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Exercises in Number Theory MAT.413

Due: November 09, 2015

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1. (1 point) Find the continued fraction expansion of $3+\sqrt{15}$.

2. Let α be irrational, let $[a_0, a_1, \ldots]$ be the continued fraxction expansion of α and let the n-th convergent of α be p_n/q_n with $\gcd(p_n, q_n) = 1$.

   (a) (1 point) Show that $q_n \geq F_n$, where F_n denotes the n-th Fibonacci number [1] .

   (b) (1 point) Show that if $a_0 > 0$, then $p_n \geq \phi^n$, where $\phi = (1+\sqrt{5})/2$.

3. (2 points) Show that $\sqrt{d^2-d} = [d-1, \overline{2, 2d-2}]$ for all integers $d \geq 2$.

   (Recall that an infinite continued fraction $[a_0, a_1, a_2, \ldots]$ such that for integers $k \geq 0$, $m \geq 1$ we have $a_{m+n} = a_n$ for all $n \geq k$ is denoted by $[a_0, a_1, \ldots, a_{k-1}, \overline{a_k, \ldots, a_{k+m-1}}]$.)

4. Let α be irrational and let p_{n-2}/q_{n-2} , p_{n-1}/q_{n-1} and p_n/q_n, with $n \geq 2$, be three consecutive convergents of α.

   (a) (2 points) Show that for at least one $i \in \{n-1, n\}$ the following holds
   $$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{2q_i^2}.$$

   (b) (2 points) Show that for at least one $i \in \{n-2, n-1, n\}$ the following holds
   $$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{\sqrt{5}q_i^2}.$$

5. (2 points) Suppose that α is a real irrational number which is a root of the nonzero polynomial $f(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$. Let $D = b^2 - 4ac$ be the discriminant of f. If C is a real number satisfying $C > \sqrt{D}$, show that there are only finitely many pairs (p, q) with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, that satisfy
   $$\left|\alpha - \frac{p}{q}\right| < \frac{1}{Cq^2}.$$

6. (1 point) Note that from Problem 4 (b) it follows that for an irrational α there are infinitely many rational numbers p/q such that
   $$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}. \tag{1}$$
   Show that $\sqrt{5}$ can not be replaced by a constant $c > \sqrt{5}$, that is show that for any $c > \sqrt{5}$ there exists irrational number α such that there are only finitely many rational numbers p/q for which (1) with $\sqrt{5}$ replaced by c holds.

7. (2 points) Show that
   $$\alpha = \sum_{n=1}^{\infty} \frac{1}{2^{n!}}$$
   is transcendental.

[1] Fibonacci numbers are defined by: $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.