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LAGRANGE'S EQUATION WITH ALMOST PRIME VARIABLES LYING IN A SHORT INTERVAL

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Abstract

We prove that every sufficiently large integer satisfying a natural congruence condition can be represented as a sum of four squares of almost primes that are close to each other.
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The famous theorem of Lagrange states that for every natural number N there exist integers $x_1, \ldots, x_4$ such that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = N. \tag{1}$$

A popular conjecture asserts that if N is sufficiently large and satisfies $N \equiv 4 \pmod{24}$ than it can be represented in the form (1) with prime variables $x_1, \ldots, x_4$. This conjecture has not been proved so far, but several authors have considered approximations to it.

Greaves [4], Plaksin [7] and Shields [8] studied equation (1) with two prime and two integer variables and proved that it is solvable if N is sufficiently large and satisfies a natural congruence condition. Brüdern and Fouvry [2] proved that every sufficiently large integer $N \equiv 4 \pmod{24}$ can be represented in the form (1) with almost prime variables of type P_{34}. (As usual we denote by P_r any integer with at most r prime factors, counted according to multiplicity, and refer to such a number as an almost prime of order r.)

Heath-Brown and Tolev [6] considered the equation (1) with the same conditions on N, but with stronger multiplicative restrictions on the variables. Theorem 1 in [6] states that (1) has a solution $x_1, \ldots x_4$ such that x_1 is a prime and $x_j = P_{101}$ for $j = 2, 3, 4$. Respectively, Theorem 2 in [6] asserts that (1) has a solution with $x_j = P_{25}$

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for $j = 1, 2, 3, 4$. Later TOLEV [9] established that (1) has a solution with one prime and three almost primes of type P_{80} and, respectively, with four almost primes of type P_{21}.

The most recent result in this direction is due to BLOMER and BRÜDERN [1]. They proved that every sufficiently large integer N satisfying $N \equiv 3 \pmod{24}$ and $N \not\equiv 0 \pmod{5}$ can be represented as a sum of three almost primes of type P_{521}. If, in addition, N is squarefree, then the variables can be taken of the type P_{371}. It is clear that this result implies a version of Theorem 1 in [6], but with almost primes of higher order.

We should mention that the problem of representations of integers as sums of five or more squares of primes was settled in 1938 by HUA [5]. He proved that all large integers N such that $N \equiv 5 \pmod{24}$ can be represented in the form

$$p_1^2 + \cdots + p_5^2 = N, \tag{2}$$

where $p_1, \ldots, p_5$ are primes. Several papers published recently were devoted to equation (2) with prime variables that are close to each other. The strongest result of this type, available in the literature at present, is due to GUANGSHI [3]. In 2005 he proved that equation (2) has a solution in primes satisfying

$$\left| \sqrt{N/5} - p_i \right| < N^{\frac{1}{2} - \frac{1}{35} + \varepsilon}, \qquad i = 1, \ldots, 5,$$

where $\varepsilon > 0$ is arbitrarily small.

In the present paper we state a result concerning the solvability of (1) in almost prime variables which are close to each other. More precisely, we have the following:

Theorem 1. Suppose that N is a sufficiently large integer satisfying $N \equiv 4 \pmod{24}$, θ and γ are constants such that

$$\frac{15}{34} < \theta < \frac{1}{2}, \qquad \gamma = \frac{50.08}{34\theta - 15}. \tag{3}$$

Then equation (1) has a solution in integers x_1, x_2, x_3, x_4 each of which has at most γ prime factors and such that

$$\left| \frac{1}{2}\sqrt{N} - x_i \right| < N^{\theta}, \qquad i = 1, 2, 3, 4.$$

This theorem is a short variant version of Theorem 2 in [6]. If we apply the refined sieve method of Tolev [9], then we will be able to replace the constant 50.08 by a slightly smaller one.

To prove our theorem we consider the sum

$$\Gamma = \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=N \\ (x_1x_2x_3x_4, B)=1}} \omega(x_1)\omega(x_2)\omega(x_3)\omega(x_4), \tag{4}$$

where (u, v) denotes the greatest common divisor of the integers u and v,

$$B = \prod_{p < N^{\frac{1}{2\gamma}}} p \tag{5}$$

(the product is taken over the primes) and $\omega(x)$ is a suitably chosen smooth and non-negative function such that $\omega(x) > 0$ exactly when $x \in \left(\frac{1}{2}\sqrt{N} - N^\theta, \frac{1}{2}\sqrt{N} + N^\theta\right)$.

In order to find a non-trivial lower bound for Γ we proceed as in Section 5 in [6]. First we show that if $d_1, \dots, d_4$ are squarefree numbers then the sum

$$\Phi(N; d_1, \dots d_4) = \sum_{\substack{x_1^2+x_2^2+x_3^2+x_4^2=N \\ x_i \equiv 0 \pmod{d_i},\ i=1,2,3,4}} \omega(x_1)\omega(x_2)\omega(x_3)\omega(x_4)$$

can be approximated (on average with respect to $d_1, \dots, d_4$ up to a certain bound) by the quantity

$$M(N; d_1, \dots d_4) = \frac{\kappa(N)\, N^{2\theta}\, \Sigma(N, d_1, d_2, d_3, d_4)}{d_1 d_2 d_3 d_4}.$$

Here $\Sigma(N, d_1, d_2, d_3, d_4)$ is the 'singular series' for our additive problem. It is defined by formula (2.45) in [2] (and also by formula (339) of [6]). Respectively, $\kappa(N)$ is the 'singular integral', which is the analogue of the quantity κ_1 from formula (24) in [6]. (However, the properties of $\kappa(N)$ are slightly different; in particular we have $N^{\theta-\frac{1}{2}} \ll \kappa(N) \ll N^{\theta-\frac{1}{2}}$.)

We denote

$$E(N, d_1, d_2, d_3, d_4) = \Phi(N, d_1, d_2, d_3, d_4) - M(N, d_1, d_2, d_3, d_4)$$

and

$$\mathcal{E}(N, D) = \sum_{d_1, \dots, d_4 \le D} \mu^2(d_1) \dots \mu^2(d_4)\, |E(N, d_1, d_2, d_3, d_4)|$$

(as usual $\mu(n)$ stands for the Möbius function). We establish the following:

Proposition 2. If $D = N^\delta$, where $\delta < \dfrac{34\theta - 15}{32}$, then for some $\varepsilon > 0$ we have

$$\mathcal{E}(N, D) \ll N^{3\theta - \frac{1}{2} - \varepsilon}.$$

To prove the Proposition we apply the Kloosterman form of the circle method. We proceed as in the proof of Proposition 3 in [6], but now the arguments are more complicated because of the choice of the function $\omega(x)$.

From this Proposition one can obtain the proof of the Theorem using the vector sieve. Working as in Section 3 in [2] we prove that if θ and γ satisfy (3) then $\Gamma > 0$. Using this inequality, the definitions of Γ and B, given respectively by (4) and (5), and having in mind the properties of $\omega(x)$ we conclude that (1) has a solution satisfying the requirements imposed in the Theorem.

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