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Exercises in Number Theory MAT.413

Due: October 19, 2015

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1. (1 point) Using the Euclidean algorithm, solve the congruence relation $555x \equiv 15 \pmod{5005}$.

2. (2 points) Solve the system of congruences

$$x \equiv 3 \pmod{10}, \quad x \equiv 8 \pmod{15}, \quad x \equiv 5 \pmod{84}.$$

3. (2 points) Let p be a positive integer greater than 1. Show that $(p-1)!+1$ is divisible by p if and only if p is a prime number.

4. (2 points) Let p be a prime number. Show that $(p-1)!+1 = p^k$ for some positive integer k if and only if $p \in \{2,3,5\}$.

5. (1 point) Show that there do not exist positive integers m,n,k such that $4mn - m - n = k^2$.

6. Let ϕ be Euler function and let μ be Möbius function.
   (a) (1 point) Determine all positive integers n such that $\phi(n) = 12$.

   (b) (2 points) Show that for every positive integer n the following holds:

$$\phi(n) = n \sum_{d \in \mathbb{N}, d \mid n} \frac{\mu(d)}{d}.$$

7. (2 points) Let $p \geq 2$ be a prime number such that $p \equiv 2 \pmod{3}$. Show that in the set

$$\{y^2 - x^3 - 1 \mid x, y \in \mathbb{Z}, 0 \leq x, y \leq p-1\}$$

there are less or equal than p elements which are divisible by p.

8. (2 points) Find all positive integers n such that $2^n - 1$ is divisible by 3 and $\frac{2^n-1}{3}$ divides $4m^2+1$ for some integer m.