Concrete Mathematics

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1. (2 points) Find a closed formula for the maximum number a_n of regions defined by n lines in the plane? (Note that $a_1 = 2$, $a_2 = 4$, $a_3 = 7$.)

2. (2 points) Find a closed formula for the sequence $(a_n)_{n\geq 0}$ given by $a_0 = 2$, $a_n = 3a_{n-1} - 4n$ for $n \geq 1$.

3. (3 points) Let $(a_n)_{n\geq 0}$ be the sequence defined by $a_0 = 0$, $a_1 = 1$ and $a_{n+1} - 3a_n + a_{n-1} = 2\cdot(-1)^n$. Prove that a_n is a perfect square for all $n \geq 0$.

4. (2 points) Assume that for some $n \in \mathbb{N}$ there are sequences of positive numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ such that the multisets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are not the same.

   Assume further that the sums $a_i + a_j$ with $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$, and the sums $b_i + b_j$ with $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$, are the same up to permutation (i.e. that the multiset (of size $\binom{n}{2}$) of pairwise sums of a_i's is equal to the multiset (of size $\binom{n}{2}$) of pairwise sums of b_i's).

   Prove that n is a power of two.

5. (2 points) Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers. Find its ordinary generating function.

6. (2 points) Let a_n be the number of sequences of 0's, 1's and 2's of length n that contain two consecutive digits that are the same. Find the ordinary generating function for $(a_n)_{n\geq 0}$.

7. (2 points) Let a_n be the number of distinct ways to choose n pieces of fruit among apples, bananas, oranges and pears such that the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is either 0 or 1. Find a_n.