Dijana Kreso (kreso@math.tugraz.at)

Exercises in Number Theory MAT.413

Due: November 16, 2015

Graz University of Technology

Note that there are two pages of this exercise sheet!

1. (a) (1 point) Find all solutions in integers x, y of the equation $x^3 + y^3 = 1729$.

   (b) (1 point) Let m be a positive integer. Show that all integers x, y such that $x^3 + y^3 = m$, satisfy $\max\{|x|, |y|\} \le 2\sqrt{m/3}$.

   (c) (2 points) If a, b, c are nonzero integers, show that the equation $ax^3 + by^3 = c$ has only finitely many solutions in integers x, y.

2. (2 points) Roth's theorem implies that for any irrational algebraic α and $\xi > 2$ there exists a constant $c(\alpha, \xi) > 0$ such that
$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha,\xi)}{q^{\xi}}.$$
for all rational numbers p/q, $q > 0$. However, Roth's proof does not give a method for finding the constant $c(\alpha, \xi)$. Use Baker's result which states that for all rational numbers p/q with $q > 0$
$$\left|\sqrt[3]{2} - \frac{p}{q}\right| > \frac{1.36 \cdot 10^{-6}}{q^{2.954}},$$
to show that all integers x, y such that $x^3 - 2y^3 = 1$, satisfy $\max\{|x|, |y|\} < 10^{127}$.

3. By Schmidt's subspace theorem, for $n \ge 2$, linearly independent linear forms $L_1, \ldots, L_n$ in n variables with algebraic coefficients, and for $\delta > 0$, the set of solutions of the inequality
$$0 < |L_1(x) \cdots L_n(x)| \le \|x\|^{-\delta} \quad \text{with } x \in \mathbb{Z}^n \tag{1}$$
is contained in a union of finitely many proper linear subspaces of $\mathbb{Q}^n$.

   (a) (1 point) Show that if $n = 2$, then there are only finitely many $x \in \mathbb{Z}^2$ such that (1) holds.

   (b) (2 points) Show that when $n \ge 3$ there can be infinitely many $x \in \mathbb{Z}^n$ such that (1) holds, by showing that this is the case when $n = 3$, $0 < \delta < 1$ and L_1, L_2, L_3 are given by
$$L_1(x) = x_1 + \sqrt{2}x_2 + \sqrt{3}x_3, \quad L_2(x) = x_2 - \sqrt{2}x_2 + \sqrt{3}x_3, \quad L_3(x) = x_1 - \sqrt{2}x_2 - \sqrt{3}x_3. \tag{2}$$

4. (2 points) Deduce Roth's theorem from Schmidt's subspace theorem.

5. Let S be a finite nonempty set of primes. By an S-unit we mean a rational number whose both numerator and denominator (when written in lowest terms) are not divisible by primes outside S. Let $m := \gcd\{p - 1 : p \in S\}$.

   (a) (1 point) Show that if there exist S-units $u_1, \ldots, u_k > 0$ such that $u_1 + \cdots + u_k = 1$, then $k \equiv 1 \pmod{m}$.

   (b) (1 point) Show that for fixed $k \equiv 1 \pmod{m}$ there exists at most finitely many solutions of the equation $u_1 + \cdots + u_k = 1$ in S-units $u_1, u_2, \ldots, u_k > 0$.

(c) (2 points) Show that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $k \equiv 1 \pmod{m}$ there exist S-units $u_1, \ldots, u_k > 0$ such that $u_1 + \cdots + u_k = 1$.

(d) **(2 extra points)** Assume that S consists of a single prime, i.e. $S = \{p\}$. Show that there exist S-units u_i's such that $u_1 + \cdots + u_k = 0$ and that this sum has no proper zero subsum if and only if $k \equiv 2 \pmod{p-1}$.