Concrete Mathematics

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Let μ be the Möbius function and φ be the Euler function.

1. (a) (1 point) Show that the Euler function φ is multiplicative.

   (b) (2 points) Show that $\sum_{d|n} \varphi(d) = n$ for $n \in \mathbb{N}$.

   (c) (1 point) Determine all positive integers n such that $\varphi(n) = 10$.

2. (2 points) Show that if $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N} \to \mathbb{C}$ are such that $g(n) = \sum_{d|n} f(d)$ is multiplicative, then f is multiplicative. (We showed the converse in lectures).

3. (2 points) Show that for any real $x \geq 1$ we have
$$\sum_{n \leq x} \mu(n) \cdot \left\lfloor \frac{x}{n} \right\rfloor = 1.$$

4. (2 points) Assume that $f : \mathbb{N} \to \mathbb{C}$ and $F : \mathbb{N} \to \mathbb{C}$ are such that
$$f(n) = \sum_{d|n} \mu(d) F(n/d)$$
for all n. Show that then $F(n) = \sum_{d|n} f(d)$, $n \in \mathbb{N}$.

   (In lectures, we showed the converse of this statement which is called Möbius inversion theorem.)

5. Let n be a positive integer and let $\tau(n)$ denote the number of positive divisors of n.

   (a) (1 point) Show that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes and $\alpha_1, \ldots, \alpha_k$ positive integers, then
$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1).$$

   (b) (2 points) Show that $\sum_{n \leq x} \tau(n) = x \ln(x) + O(x)$.

6. (2 points) Let $a, b \in \mathbb{N}$ be such that $b^n + n$ is a multiple of $a^n + n$ for all positive integers n. Prove that $a = b$.