# On estimating divisor sums over quadratic polynomials

ELAZ 2016, Strobl am Wolfgangsee

Kostadinka Lapkova

Alfréd Rényi Institute of Mathematics, Budapest (currently)
Graz University of Technology (from next week on)

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# Introduction

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- $f$ - linear polynomial : classical problem, Dirichlet hyperbola method;
- $f$ - quadratic polynomial : Dirichlet hyperbola method still works;
- $\deg(f) \geq 3$ : the hyperbola method does not work any more, error term larger than the main term.

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for the sum $T(f, N)$, when $f$ is a quadratic polynomial.
These explicit upper bounds have applications in certain Diophantine sets problems.

# Irreducible $f$ : asymptotic formulae for $T(f;N)$

- Scourfield, 1961 (first published):

$$T(an^2 + bn + c; N) \sim C_1(a, b, c) N \log N, \text{when } N \to \infty .$$

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- Hooley, 1963: $f(n)=n^2+c\,;$
- McKee, 1995, 1999: $f(n)=n^2+bn+c\,.$

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More precise work on the coefficient $C_1$ and the error terms for polynomials of special type:

- Hooley, 1963: $f(n) = n^2 + c\,;$
- McKee, 1995, 1999: $f(n) = n^2 + bn + c\,.$
- Corollary:

$$T(n^2+1;N) = \frac{3}{\pi}N\log N + \mathcal{O}(N)\,.$$

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**Theorem (L,2016)**

*For any integer $N \ge 1$ we have*

$$T(n^2+1;N) = \sum_{n=1}^{N} \tau(n^2+1) < \frac{12}{\pi^2} N \log N + 4.332 \cdot N.$$

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*Let $f(n) = n^2 + 2bn + c \in \mathbb{Z}[n]$, such that $\delta := b^2 - c$ is non-zero and square-free, and $\delta \not\equiv 1 \pmod 4$. Assume also that for $n \geq 1$ the function $f(n)$ is positive and non-decreasing. Then for any integer $N \geq 1$ there exist positive constants $C_1, C_2$ and $C_3$, such that*

$$\sum_{n=1}^{N} \tau(n^2 + 2bn + c) < C_1 N \log N + C_2 N + C_3.$$

# Irreducible $f$ : explicit upper bound for $T(f;N)$

**Theorem (L, 2016, cont.)**

*Let $A$ be the least positive integer such that $A \geq \max(|b|, |c|^{1/2})$, let $\xi = \sqrt{1+2|b|+|c|}$ and $\varkappa = g(4|\delta|)$ for $g(q) = 4/\pi^2\sqrt{q}\log q + 0.648\sqrt{q}$. Then we have*

$$C_1 = \frac{12}{\pi^2}\left(\log\varkappa + 1\right),$$

$$C_2 = 2\left[\varkappa + (\log\varkappa + 1)\left(\frac{6}{\pi^2}\log\xi + 1.166\right)\right],$$

$$C_3 = 2\varkappa A.$$

# Reducible $f$: asymptotic formulae for $T(f;N)$

- Ingham, 1927: For a fixed positive integer $k$

$$\sum_{n=1}^{N} \tau(n)\tau(n+k) \sim \frac{6}{\pi^2}\sigma_{-1}(k)N\log^2 N \text{ , as } N \to \infty \, ;$$

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- Dudek, 2016:

$$T(n^2-1;N) = \sum_{n \le N} \tau(n^2-1) \sim \frac{6}{\pi^2} N \log^2 N \text{ , as } N \to \infty\,.$$

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**Theorem (L, 2016)**

*Let $b < c$ be integers with the same parity. Then we have the asymptotic formula*

$$\sum_{c<n\le N} \tau\left((n-b)(n-c)\right) \sim \frac{6}{\pi^2} N \log^2 N \,,$$

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Proof: the method of Dudek with some information from [Hooley, 1958] about the number of solutions of quadratic congruences and the representation of a Dirichlet series.

# Reducible $f$: explicit upper bound for $T(f;N)$

If we write

$$T(n^2-1;N) \le C_1 N \log^2 N + \dots ,$$

we have

- Elsholtz, Filipin and Fujita, 2014 : $C_1 \le 2$;
- Trudgian, 2015 : $C_1 \le 12/\pi^2$;
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Bliznac and Filipin, 2016 :

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Remark: C.-Tr. and Bl.-F. method is different than ours.

# Reducible $f$: explicit upper bound for $T(f;N)$

## Theorem (L, 2016)

*Let $b < c$ be integers with the same parity and $\delta = (b-c)^2/4$ factor as $\delta = 2^{2t'}\Omega^2$ for integers $t' \geq 0$ and odd $\Omega \geq 1$. Assume that $\sigma_{-1}(\Omega) \leq 4/3$. Let $c^* = \max(1, c+1)$ and $X = \sqrt{f(N)}$. Then for any integer $N \geq c^*$ we have*

$$\sum_{c^* \leq n \leq N} \tau\left((n-b)(n-c)\right) < 2N\left(\frac{3}{\pi^2}\log^2 X + \left(\frac{6}{\pi^2} + C(\Omega)\right)\log X\right) + 2C(\Omega)N + 2X\left(\frac{6}{\pi^2}\log X + C(\Omega)\right),$$

*where*

$$C(\Omega) = 2\sum_{d|\Omega} \frac{1}{d}\left(2\sigma_0(\Omega/d) - 1.749 \cdot \sigma_{-1}(\Omega/d) + 1.332\right).$$

# Reducible $f$: explicit upper bound for $T(f;N)$

In a standard way we obtain

$$\sum_{n\le N}\tau(f(n)) \le 2N \sum_{d\le\sqrt{f(N)}} \rho_\delta(d)/d + 2 \sum_{d\le\sqrt{f(N)}} \rho_\delta(d)\,,$$

where

$$\rho_k(d) := \#\left\{0\le x<d\ :\quad x^2\equiv k \pmod{d}\right\}$$

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On the other hand the condition

$$\sigma_{-1}(\Omega)=\sum_{d|\Omega}\frac{1}{d}\leq 4/3$$

is fulfilled for example for $\Omega=1$, i.e. $\delta=2^{2t'}$ for integer $t'\geq 0$. These are the cases

$$f(n)=n^2-2^{2t'}\,.$$

# Reducible $f$: explicit upper bound for $T(f;N)$

A corollary of the previous theorem:

**Theorem (L, 2016)**

*For any integer $N \geq 1$ we have the following claims:*

i) *For any integer $t' \geq 0$ we have*

$$\sum_{\lambda \leq N} \frac{\rho_{2^{2t'}}(\lambda)}{\lambda} < \frac{3}{\pi^2} \log^2 N + 2.774 \cdot \log N + 2.166 \,.$$

ii)

$$\sum_{n=1}^{N} \tau(n^2 - 1) < N \left( \frac{6}{\pi^2} \log^2 N + 5.548 \cdot \log N + 4.332 \right) \,.$$

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Remark: This claim recreates the right main term as in the lemmas of Cipu-Trudgian for $f(n) = n^2 - 1$ and Bliznac-Filipin for $f(n) = n^2 - 4$.

# Proofs: the convolution method for $\rho_\delta(d)$

For **irreducible** $f(n) = n^2 + 2bn + c$ we have

**Lemma**

*Let* $\delta = b^2 - c$ *be square-free,* $\delta \not\equiv 1 \pmod 4$ *and* $\chi(n) = \left(\frac{4\delta}{n}\right)$ *for the Jacobi symbol* $\left(\frac{\cdot}{\cdot}\right)$. *Then*

$$\rho_\delta = \mu^2 * \chi \,.$$

# Proofs: the convolution method for $\rho_\delta(d)$

## The convolution method

Estimate the sums

$$\sum_{d\le x} \rho_\delta(d) = \sum_{l\le x} \mu^2(l) \sum_{m\le x/l} \chi(m)\,,$$

$$\sum_{d\le x} \rho_\delta(d)/d = \sum_{l\le x} \mu^2(l)/l \sum_{m\le x/l} \chi(m)/m$$

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For example

- $\sum_{m\leq x/l} \chi(m)$ : effective Pólya-Vinogradov inequality;
- $\sum_{l\leq x} \mu^2(l)/l$ : effective inequality from [Ramaré, 2016].

# Proofs: the convolution method for $\xi_\delta(d)$

For **reducible** $f(n) = (n-b)(n-c)$ with $\delta = (b-c)^2/4 = 2^{2t'}\Omega^2$ for odd $\Omega \geq 1$ the function $\rho_\delta$ can be expressed as a more complicated sum over $d \mid \Omega$ of functions

$$\xi_d = \mu^2 * \chi_d \, ,$$

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# Proofs: the convolution method for $\xi_\delta(d)$

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where $\chi_d$ is the principal character modulo $2\Omega/d$.
**Hooley, 1958** : Analyses on the Dirichlet series $\sum_{\lambda=1}^{\infty} \rho_n(\lambda)/\lambda^s$ for a general positive integer $n$.

# Applications for $D(m)$-sets

For integer $m \neq 0$ a set of $n$ positive integers $\{a_1, \ldots, a_n\}$ is called a

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$a_i a_j + m$ is a perfect square for all $i, j$ with $1 \leq i < j \leq n$.

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Application of explicit upper bounds of $T(f(n); N)$:

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- $T(n^2 - 4^{t'}; N)$ : $D(4^{t'})-$sets for $t' \geq 1$ [Bliznac-Filipin, 2016 for D(4)-quintuples];

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- $T(n^2 + k; N)$ : $D(-k)-$sets for $k \neq 0$.

Thank you for your attention!