On the k-free values of the polynomial $xy^k + C$

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Introduction

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$$N_f(H,k) := \#\{n \leq H : f(n) \text{ is } k\text{-free}\}.$$

- (Ricci, 1933) $k \geq d$, d is the degree of $f(x)$;
- (Hooley, 1967) $k = d - 1$;
- (Browning, 2011) $k \geq (3d+1)/4$.

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Positive lower bound of the expected order of magnitude for general inhomogeneous polynomials:

- (Hooley, 2009) $k \geq 3d/4 - 1$;
- (Browning, 2011) $k > 39d/64$.

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Theorem(Hooley, 1976)

For the irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree d there exist constants $\delta = \delta(d)$, $0 < \delta < 1$, and $c_f > 0$, such that the following asymptotic formula holds:

$$N_f(H, d-1) = c_f H + \mathcal{O}\left(\frac{H}{\log^{\delta} H}\right).$$

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Can we get a positive power saving in the error term?

k-free values of $xy^k + C$

Theorem (L.,2015)

Let $f(x,y) = xy^k + C \in \mathbb{Z}[x,y]$ for $k \geq 2$ and $C \neq 0$. Let $S(H)$ count the k-free values of $f(x,y)$ when $1 \leq x,y \leq H$. Then, for some real $\delta = \delta(k) > 0$, we have

$$S(H) = c_{f,k} H^2 + \mathcal{O}\left(H^{2-\delta}\right),$$

where

$$c_{f,k} = \prod_p \left(1 - \frac{\rho(p^k)}{p^{2k}}\right)$$

and

$$\rho(m) = \#\left\{(\mu,\nu) \in (\mathbb{Z}/m\mathbb{Z})^2 : \quad m \mid f(\mu,\nu)\right\}.$$

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Actually $\delta(k) = 1/(7k)$ for $k \geq 3$ and for $k = 2$ the error term is a bit worse: $\mathcal{O}(H^{1.979})$.

Proof of Theorem 1

Use the identity

$$\sum_{d^k|n} \mu(d) = \begin{cases} 1 & , n \text{ is } k\text{-free;} \\ 0 & , \text{ otherwise.} \end{cases}$$

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Then we can write

$$\begin{aligned} S(H) &= \sum_{\substack{1\le x,y\le H \\ xy^k+C-k\text{-free}}} 1 = \sum_{1\le x,y\le H} \sum_{d^k|f(x,y)} \mu(d) \\ &= \sum_{1\le d\ll H^{1+1/k}} \mu(d) S(d^k, H), \end{aligned}$$

where

$$S(d^k, H) = \sum_{\substack{1\le x,y\le H \\ d^k|f(x,y)}} 1\,.$$

Proof of Theorem 1 (Cont.)

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For the interval $H^{1-\delta} < d \le H^{1+\delta}$ of the sum S_3 we further bound trivially the contributions when $1 \le x \le H^{\eta}$ for small enough $\eta > 0$ and $1 \le y \le H^{1-2\delta}$.

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We finally need to estimate the sum

$$\tilde{S}_3 = \sum_{H^{1-\delta} < d \le H^{1+\delta}} \sum_{\substack{H^{\eta} < x \le H \\ H^{1-2\delta} < y \le H}} \sum_{xy^k + C = ad^k} 1\,.$$

Counting solutions

In other words we need to count the solutions of the Diophantine equation

$$xy^k - ad^k = -C$$

for

$$\begin{aligned} H^{1-\delta} < \quad & d \quad \le H^{1+\delta}, \\ H^{\eta} < \quad & x \quad \le H, \\ H^{1-2\delta} < \quad & y \quad \le H, \\ 1 \le \quad & a \quad \le H^{k+1}/d^k \le H^{1+k\delta}. \end{aligned}$$

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Write

- $x \sim X$ when $X < x < 2X$;
- $x \asymp X$ when there are constants $A, B > 0$, independent of X, such that $AX \leq |x| \leq BX$.

Theorem of Reuss, 2014

Let $D, Y, z > 1$ and $\varepsilon > 0$. Let k, ℓ, h be integers such that $1 \le \ell < k$ and $h \neq 0$. Let

$$\mathcal{N}(z; D, Y) := \left\{ (d, y, a, x) \in \mathbb{N}^4 : d \sim D, y \sim Y, a \sim A, x \sim X, x^\ell y^k - a^\ell d^k = h \right\},$$

where $X^\ell Y^k = A^\ell D^k = z$. Let $M > 1$ be defined by

$$\log M = \frac{9}{8} \frac{\log(DY) \log(AX)}{\log z},$$

and suppose the following conditions are satisfied:

1. $\log(DY) \asymp \log(AX) \asymp \log z$;
2. $\ell \ge 2$, or $DY \gg_{k,\ell,h} z^{1/k}$.

Then, if z is large enough in terms of ε,

$$\mathcal{N}(z; D, Y) \ll_{\varepsilon,k,\ell,h} z^\varepsilon \min \left\{ (DYM)^{1/2} + D + Y, (AXM)^{1/2} + A + X \right\}.$$

Proof of Theorem 1(Cont.)

From Reuss' theorem, and choosing $\delta = 1/(7k)$, we get

$$\tilde{S}_3 \ll H^{\varepsilon + G_k},$$

where

$$G_k = \left(1 + \frac{1}{14k}\right)\left(1 + \frac{9 \cdot 17}{7 \cdot 8}\frac{1}{k+1}\right).$$

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We have $G_k < 2$ for any $k \geq 2$.
For $k = 2$ the error term $H^{\varepsilon + G_k}$ is the largest, for $k \geq 3$ the expression G_k is smaller than $2 - 1/(7k)$.

Prime arguments

Conjecture (Erdős, 1953)

For the irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of degree d with *no fixed* $(d-1)$*-th power prime divisor* the set $f(\mathbb{P}) = \{f(p), p - \text{ prime}\}$ contains infinitely many $(d-1)$-free values.

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We resolve the conjecture for our specific polynomial $xy^k + C$.

Theorem for prime arguments

Theorem (L.,2015)

Let $f(x,y) = xy^k + C \in \mathbb{Z}[x,y]$ *for* $k \geq 2$ *and* $C \neq 0$. *Let* $S'(H)$ *count the* k*-free values of* $f(p,q)$ *for prime numbers* $1 < p, q \leq H$. *Then, for any real* $K > 2$, *we have the asymptotic formula*

$$S'(H) = c'_{f,k}\pi(H)^2 + \mathcal{O}\left(\frac{H^2}{(\log H)^K}\right),$$

where

$$c'_{f,k} = \prod_p \left(1 - \frac{\rho'(p^k)}{\varphi(p^k)^2}\right)$$

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$$\rho'(m) = \#\left\{(\mu,\nu) \in (\mathbb{Z}/m\mathbb{Z})^2 : \quad (\mu,m) = (\nu,m) = 1 \text{ and } m \mid f(\mu,\nu)\right\}.$$

Proof of Theorem 2

We split the sum $S'(H)$ into three parts:

$$S'(H) = \sum_{1 \le d \ll H^{1+1/k}} \mu(d) S'(d^k, H) = S'_1 + S'_2 + S'_3,$$

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- $H^{1/k} < d \ll H^{1+1/k}$ in S'_3 - use the bounds from Theorem 1 .

Speculations

The determinant method estimates

Reuss' equation

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General equation

$x^k y^\ell - a^{d-1} b = h$ for $k \geq \ell \geq 1$ and $k + \ell = d,\ h \neq 0\,?$

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Then the power-saving in the error term can solve the two-dimensional Erdős' conjecture!

Thank you for your attention!