F_m\mid F_n$. Conclude that if $n\ge5$ and F_n is a prime, then n is a prime.

7. (1 point) Let $a(r,n)$ denote the number of solutions of the equation $x_1+x_2+\cdots+x_r=n$ in nonnegative integers x_i. Show that $a(r,n)=\binom{n+r-1}{r-1}$.

8. (2 points) Show that there exists a sequence $(a_n)_{n\ge0}\subseteq\mathbb{N}$ in which every positive integer appears exactly once and for any $n\in\mathbb{N}$ we have that $a_0+a_1+\cdots+a_{n-1}$ is a multiple of n.

9. (1 point) Find a closed formula for a_n if $a_0=1$ and $a_{n+1}=2a_n+\sqrt{3a_n^2-2}$, $n\ge0$.