Class number one problem for real quadratic fields of certain type

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Gauss conjectures:

1. If $d < 0$ and $|d| \to \infty$, then $h(d) \to \infty$. (solved)
2. There are infinitely many $d > 0$, for which $h(d) = 1$. (open)

Dirichlet class number formula

For positive d we have

$$h(d)\log\epsilon_d = d^{1/2}L(1,\chi_d),$$

where ϵ_d is the fundamental unit of K and $\chi_d = \left(\frac{\cdot}{d}\right)$.

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Siegel's theorem

$L(1,\chi_d) \gg_\epsilon |d|^{-\epsilon}$.

If ϵ_d is small, then $h(d)\to\infty$.

Richaud-Degert (R-D) discriminants:

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$\Rightarrow$ R-D class number tends to infinity with $d \to \infty$.

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Biró solves the class number one problem in the following cases:

Theorem (Biró 2003)

- *Yokoi's conjecture is true : Let* $d = n^2 + 4$. *Then* $h(d) > 1$ *if* $n > 17$;
- *Chowla's conjecture is true : Let* $d = 4n^2 + 1$. *Then* $h(d) > 1$ *if* $n > 13$.

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Until now not known results for two-parameter R-D discriminants without GRH.

We use Biró's methods, without any computer work, to obtain

Theorem

If $d = (an)^2 + 4a$ *is square-free for* a *and* n *– odd positive integers such that* 43.181.353 *divides* n*, then* $h(d) > 1$*.*

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The parameter 43.181.353 :

$$h(-43.181.353) = 2^9.3\,.$$

Main identity

$$qh(-q)h(-qd) = \frac{n}{6}\left(a + \left(\frac{a}{q}\right)\right)\prod_{p|q}(p^2 - 1)\,,$$

where $q \equiv 3 \pmod 4$ is squarefree, $q \mid n$, $(q, a) = 1$ and $h(d) = h((an)^2 + 4a) = 1$.

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! Take $h(-q)$ with big 2-part

Another possible choice of parameter is 5.359.541:

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If $d = (an)^2 + 4a$ *is square-free for* a *and* n *– odd positive integers such that* 5.359.541 *divides* n*, then* $h(d) > 1$.

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If $n \geq 1$ is integer, then there are infinitely many imaginary quadratic fields with discriminant of only two prime divisors and an element of order 2^n in their class group.

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Using application of the circle method from Balog&Ono[1]:

Theorem

Let $n \geq 1$ be an integer. There are infinitely many imaginary quadratic fields with discriminant of only three prime divisors, each of which is congruent to 3 modulo 8, such that in their class group there is an element of order 2^n.

Theorem

There exists an infinite family of parameters q, which have exactly three distinct prime factors, with the following property. If $d = (an)^2 + 4a$ is square-free for a and n – odd positive integers, and q divides n, then $h(d) > 1$.

Problem

Solve the class number one problem for all R-D discriminants of square-free $d = (an)^2 + 4a$, *a and n – odd positive integers.*

- Partial zeta function at 0 after Biró&Granville[3] for the particular R-D discriminant.
- Results with computer for some residue classes of a, computer work on pregress.

Theorem

Let Δ, ℓ be positive integers for which $16\ell^2 \mid \Delta$ and $(15, \Delta) = 1$. Let $\mathcal{P}_1, \mathcal{P}_2$ be infinite sets of primes satisfying Siegel-Walfisz condition for Δ such that every $p \in \mathcal{P}_1$ is $\equiv -5 \pmod{\Delta}$ and every $r \in \mathcal{P}_2$ is $\equiv 3 \pmod{\Delta}$. If $R_d(X)$ denotes the number of positive integers $d \leq X$ of the form

$$d = p_1 p_2 p_3 = 4m^{2\ell} - n^2 \,,$$

where $p_1 \in \mathcal{P}_1$ and $p_2, p_3 \in \mathcal{P}_2$ are distinct and satisfy

$$p_1 \leq x \,, p_1 \in \mathcal{P}_1 \;; \quad x^{1/4} < p_2 \leq x^{1/2} \,, x^{3/4} < p_2 p_3 \leq x \quad \text{and} \quad p_2, p_3 \in \mathcal{P}_2 \,.$$

with $x = \sqrt{X}$, then

$$R_d(X) \gg \frac{X^{1/2+1/(2\ell)}}{\log^2 X} \,.$$

- $4m^\ell = p_1 + p_2 p_3$

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Thank you for your attention!