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Exercises in Number Theory MAT.413

Due: January 25, 2016

Graz University of Technology

Note that there are two pages of this exercise sheet!

Let μ be the Möbius function, φ be the Euler function and ζ the Riemann zeta function.

1. (a) (1 point) Show that $\sum_{d|n} \mu(d) = 0$ for $n \in \mathbb{N}$ with $n > 1$.

   (b) (1 point) Show that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$ for any $n \in \mathbb{N}$.

   (c) (2 points) Show that for any real $x \geq 1$ we have
$$\sum_{n \in \mathbb{N},\ n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1.$$

   (d) (2 points) Show that for any real $x \geq 1$ we have
$$\left| \sum_{n \in \mathbb{N},\ n \leq x} \frac{\mu(n)}{n} \right| \leq 1.$$

2. (1 point) Show that $\sum_{d|n} \varphi(d) = n$ with $d, n \in \mathbb{N}$.

3. (1 point) Show that the product of Dirichlet series of two arithmetical functions f and g is the Dirichlet series of the convolution of f and g.

4. (2 points) Show that for any complex number s with the real part > 1 we have
$$\frac{1}{\zeta(s)} = \prod_{\text{prime } p} (1 - p^{-s}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

5. (2 points) Show that for any complex number s with the real part > 2 we have
$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}.$$

6. (2 points) Let $a : \mathbb{N} \to \mathbb{C}$ be an arbitrary sequence of complex numbers, and let
$$A(x) := \sum_{n \leq x} a(n), \quad A(0) = 0,$$
   be the associated summatory function. Show Abel's identity, which states that for any continuously differentiable function $f : [x, y] \to \mathbb{C}$, $0 < y < x$ we have
$$\sum_{y < n \leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t) dt.$$

7. (2 points) Let p_n be the n-th prime. Show that the Prime Number Theorem implies
$$\lim_{n \to \infty} \frac{p_n}{n \log n} = 1.$$

8. **(2 extra points)** Show that the series

$$\sum_{\text{p prime}} \frac{1}{p}$$

diverges.

Hint: Either use Abel's identity and Prime Number Theorem or assume to the contrary and deduce that the series $\sum \log(1-p^{-s})$ converges uniformly for $1 \le \sigma \le 2$, where $\sigma := \Re(s)$ is the real part of s.

9. **(2 extra points)** Show that $\sum_{n\le x} \mu(n) = o(x)$, that is that $\sum_{n\le x} \mu(n)/x \to 0$ when $x \to \infty$.

10. **(1 extra points)** Let p_n be the n-th prime. Assume that

$$\lim_{n\to\infty} \frac{p_n}{n\log n} = 1,$$

and show that the Prime Number Theorem holds.